LS $\arg\min_{w \in \mathbb{R}^d} (Xw - Y)^2$ $\quad w = (X^T X)^{-1} X^T y$

Ridge " $+ \lambda \|w\|_2^2$ $\quad w = (X^T X + \lambda I)^{-1} X Y$

Lasso $\arg\min_{w \in \mathbb{R}^d} (Xw - Y)^2 + \lambda |w|_1$

Lecture 10: Convexity

- When is an optimization (or learning) easy/fast to solve?

Recap: Ridge vs. Lasso

- **Ridge**

$$\text{minimize}_w \quad \sum_{i=1}^{n} (w^T x_i - y_i)^2 + \lambda \|w\|_2^2$$

  - Very fast:
    - Closed form solution if used with linear models
    - Even with other loss functions, optimization is fast for squared ℓ_2 regularization, because $\|w\|_2^2$ is **convex and smooth**

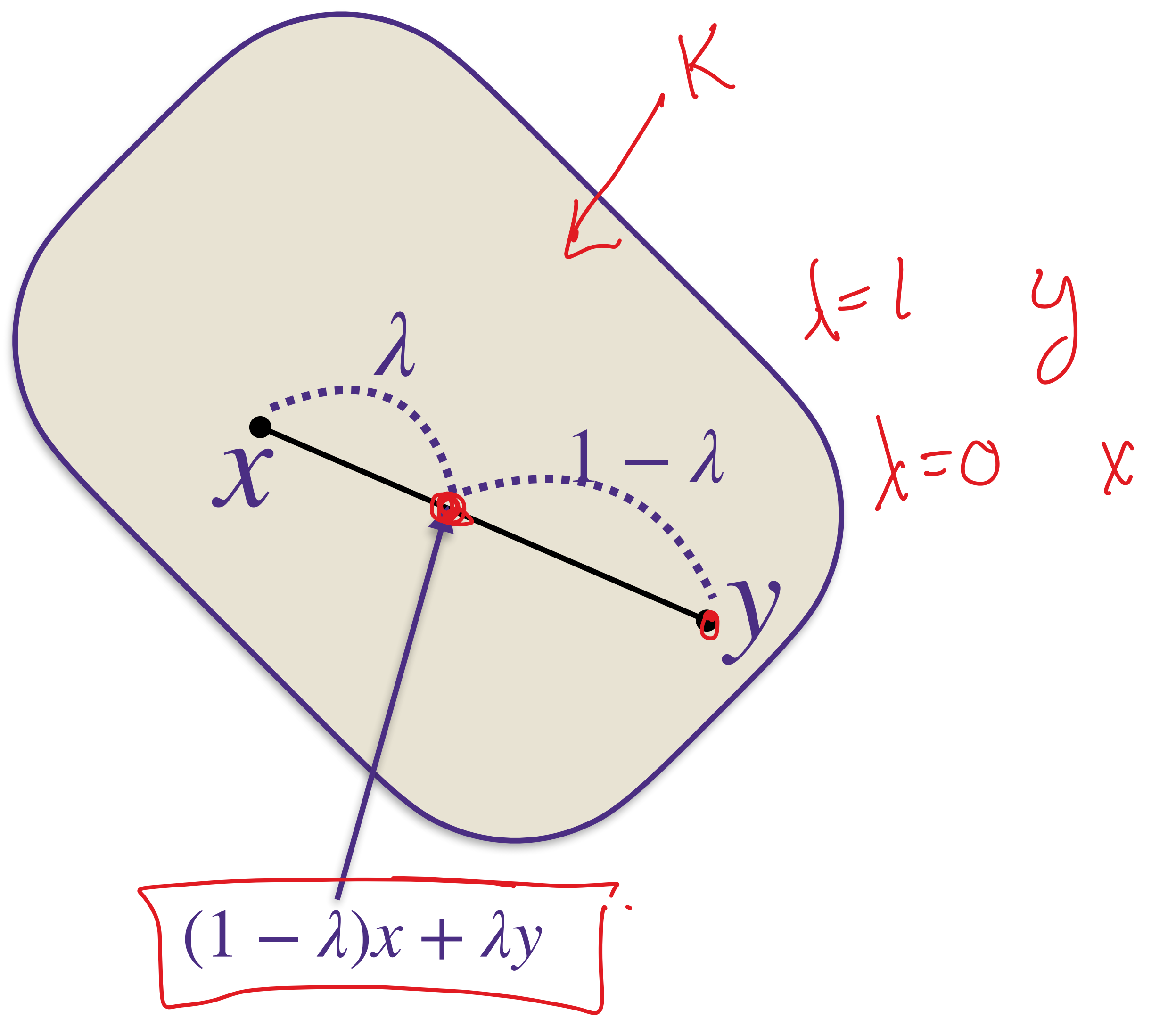
- **Lasso**

$$\text{minimize}_w \quad \sum_{i=1}^{n} (w^T x_i - y_i)^2 + \lambda \|w\|_1$$

  - Slower than Ridge:
    - Requires iterative optimization algorithm like sub-gradient descent
    - In particular, it is slower because $\|w\|_1$ is **convex but non-smooth**

What is a convex set?

A set $K \subset \mathbb{R}^d$ is convex if $(1-\lambda)x + \lambda y \in K$ for all $x, y \in K$ and $\lambda \in [0,1]$



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