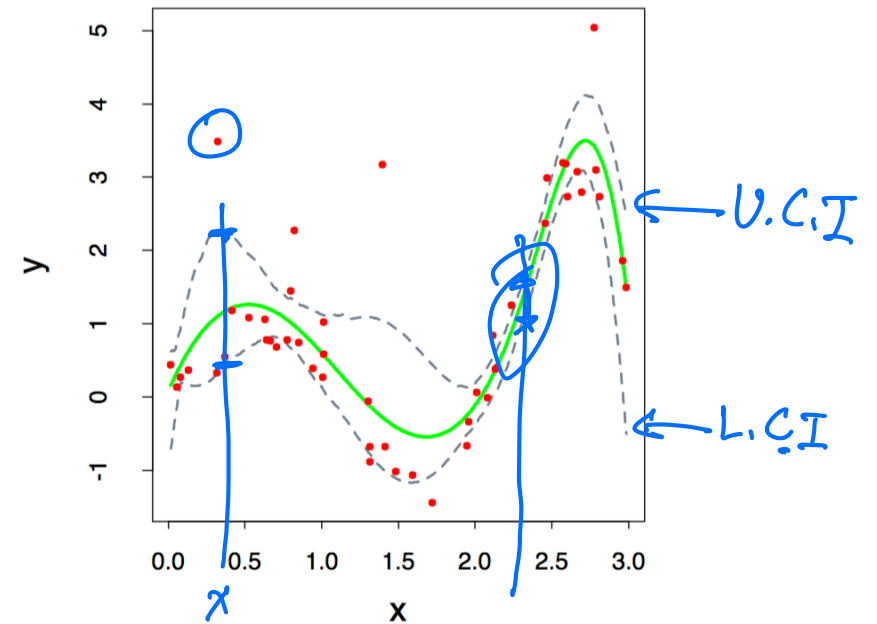
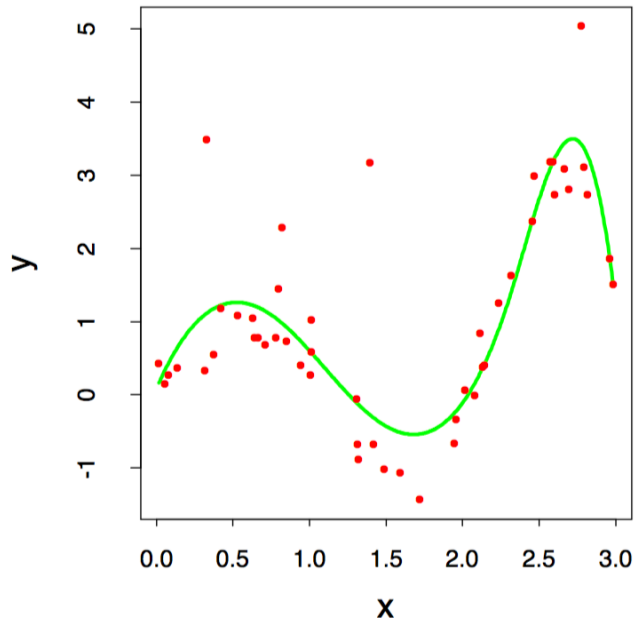


Lecture 10:

Bootstrap and Kernel Methods





Bootstrap

- Measuring uncertainty of your model



Confidence interval for statistics $\hat{\theta} = g(\mathcal{D})$

- **Consider a simple statistical problem:** mean estimation given access to an oracle that gives fresh samples, and we are interested in estimating mean from n samples
- We can first collect one dataset $\mathcal{D}_n = \{z_1, \dots, z_n\} \in \mathbb{R}^n$ sampled i.i.d. from that distribution
- And compute the mean estimate using our estimator of interest, say,

$$\mathcal{R} \Rightarrow \boxed{\hat{\theta}} \leftarrow g(\mathcal{D}_n) = \frac{1}{n} \sum_{i=1}^n z_i$$

empirical mean

- **Now consider a follow-up question:** we are interested in estimating how **uncertain** we are about this estimate. This can be done using **confidence intervals**.
- This course is not a statistics course, so we do not go into a great detail on confidence interval. We will only be using it to learn **bootstrapping**.

- We have a mean estimate using our estimator of interest, say,

$$\hat{\theta} \leftarrow g(\mathcal{D}_n) = \frac{1}{n} \sum_{i=1}^n z_i.$$

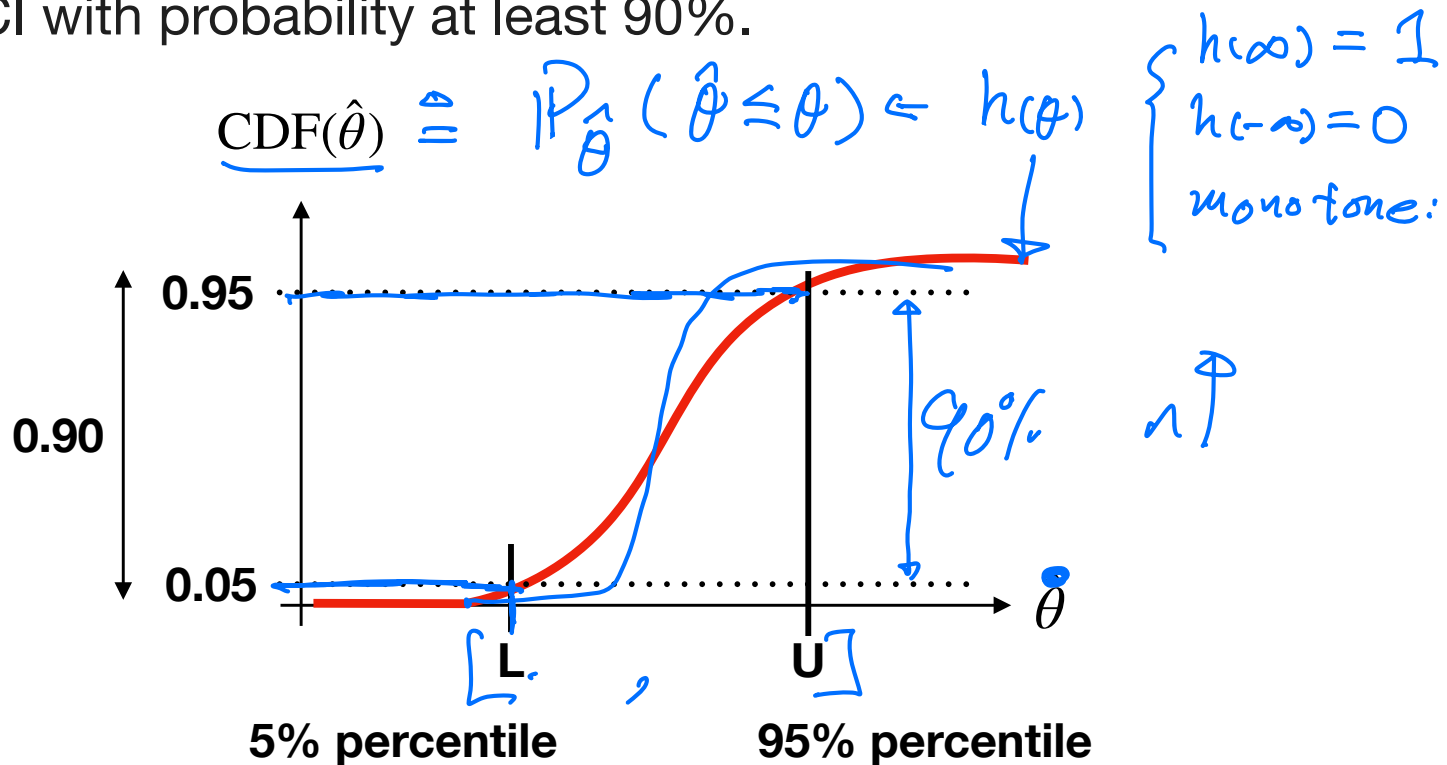
Handwritten notes: "random" with arrows pointing to $\hat{\theta}$ and \mathcal{D}_n .

and an oracle sampler from some distribution $P_Z \sim z_i$.

- If we know the distribution of our estimate $\hat{\theta}$, then one would output a, for example, 90% **confidence interval** of

$$CI = [L, U]$$

- The main idea is to ensure that the actual mean, say θ^* , of the distribution P_Z falls in the CI with probability at least 90%.



Confidence interval for statistics $\hat{\theta} = g(\mathcal{D})$

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$$\hat{\theta} \leftarrow g(\mathcal{D}_n) = \frac{1}{n} \sum_{i=1}^n z_i$$

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$CDF_n(\hat{\theta}_n)$

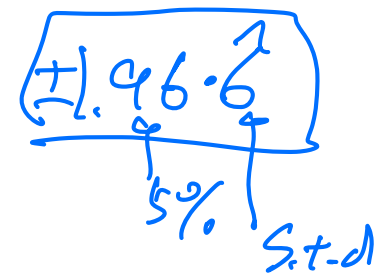
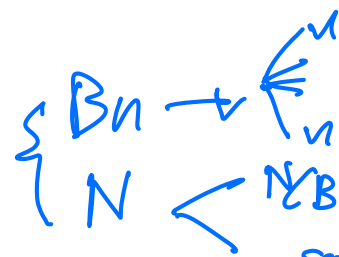
- The problem is that we typically do not know the distribution (i.e., CDF) of $\hat{\theta}$, so what can we do?
 - This is easy if we have an oracle and can keep sampling new datasets \mathcal{D}_n so that we can get multiple “samples” of $\hat{\theta}_n$ to estimate its distribution.

Confidence interval with oracle access to sampler

- “sample” many fresh datasets $\{\mathcal{D}_n^{(1)}, \dots, \mathcal{D}_n^{(B)}\}$, each of size n
- compute estimates $\{\hat{\theta}^{(1)} = g(\mathcal{D}_n^{(1)}) \dots, \hat{\theta}^{(B)} = g(\mathcal{D}_n^{(B)})\}$ for each dataset

CDF of an unknown True distribution

$$F_{\theta}(\theta) = P_{\mathcal{D}}(g(\mathcal{D}) \leq \theta)$$



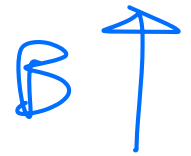
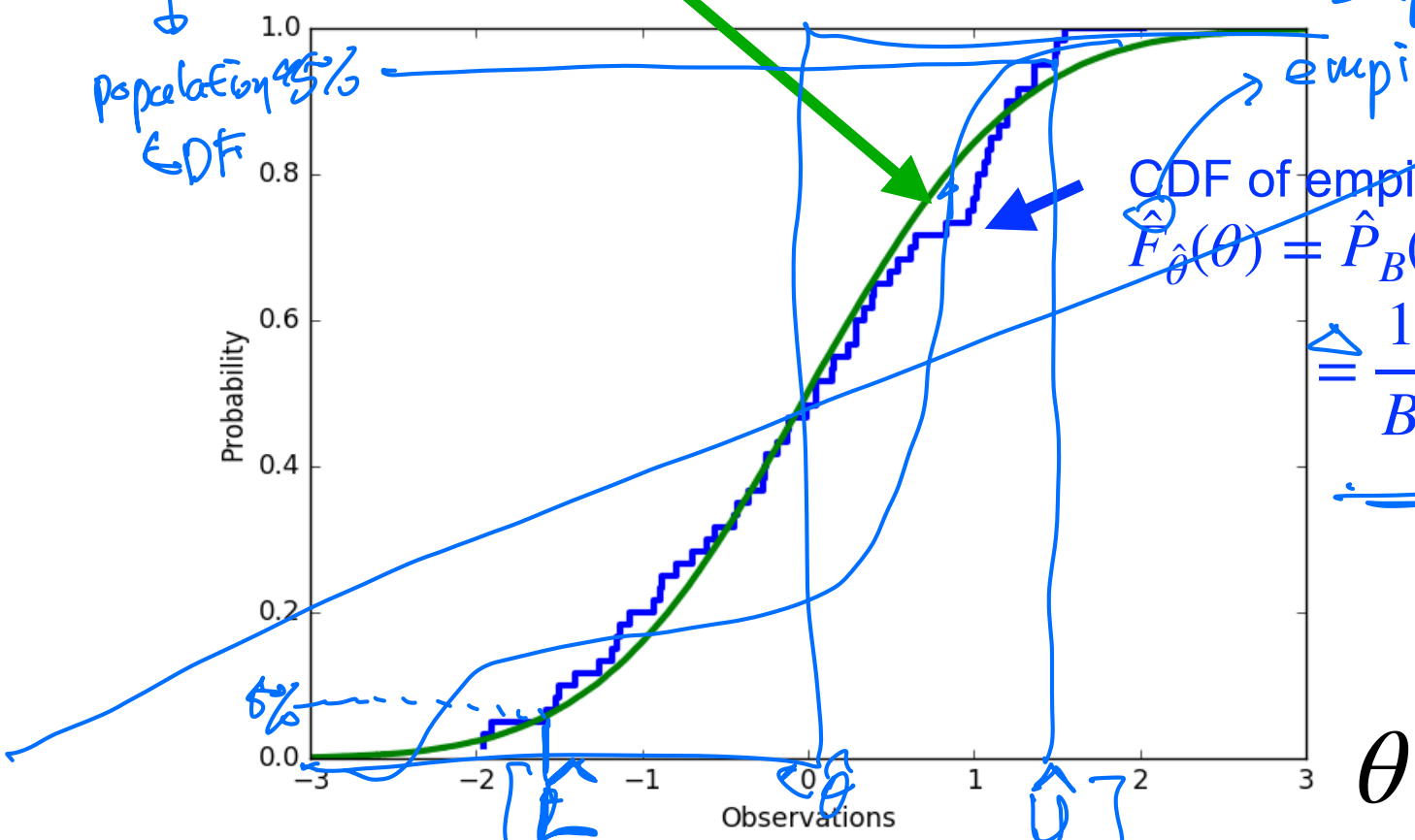
Population 95% CDF

empirical CDF

CDF of empirical observed distribution

$$\hat{F}_{\hat{\theta}}(\theta) = \hat{P}_B(g(\mathcal{D}) \leq \theta)$$

$$\hat{=} \frac{1}{B} \sum_{b=1}^B \mathbf{I}\{\hat{\theta}^{(b)} \leq \theta\}$$



Set-up for regression/classification

We have a dataset of n samples drawn i.i.d. with (CDF) F_Z

$$\mathcal{D} = \{z_1, \dots, z_n\} \sim F_Z$$

↳ $(x_1, y_1) \dots (x_n, y_n)$.

We compute a *statistic* of the data to get $g(\mathcal{D})$

Goal: Estimate uncertainty around $g(\mathcal{D})$.

In other words, what's the distribution of $g(\mathcal{D})$ over different random samples of \mathcal{D} ?

model-
Parameter $\theta(\mathcal{D})$

$$y = f_{\theta(\mathcal{D})}(x) \in \mathbb{R}$$

↑

Set-up for regression/classification

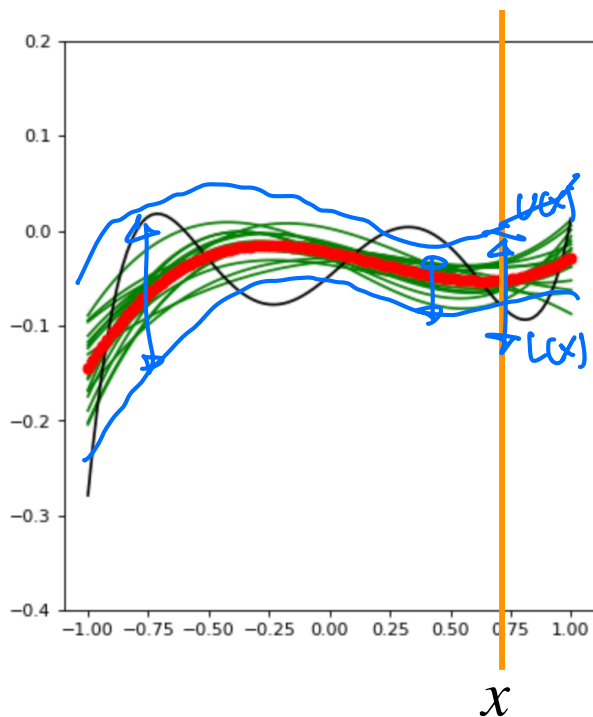
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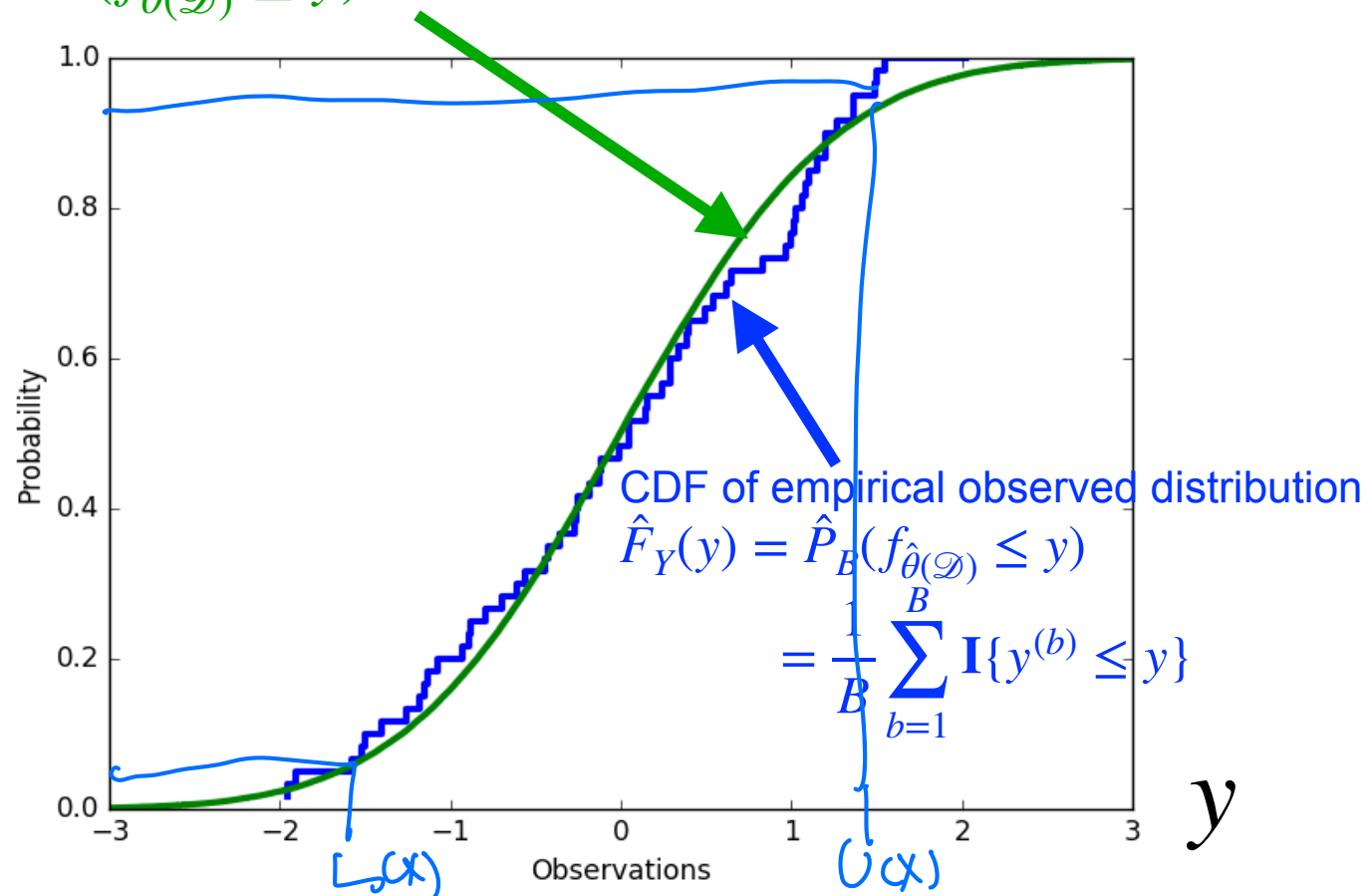
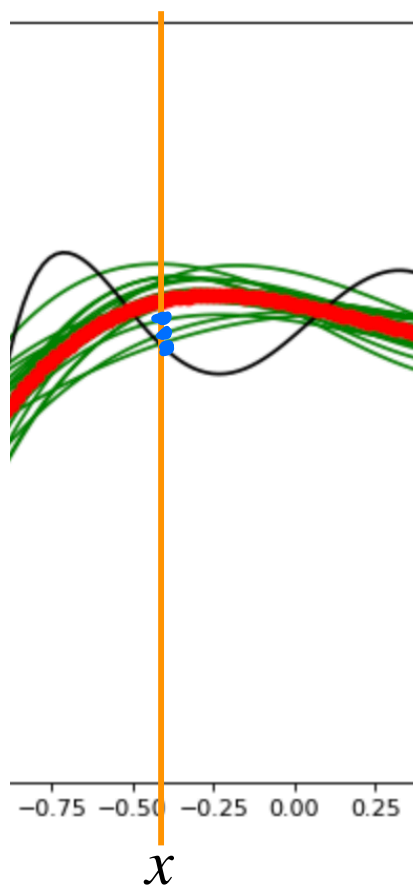
For example, in the example of regression, we might care about $g(\mathcal{D}) = f_{\hat{\theta}(\mathcal{D})}(x) = y$ where $\hat{\theta}(\mathcal{D})$ is the model parameter trained on the train-set \mathcal{D} , and $f_{\hat{\theta}(\mathcal{D})}(x)$ is our prediction for input x .

Confidence interval with oracle access to sampler

- “sample” many fresh datasets $\{\mathcal{D}^{(1)}, \dots, \mathcal{D}^{(B)}\}$, each of size n
- train models $\{\hat{\theta}^{(1)} \dots, \hat{\theta}^{(B)}\}$ for each dataset
- compute the prediction $\{y^{(1)} = f_{\hat{\theta}^{(1)}}(x), \dots, y^{(B)} = f_{\hat{\theta}^{(B)}}(x)\}$

CDF of an unknown True distribution

$$F_Y(y) = P(f_{\hat{\theta}(\mathcal{D})} \leq y)$$



Bootstrapping: doing something seemingly impossible



- **pull oneself up by one's bootstraps** — improve one's position by one's own efforts without help from others.
- In our context: A general method of calculating uncertainty estimates without any additional data. (Efron, 1979)

Bootstrap: Key idea

- **Goal:** Estimate uncertainty around an estimate $\hat{\theta} = g(\mathcal{D})$ from a dataset $\mathcal{D}_n = \{z_1, z_2, \dots, z_n\} \sim F_Z$
- This would be easy if we had multiple datasets \mathcal{D} drawn fresh from F_Z
- **Idea:** pretend that we have fresh draws from F_Z by **sampling with replacement** from \mathcal{D}
- Concretely,
 - for $b = 1, \dots, B$,

$$\mathcal{D}_n^{(b)} = \{z_1^{(b)}, \dots, z_n^{(b)}\}$$

Bootstrap: Key idea

- Goal: Estimate uncertainty around an estimate $\hat{\theta} = g(\mathcal{D})$ from a dataset $\mathcal{D}_n = \{z_1, z_2, \dots, z_n\} \sim F_Z$
- This would be easy if we had multiple datasets \mathcal{D} drawn fresh from F_Z
- Idea: pretend that we have fresh draws from F_Z by **sampling with replacement** from \mathcal{D}
- Concretely,
 - for $b = 1, \dots, B$,
 - draw n samples with replacement from \mathcal{D} , and call them $\mathcal{D}^{(b)} = \{z_1^{(b)}, \dots, z_n^{(b)}\} \sim \hat{F}_{Z,n}$
 - compute the estimate $\hat{\theta}^{(b)} \leftarrow g(\mathcal{D}^{(b)})$
 - use $\{\hat{\theta}^{(b)}\}_{b=1}^B$ to compute the confidence interval

Bootstrap: step 1: sampling with replacement

Real World

Bootstrap World

Real data \mathcal{D} gives you an estimate $\hat{\theta}$

Bootstrapping gives many estimates $\{\hat{\theta}^{(1)} \dots, \hat{\theta}^{(B)}\}$

$$\mathcal{D} = \{z_1, z_2, \dots, z_n\} \sim F_Z$$
$$\hat{\theta} \leftarrow g(\mathcal{D})$$

$$\mathcal{D}^{(1)} = \{z_1^{(1)}, \dots, z_n^{(1)}\} \sim \hat{F}_{Z,n}$$
$$\hat{\theta}^{(1)} \leftarrow g(\mathcal{D}^{(1)})$$

⋮

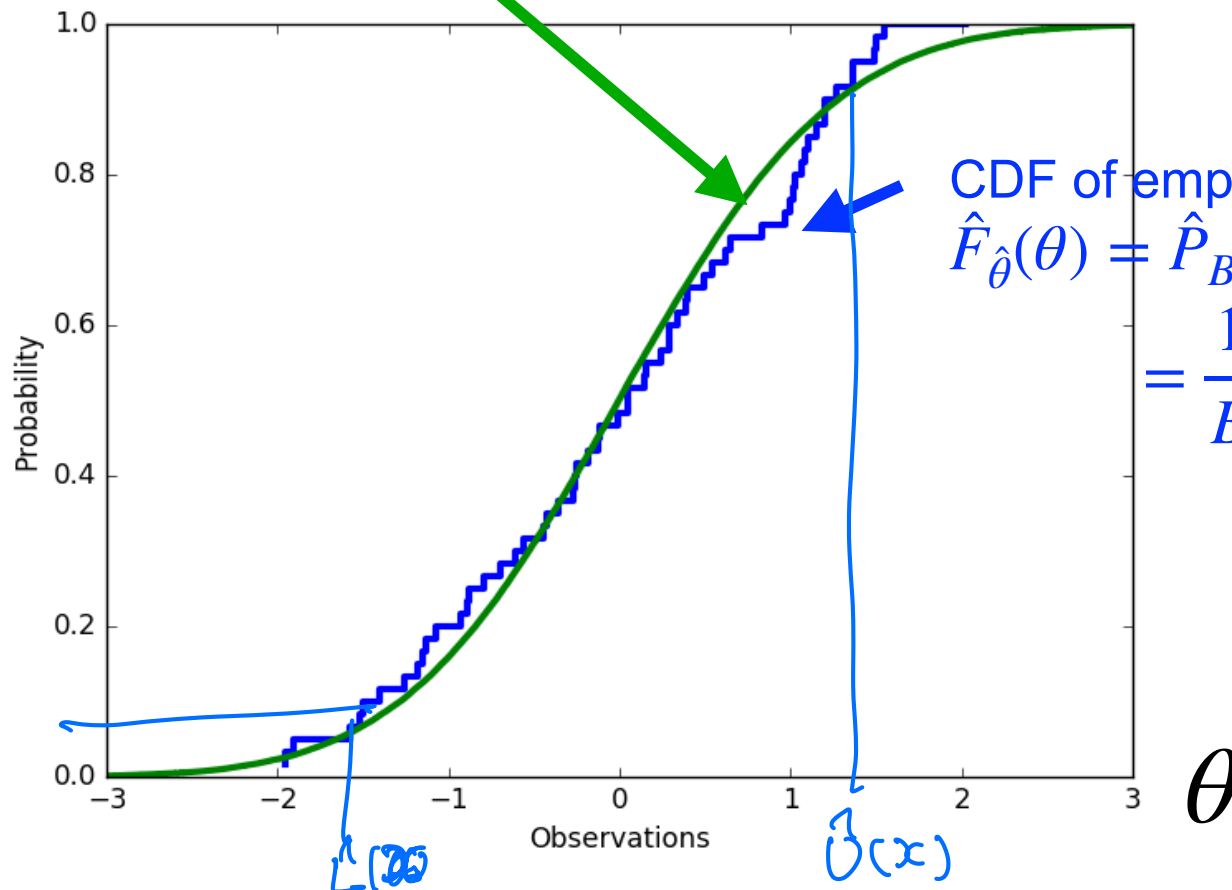
$$\mathcal{D}^{(B)} = \{z_1^{(B)}, \dots, z_n^{(B)}\} \sim \hat{F}_{Z,n}$$
$$\hat{\theta}^{(B)} \leftarrow g(\mathcal{D}^{(B)})$$

Bootstrap: step 2: confidence interval

- Bootstrap “sample” many fresh datasets $\{\mathcal{D}^{(1)}, \dots, \mathcal{D}^{(B)}\}$, each of size n
- compute estimates $\{\hat{\theta}^{(1)} = g(\mathcal{D}^{(1)}) \dots, \hat{\theta}^{(B)} = g(\mathcal{D}^{(B)})\}$ for each dataset

CDF of an unknown True distribution

$$F_{\hat{\theta}}(\theta) = P_{\mathcal{D}}(g(\mathcal{D}) \leq \theta)$$



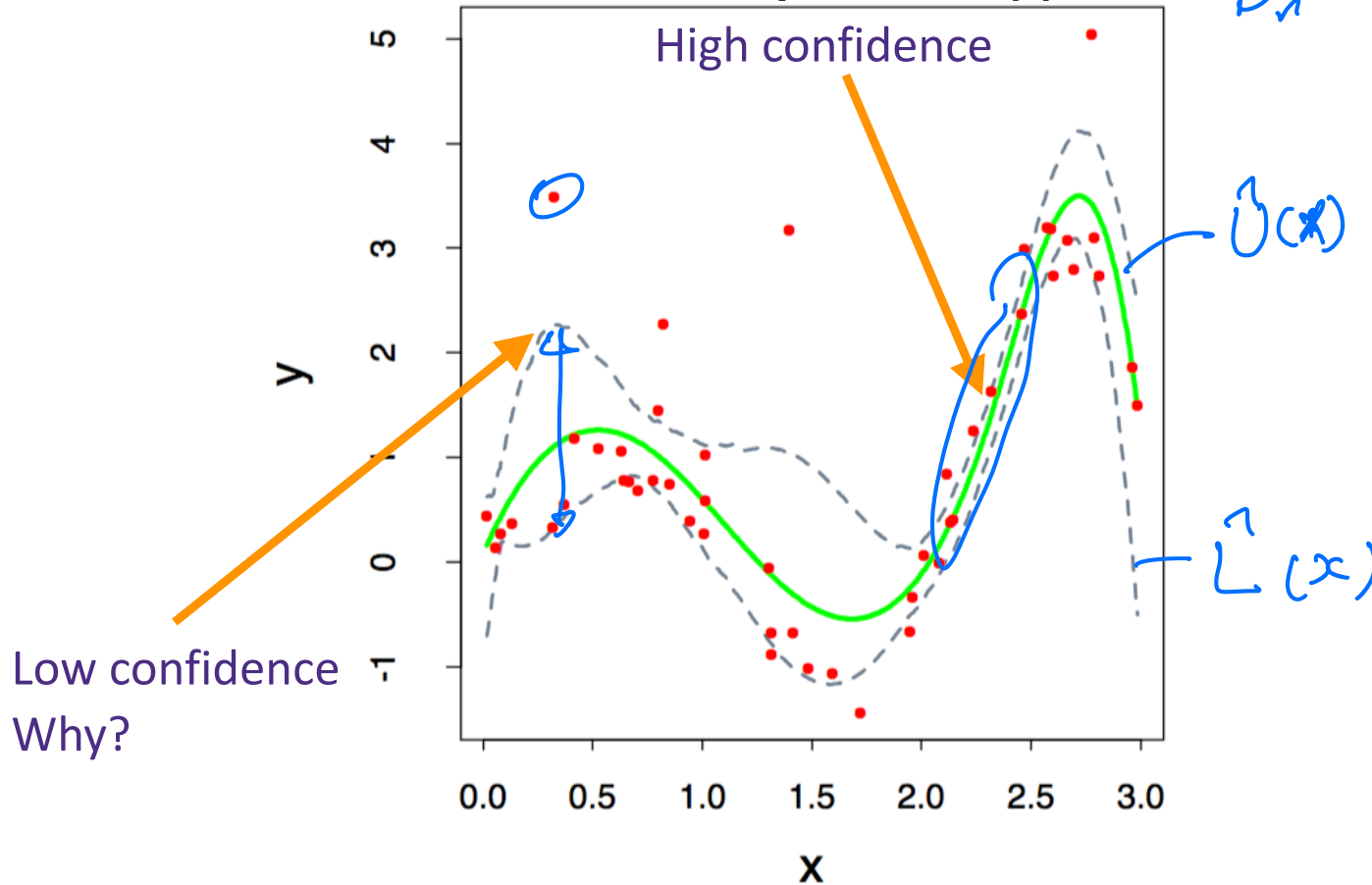
CDF of empirical observed distribution

$$\hat{F}_{\hat{\theta}}(\theta) = \hat{P}_B(g(\mathcal{D}) \leq \theta)$$

$$= \frac{1}{B} \sum_{b=1}^B \mathbf{I}\{\hat{\theta}^{(b)} \leq \theta\}$$

Example for 1-d regression

Example of 5% and 95% percentile curves for predictor $f(x)$

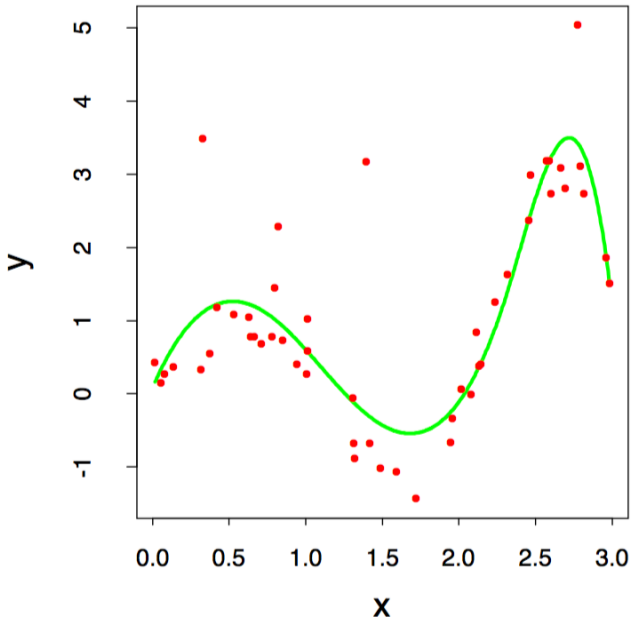


The green line is $f_{\mathcal{D}}(x)$, and the black dashed lines are the 5% and 95% percentiles at each x

bootstrap

n, B

training a single predictor

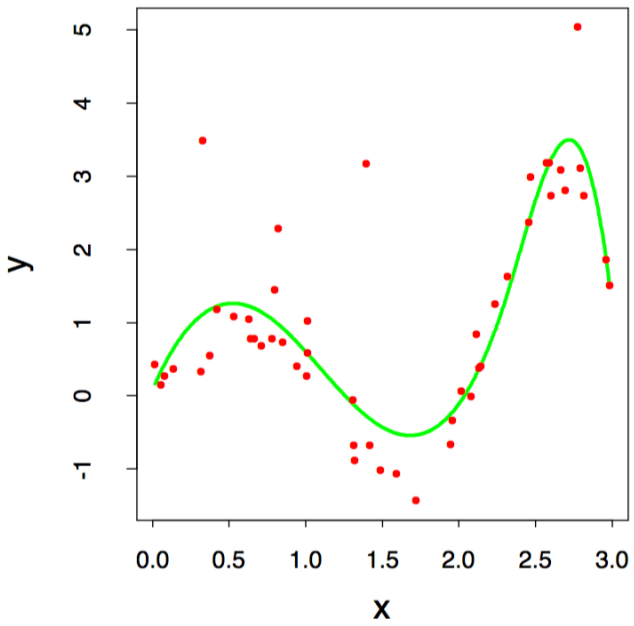


multiple bootstrapped predictors

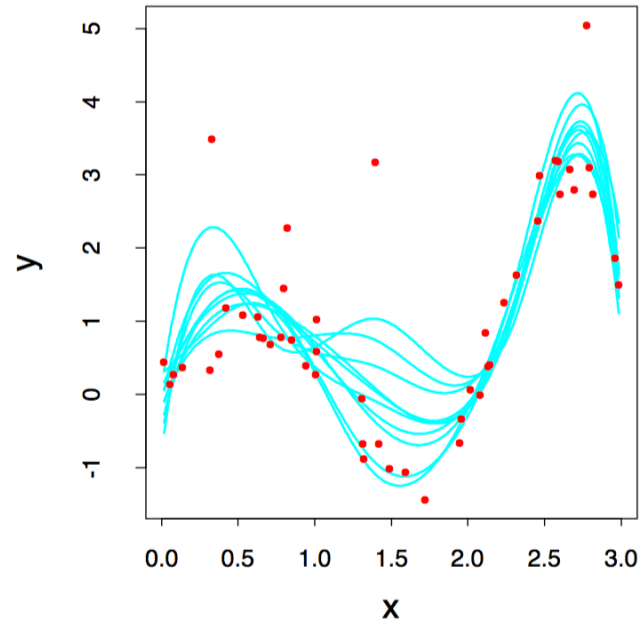
90% confidence interval

bootstrap

training a single predictor



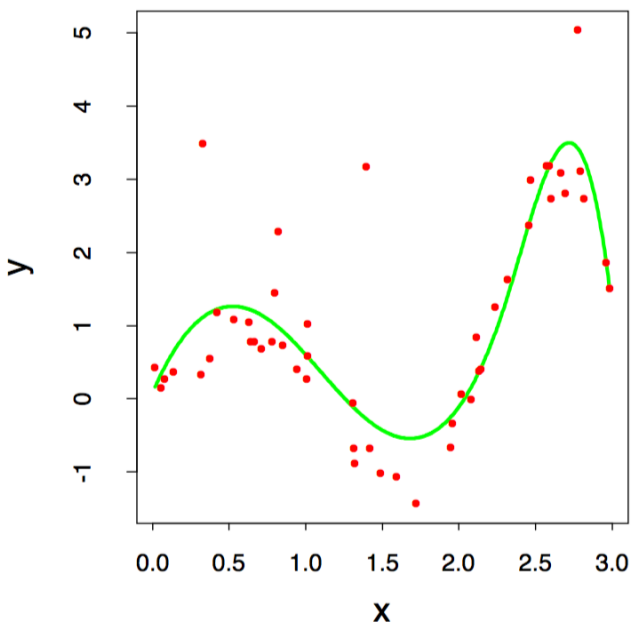
multiple bootstrapped predictors



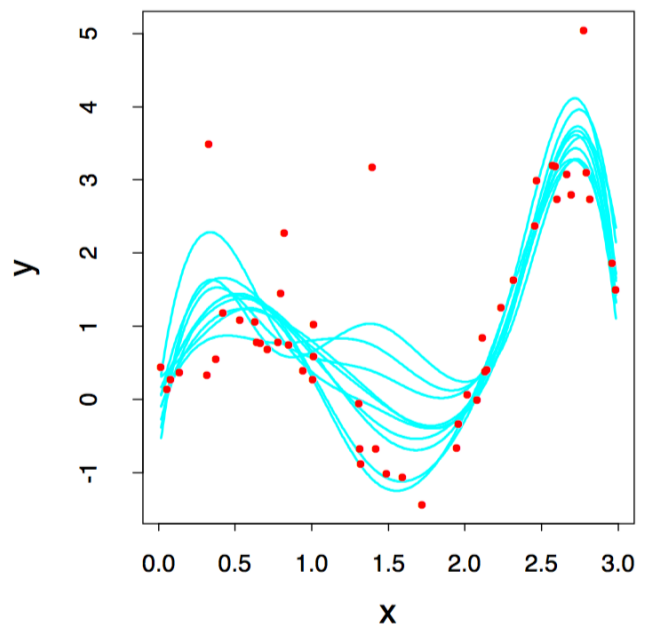
90% confidence interval

bootstrap

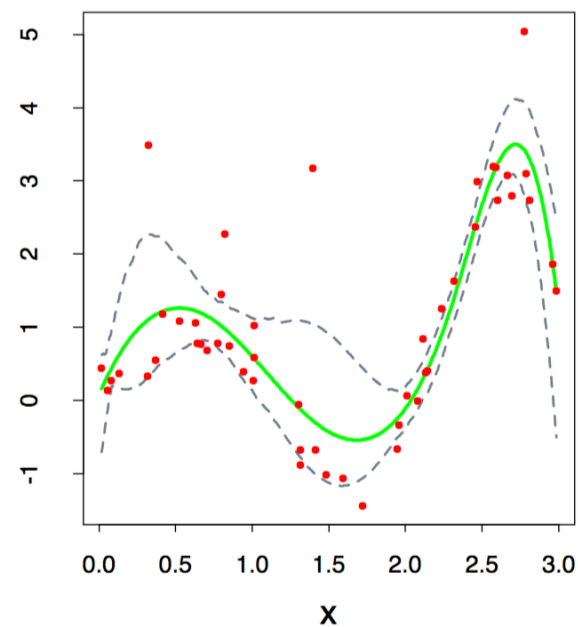
training a single predictor



multiple bootstrapped predictors



90% confidence interval



Takeaways

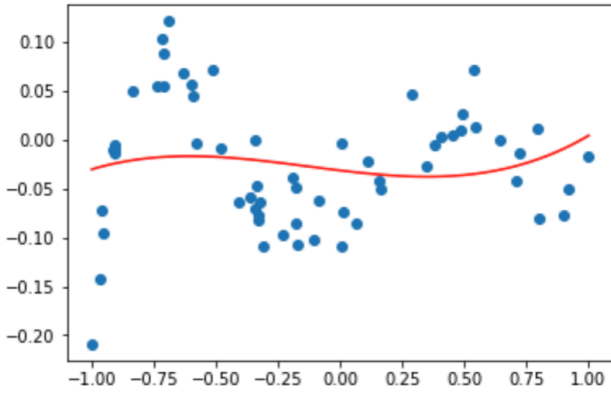
Advantages:

- Very simple to use and generally applicable – build a confidence interval around anything
- Appears to give meaningful results even when the amount of data is very small
- Very strong asymptotic theory (as number of examples goes to infinity)

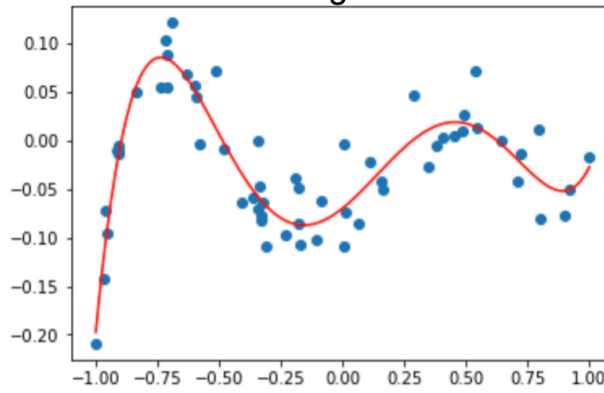
Disadvantages:

- Very few meaningful finite-sample guarantees
- Potentially computationally intensive
- Reliability hinges on test statistic and rate of convergence of empirical CDF to true CDF, which is unknown
- Poor performance on “extreme statistics” (e.g., the max)

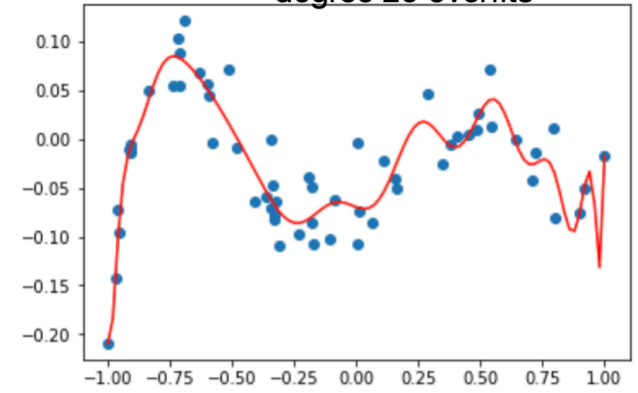
degree 3



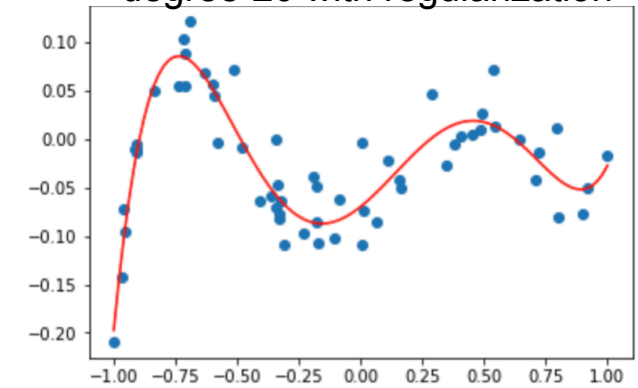
degree 5



degree 20 overfits



degree-20 with regularization



Kernel methods

- conquering extremely high-dimensional non-linear features

x

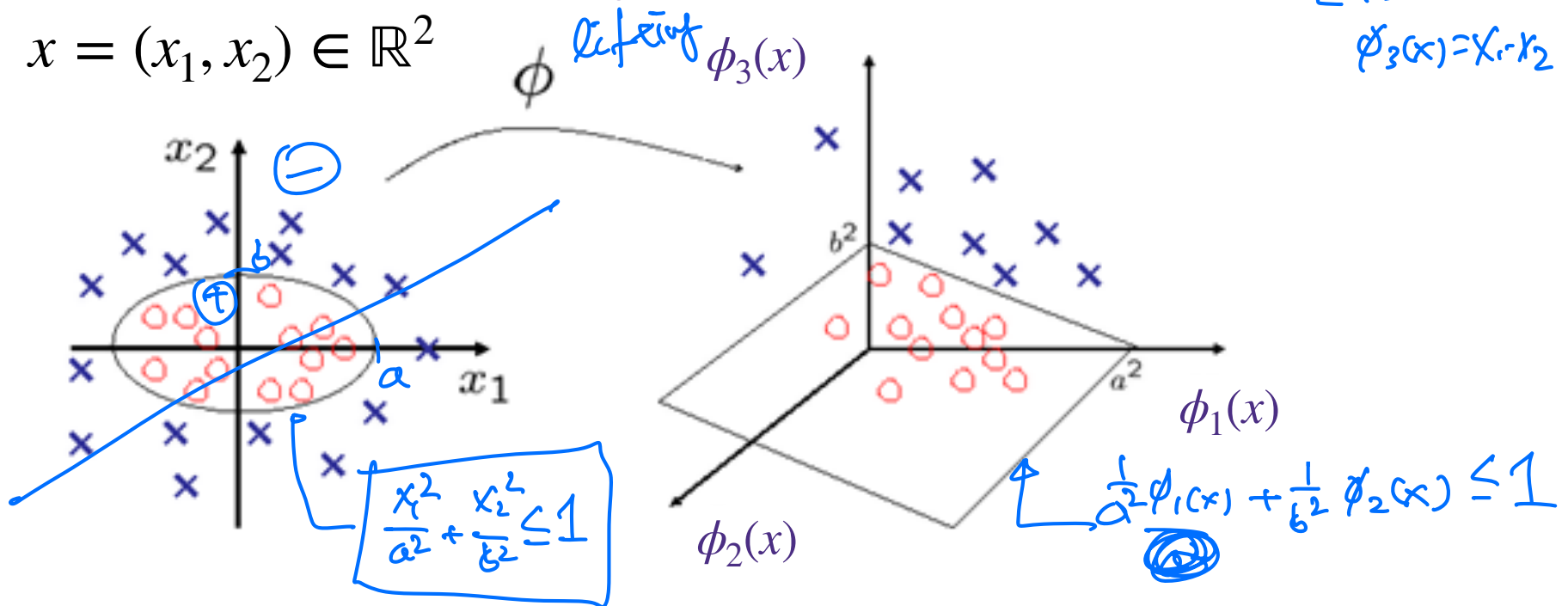
W

What if the data is not linearly separable?

- Use features, for example,

$$x = (x_1, x_2) \in \mathbb{R}^2$$

$$\phi(x) = \begin{bmatrix} \phi_1(x) = x_1^2 \\ \phi_2(x) = x_2^2 \\ \phi_3(x) = x_1 \cdot x_2 \end{bmatrix}$$



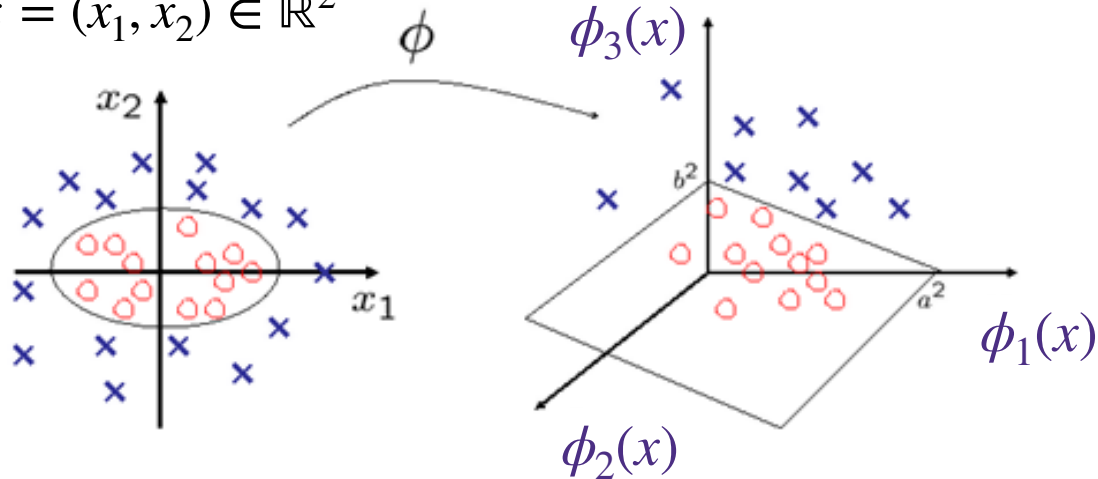
This data is not linearly separable

Can you suggest some features $\phi_1(x_1, x_2)$, $\phi_2(x_1, x_2)$, $\phi_3(x_1, x_2)$ such that this data is linearly separable in this 3-dimensional space?

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Can you suggest some features

$\phi_1(x_1, x_2)$, $\phi_2(x_1, x_2)$, $\phi_3(x_1, x_2)$ such that this data is linearly separable in this 3-dimensional space?

- Generally, in high dimensional feature space, it is easier to linearly separate different classes
- However, it is hard to know which feature map will work for given data
- So the rule of thumb is to use high-dimensional features and hope that the algorithm will automatically pick the right set of features
- What is wrong with this approach?

Creating Features

- Feature mapping $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^p$ maps original data into a rich and high-dimensional feature space (usually $d \ll p$)

For example, in $d=1$, one can use

$$\phi(x) = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_k(x) \end{bmatrix} = \begin{bmatrix} x \\ x^2 \\ \vdots \\ x^k \end{bmatrix} \quad \left. \vphantom{\begin{bmatrix} x \\ x^2 \\ \vdots \\ x^k \end{bmatrix}} \right\} \text{poly}$$

For example, for $d>1$,

one can generate vectors $\{u_j\}_{j=1}^p \subset \mathbb{R}^d$

and define features:

$$\phi_j(x) = \cos(u_j^T x) = \phi(x) \begin{bmatrix} \cos(u_1^T x) \\ \cos(u_2^T x) \\ \vdots \end{bmatrix}$$

$$\phi_j(x) = (u_j^T x)^2$$

$$\phi_j(x) = \frac{1}{1 + \exp(u_j^T x)}$$

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$$\phi_j(x) = \frac{1}{1 + \exp(u_j^T x)}$$

- Feature space can get really large really quickly!
- How many coefficients/parameters are there for degree- k polynomials for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$? $\rightarrow p \approx O\left(\binom{d+k}{k}\right) \approx d^k$
- At a first glance, it seems inevitable that we need memory (to store the features $\{\phi(x_i) \in \mathbb{R}^p\}_{i=1}^n$) and run-time that increases with p where $d < n \ll p$

How do we deal with high-dimensional lifts/data?

Complex: $\mathbb{R}^d \xrightarrow{K} \mathbb{R}^n$

A fundamental trick in ML: use kernels

A function $K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a **kernel** for a map $\phi(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^p$
if $K(x, x') \triangleq \phi(x)^T \phi(x') = \phi(x) \cdot \phi(x') = \langle \phi(x), \phi(x') \rangle$

This notation is for dot product (which is the same as inner product)

$$K(x_i, x_j) = k_{ij} \text{ easier } \phi(x_i), \phi(x_j)$$

- Main idea: computing inner products can be much more efficient than explicitly computing the (very high-dimensional) features

How do we deal with high-dimensional lifts/data?

A fundamental trick in ML: use kernels

A function $K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a **kernel** for a map $\phi(\cdot)$ if $K(x, x') \triangleq \phi(x)^T \phi(x') = \phi(x) \cdot \phi(x') = \langle \phi(x), \phi(x') \rangle$

This notation is for dot product (which is the same as inner product)

- So, if we can represent our
 - training algorithms and
 - decision rules for prediction
- as functions of dot products of feature maps (i.e. $\{\phi(x) \cdot \phi(x')\}$)
and if we can find a kernel for our feature map such that

$$K(x, x') = \phi(x)^T \phi(x')$$

then we can avoid explicitly computing and storing (high-dimensional) $\{\phi(x_i)\}_{i=1}^n$
and instead only work with the kernel matrix of training data $\{K(x_i, x_j)\}_{i,j \in \{1, \dots, n\}}$

[Homework 3 Problem A2 tests this skill]

Kernels are much more efficient to compute than features

- As illustrating examples, consider polynomial features of degree exactly k

- $\phi(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for $k = 1$ and $d = 2$, then $K(x, x') = x_1x_1' + x_2x_2'$

- $\phi(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \\ x_2x_1 \end{bmatrix}$ for $k = 2$ and $d = 2$, then $K(x, x') = (x^T x')^2$

$\phi(x)^T \cdot \phi(x')$
 $[x_1 \ x_2] [x_1' \\ x_2']$

$[x_1^2 \ x_2^2 \ x_1x_2 \ x_2x_1] \begin{bmatrix} x_1'^2 \\ x_2'^2 \\ x_1'x_2' \\ x_2'x_1' \end{bmatrix}$

4 terms

$$= x_1^2 x_1'^2 + x_2^2 x_2'^2 + 2x_1 x_2 x_1' x_2'$$

$$= (x_1 x_1' + x_2 x_2')^2$$

2 terms

ϕ
 $p = d^k \rightarrow \text{kernel}$
 $O(d)$ compute.

Kernels are much more efficient to compute than features

- As illustrating examples, consider polynomial features of degree exactly k

- $\phi(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for $k = 1$ and $d = 2$, then $K(x, x') = x_1x'_1 + x_2x'_2$

- $\phi(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \\ x_2x_1 \end{bmatrix}$ for $k = 2$ and $d = 2$, then $K(x, x') = (x^T x')^2$

- Note that for a data point x_i , **explicitly** computing the feature $\phi(x_i)$

takes memory/time $p = d^k \rightarrow d^k n$

- For a data point x_i , if we can make predictions by only computing the kernel, then computing $\{K(x_i, x_j)\}_{j=1}^n$ takes memory/time dn

- The features are **implicit** and accessed only via kernels, making it efficient

Examples of popular Kernels

- Polynomials of degree exactly k

$$K(x, x') = (x^T x')^k$$

- Polynomials of degree up to k

$$K(x, x') = (1 + x^T x')^k$$

- Gaussian (squared exponential) kernel
(a.k.a RBF kernel for Radial Basis Function)

$$K(x, x') = \exp\left(-\frac{\|x - x'\|_2^2}{2\sigma^2}\right)$$

- Sigmoid

$$K(x, x') = \tanh(\gamma x^T x' + r)$$

- All these kernels are efficient to compute, but the corresponding features are in high-dimensions

odd terms

$\rightarrow \phi : p = \infty$

$= \sum_{i=1}^{\infty} \dots$

Example: feature vs. kernel

- Ridge regression with feature map $\phi(\cdot) \in \mathbb{R}^p$

$$\hat{w}_t = \sum_{i=1}^n \alpha_i \phi(x_i)$$

Handwritten notes: $\alpha_1, \dots, \alpha_n$

- Solve for $\hat{w} = \arg \min_{w \in \mathbb{R}^p} \frac{1}{2} \sum_{i=1}^n (y_i - w^T \phi(x_i))^2 + \lambda \frac{1}{2} \|w\|_2^2$: Ridge

Handwritten notes: $w \in \mathbb{R}^p$, w^T , ϕ , p

- Slow when $p \gg d$

- For now, suppose we are solving this using gradient descent with

- $w_0 = 0$ and

- $w_{t+1} \leftarrow w_t + \eta \sum_{i=1}^n \phi(x_i) (y_i - w_t^T \phi(x_i)) - \eta \lambda w_t = (1 - \eta \lambda) w_t + \eta \sum_{i=1}^n \phi(x_i) (y_i - w_t^T \phi(x_i))$

Handwritten notes: η , $\sum_{i=1}^n$, $\phi(x_i)$, $(y_i - w_t^T \phi(x_i))$, $(1 - \eta \lambda) w_t$

Claim: For any t , w_t can be represented as $w_t = \sum_{i=1}^n \alpha_i \phi(x_i)$

Handwritten notes: \mathbb{R}^p , \mathbb{R}^p , α_i , $\phi(x_i)$

for some n -dimensional parameter $\alpha = (\alpha_1, \dots, \alpha_n)$

- Prediction $\hat{y}_{\text{new}} = \hat{w}^T \phi(x_{\text{new}}) = \sum_{i=1}^n \alpha_i \phi(x_i)^T \phi(x_{\text{new}})$

Handwritten notes: \hat{w} , $\phi(x_{\text{new}})$, α_i , $\phi(x_i)^T$, $\phi(x_{\text{new}})$

$\phi(x_i)$'s

$K(x_i, x_{\text{new}})$

Kernel ridge regression

- Ridge regression with feature map $\phi(\cdot) \in \mathbb{R}^p$

$$\hat{w} = \arg \min_{w \in \mathbb{R}^p} \frac{1}{2} \|y - \phi(X)w\|_2^2 + \lambda \frac{1}{2} \|w\|_2^2$$

- $$w = \sum_{i=1}^n \alpha_i \phi(x_i) = \phi(X)^T \alpha$$

with now an n -dimensional parameter $\alpha = (\alpha_1, \dots, \alpha_n)$, instead of p

- Let $K \in \mathbb{R}^{n \times n}$ be the kernel matrix, where $K_{i,j} = \phi(x_i)^T \phi(x_j) \Rightarrow K = \phi(X) \phi(X)^T$

- The new objective is $\frac{1}{2} \|y - \phi(X) \phi(X)^T \alpha\|_2^2 + \lambda \frac{1}{2} \|\phi(X)^T \alpha\|_2^2$

$$\nabla_{\alpha} =$$

$$\frac{1}{2} \|y - K\alpha\|_2^2 + \frac{1}{2} \lambda \alpha^T K \alpha$$

Thus, $\hat{\alpha}_{\text{kernel}} = (K + \lambda I_{n \times n})^{-1} y$

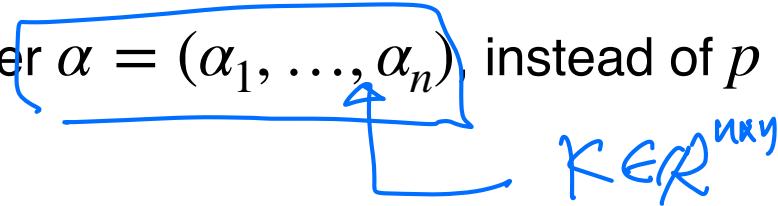
- $$\text{Prediction } \hat{y}_{\text{new}} = \hat{w}^T \phi(x_{\text{new}}) = \sum_{i=1}^n \alpha_i \phi(x_i)^T \phi(x_{\text{new}}) = \sum_{i=1}^n K(x_i, x_{\text{new}}) \alpha_i$$

Kernel regression

- This holds for more general class of algorithms other than GD:

$$w = \sum_{i=1}^n \alpha_i \phi(x_i) = \phi(X)^T \alpha$$

with now an n -dimensional parameter $\alpha = (\alpha_1, \dots, \alpha_n)$ instead of p

 KER^{ny}

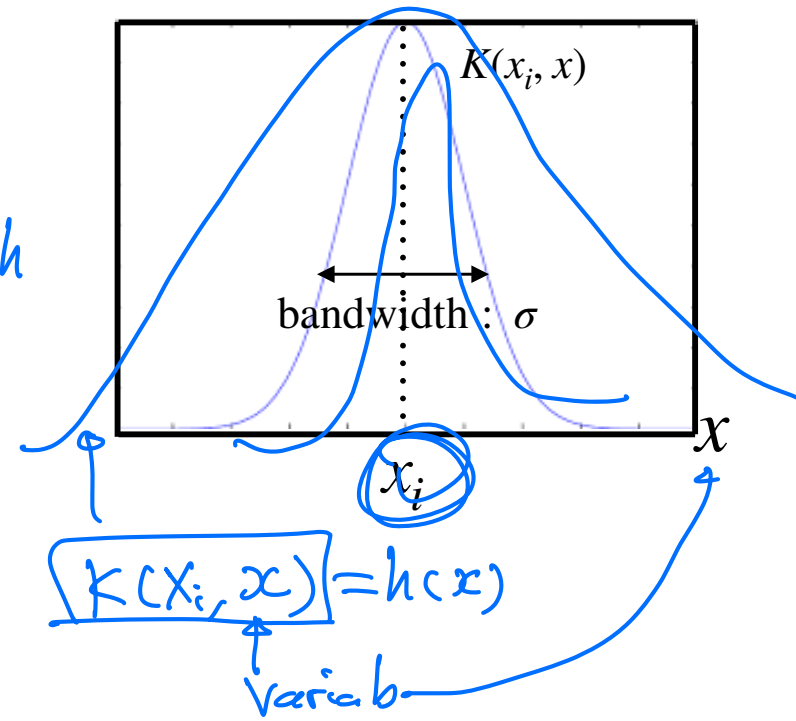
- Kernel method is not limited to linear Ridge regression, but also applies to a broad class of methods including the kernel logistic regression

[Homework 3 Problem A3 implements kernel regression for polynomial kernel and RBF kernel]

$$\text{RBF kernel } k(x_i, x) = \exp\left\{-\frac{\|x_i - x\|_2^2}{2\sigma^2}\right\}$$

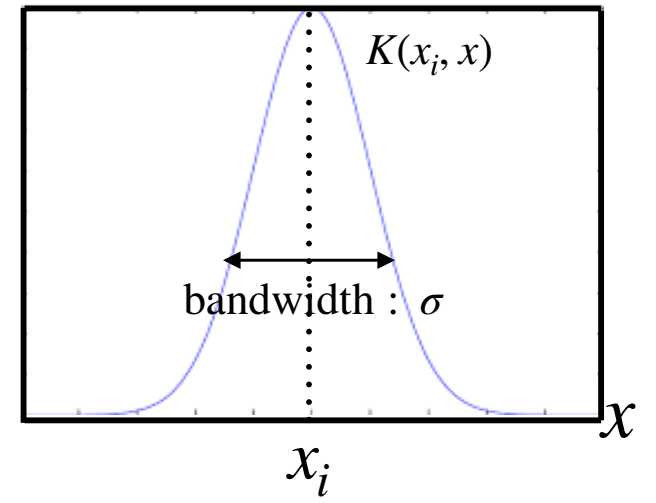
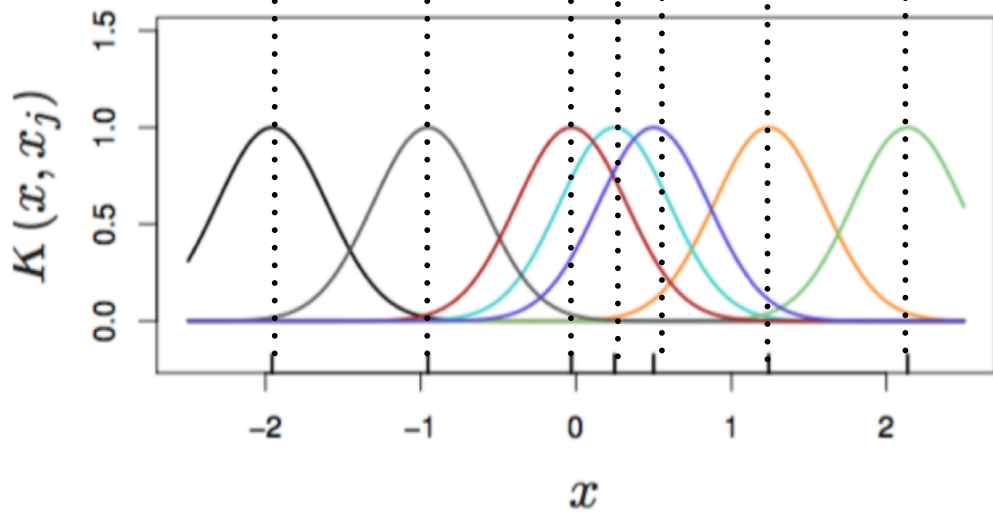
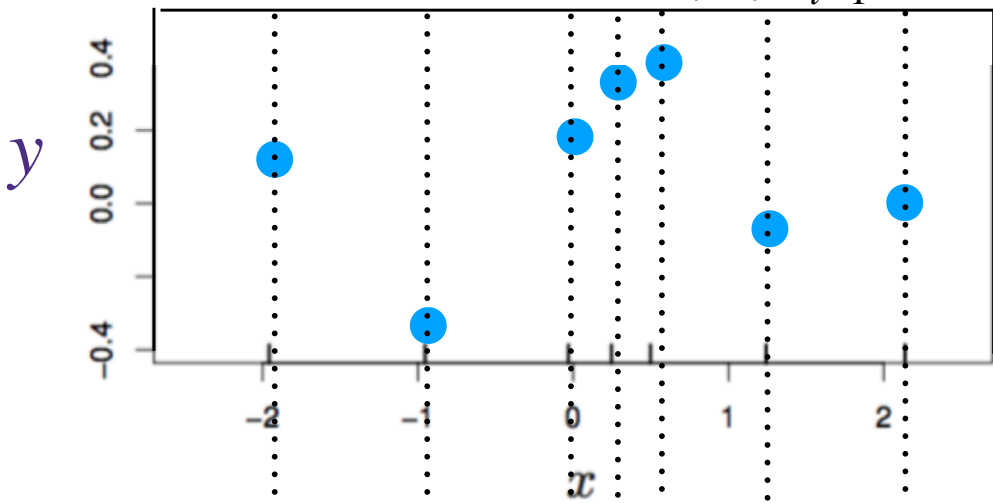
bandwidth

$K(x_i, x)$



RBF kernel $k(x_i, x) = \exp\left\{-\frac{\|x_i - x\|_2^2}{2\sigma^2}\right\}$

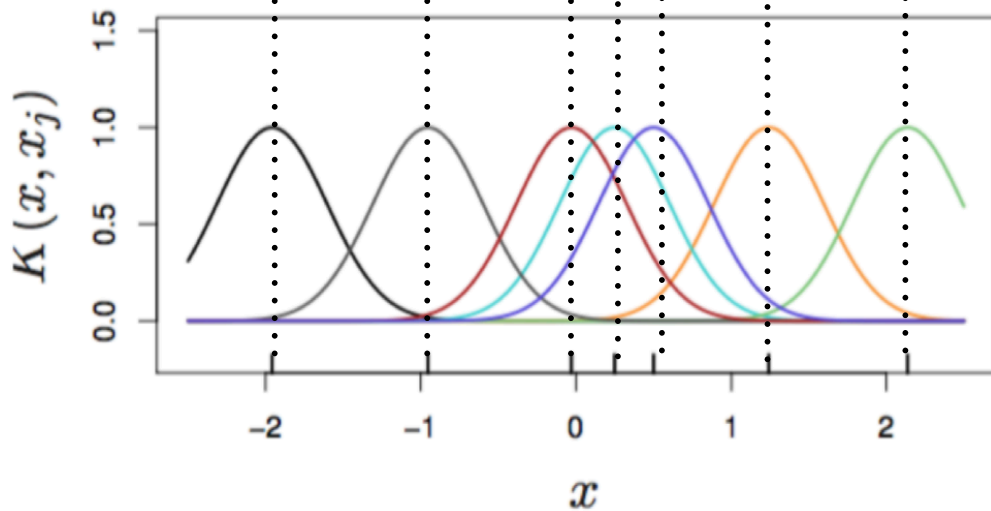
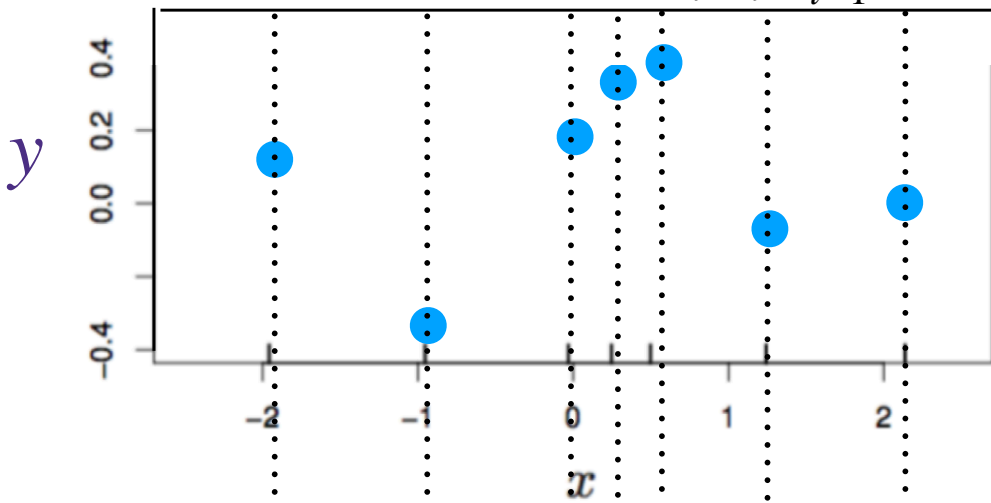
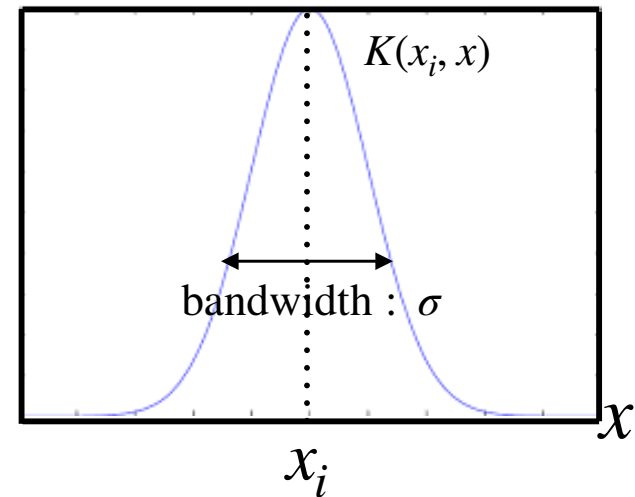
samples $\{(x_i, y_i)\}_{i=1}^n$



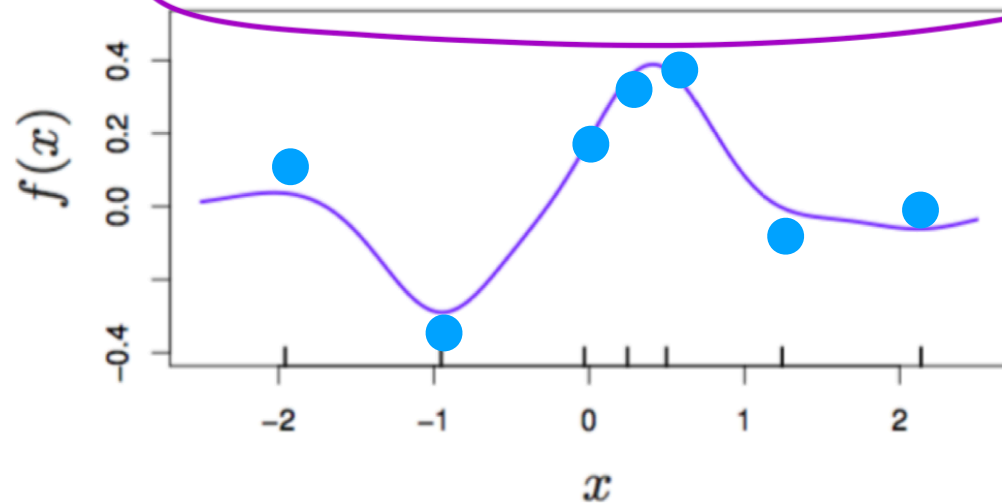
- predictor $f(x) = \sum_{i=1}^n \alpha_i K(x_i, x)$ is taking weighted sum of n kernel functions centered at each sample points

$$\text{RBF kernel } k(x_i, x) = \exp\left\{-\frac{\|x_i - x\|_2^2}{2\sigma^2}\right\}$$

samples $\{(x_i, y_i)\}_{i=1}^n$



$$f(x) = \alpha_0 + \sum_j \alpha_j K(x, x_j)$$



- predictor $f(x) = \sum_{i=1}^n \alpha_i K(x_i, x)$ is taking weighted sum of n kernel functions centered at each sample points

Why do we need regularization when using kernels?

- Typically, $p \gg d$ and $\mathbf{K} \succ 0$. Why?
- So \mathbf{K} is invertible and $\hat{\alpha} = (\mathbf{K} + \lambda \mathbf{I}_{n \times n})^{-1} \mathbf{y}$ is well defined.
- What if $\lambda = 0$? What goes wrong?

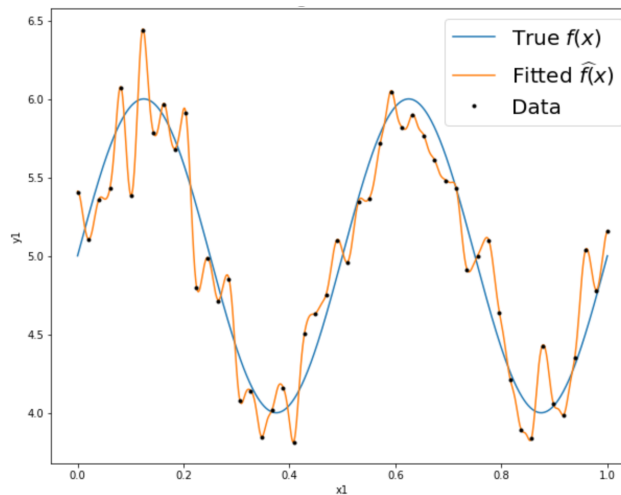
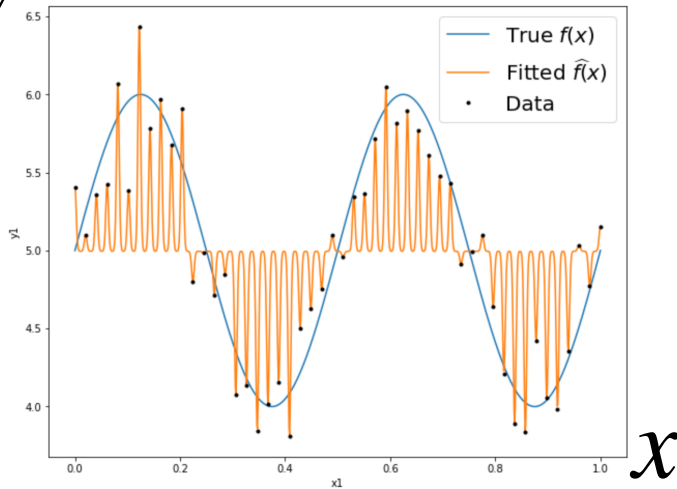
$$\arg \min_{\alpha} \|\mathbf{y} - \mathbf{K}\alpha\|_2^2$$

RBF kernel $k(x_i, x) = \exp\left\{-\frac{\|x_i - x\|_2^2}{2\sigma^2}\right\}$

- $\mathcal{L}(\alpha) = \|\mathbf{K}\alpha - \mathbf{y}\|_2^2 + \lambda\|\mathbf{w}\|_2^2$
- The bandwidth σ^2 of the kernel regularizes the predictor, and the regularization coefficient λ also regularizes the predictor

$\sigma = 10^{-3} \quad \lambda = 10^{-4}$

$\sigma = 10^{-2} \quad \lambda = 10^{-4}$

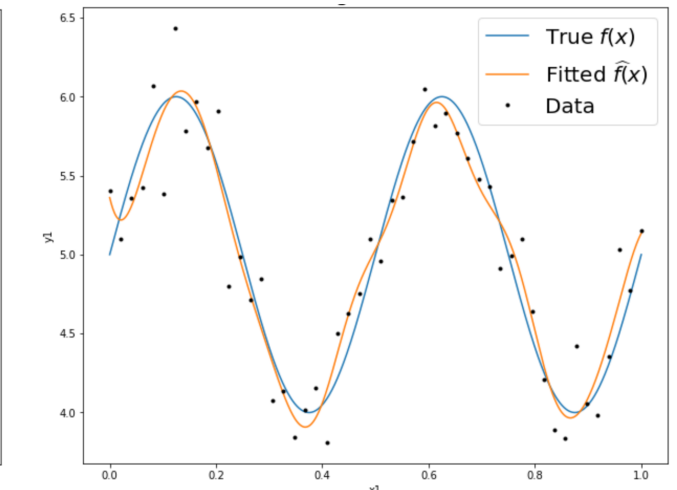
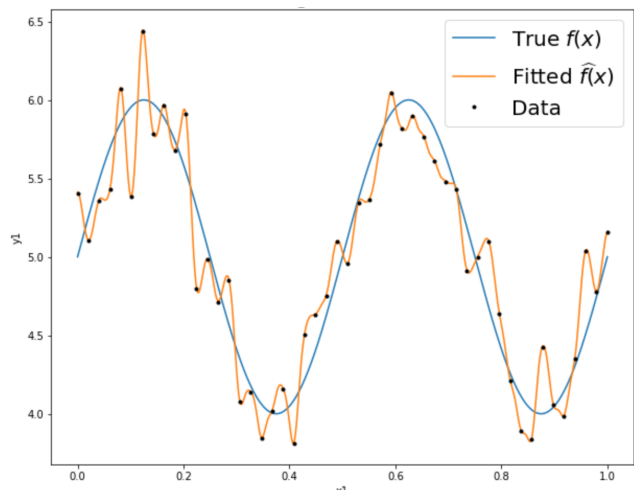
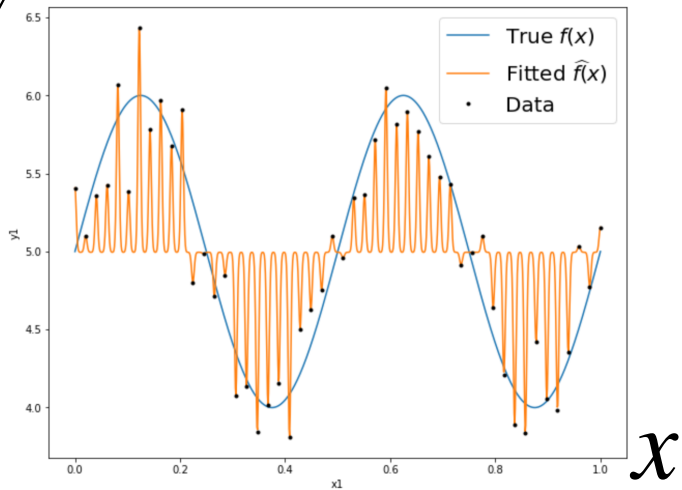


$$\hat{f}(x) = \sum_{i=1}^n \hat{\alpha}_i K(x_i, x)$$

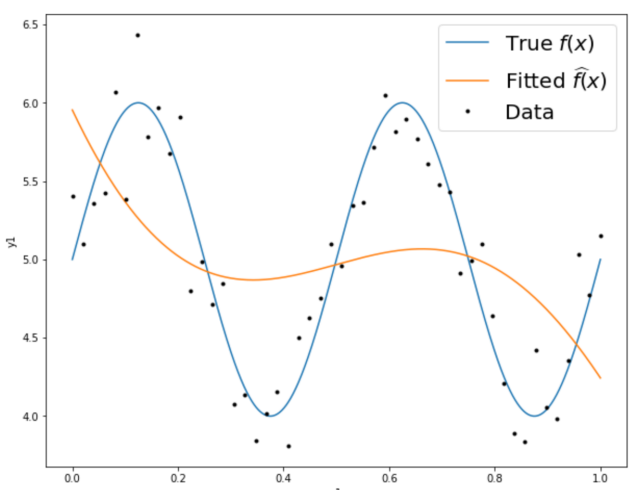
RBF kernel $k(x_i, x) = \exp\left\{-\frac{\|x_i - x\|_2^2}{2\sigma^2}\right\}$

- $\mathcal{L}(\alpha) = \|\mathbf{K}\alpha - \mathbf{y}\|_2^2 + \lambda\|w\|_2^2$
- The bandwidth σ^2 of the kernel regularizes the predictor, and the regularization coefficient λ also regularizes the predictor

y



$\sigma = 10^{-0} \quad \lambda = 10^{-4}$



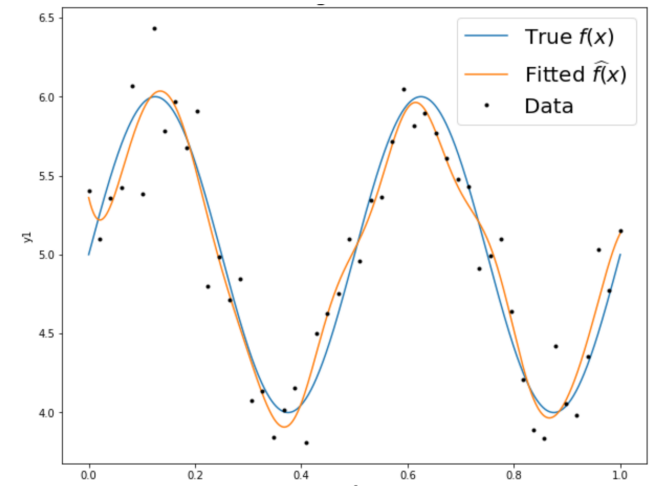
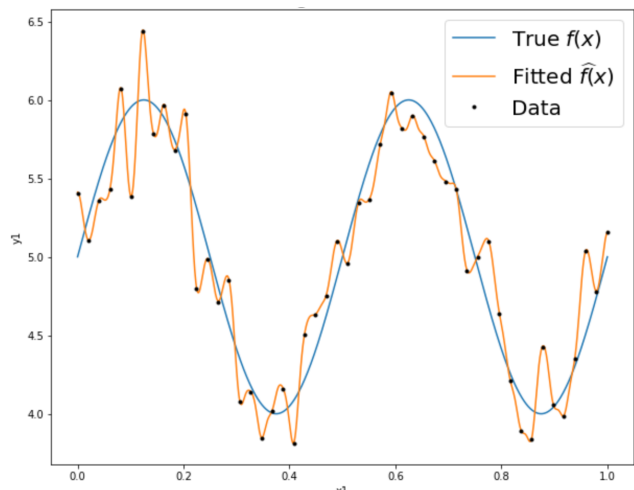
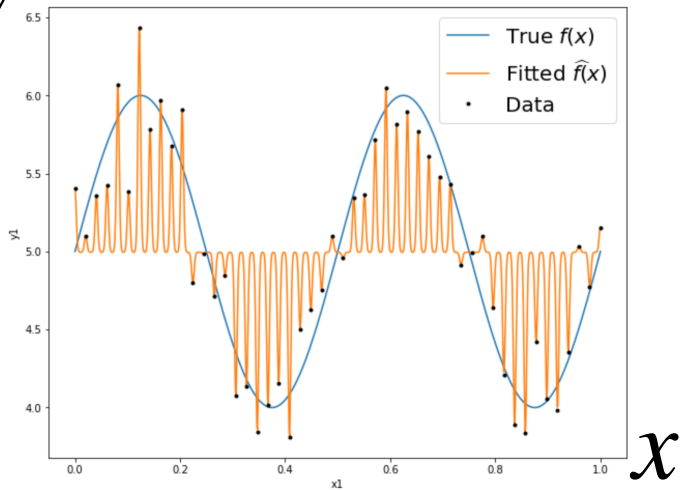
x

$$\hat{f}(x) = \sum_{i=1}^n \hat{\alpha}_i K(x_i, x)$$

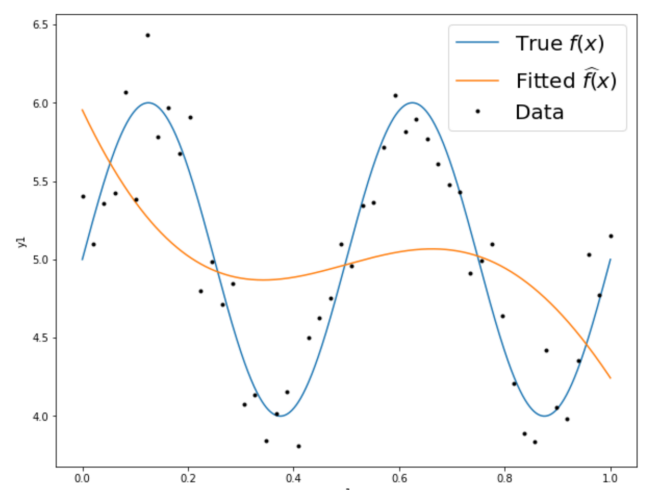
RBF kernel $k(x_i, x) = \exp\left\{-\frac{\|x_i - x\|_2^2}{2\sigma^2}\right\}$

- $\mathcal{L}(\alpha) = \|\mathbf{K}\alpha - \mathbf{y}\|_2^2 + \lambda\|\mathbf{w}\|_2^2$
- The bandwidth σ^2 of the kernel regularizes the predictor, and the regularization coefficient λ also regularizes the predictor

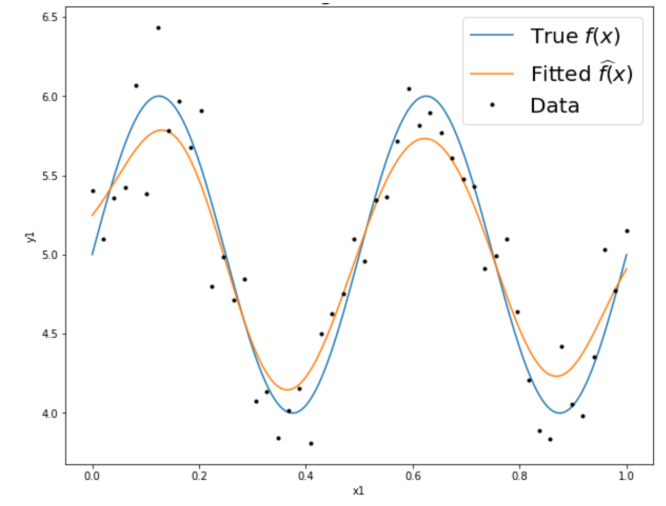
y



$\sigma = 10^{-0} \quad \lambda = 10^{-4}$



$\sigma = 10^{-1} \quad \lambda = 10^{-0}$



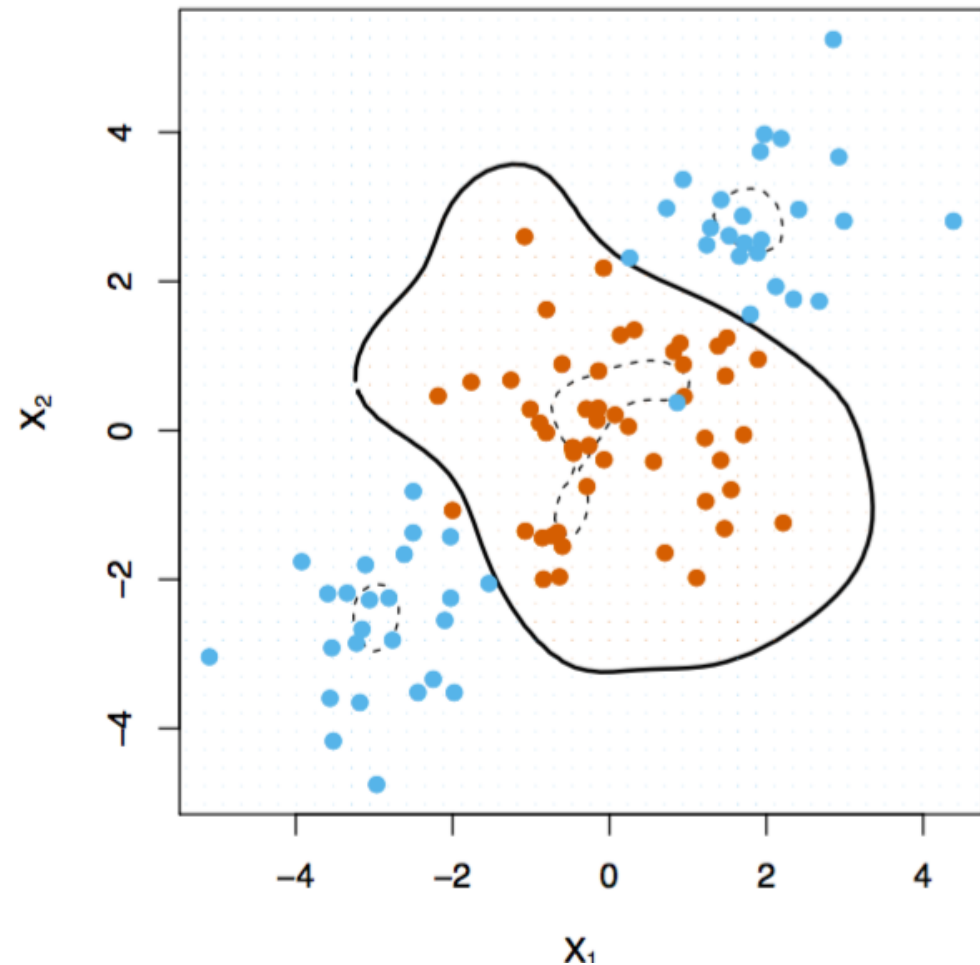
$$\hat{f}(x) = \sum_{i=1}^n \hat{\alpha}_i K(x_i, x)$$

x

RBF kernel and random features

$$k(x_i, x) = \exp \left\{ - \frac{\|x_i - x\|_2^2}{2\sigma^2} \right\}$$

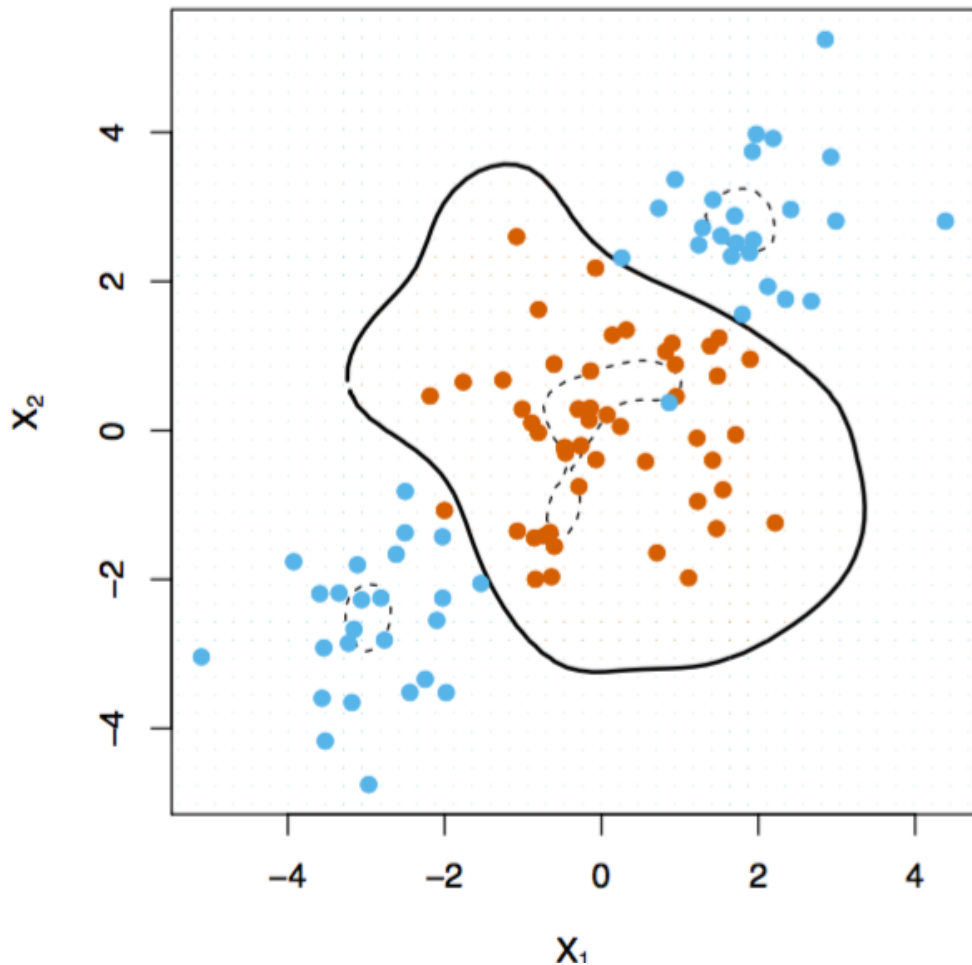
Bandwidth σ is large enough



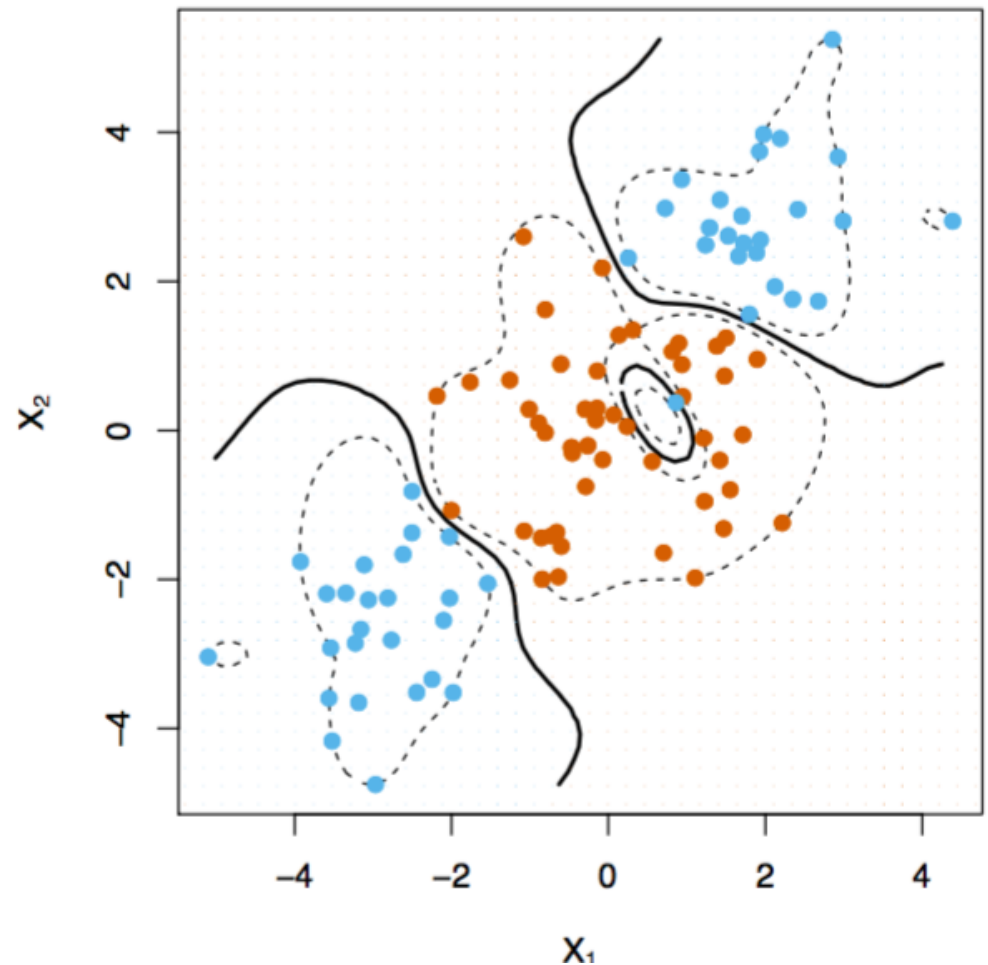
RBF kernel and random features

$$k(x_i, x) = \exp\left\{-\frac{\|x_i - x\|_2^2}{2\sigma^2}\right\}$$

Bandwidth σ is large enough



Bandwidth σ is small



Fixed Feature V.S. Learned Feature

- Kernel method works well if we choose a good kernel such that the data is linearly separable in the corresponding (possibly infinite dimensional) feature space
- In practice, it is hard to choose a good kernel for a given problem
- Can we **learn** the feature mapping $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^p$ from data also?

Questions?
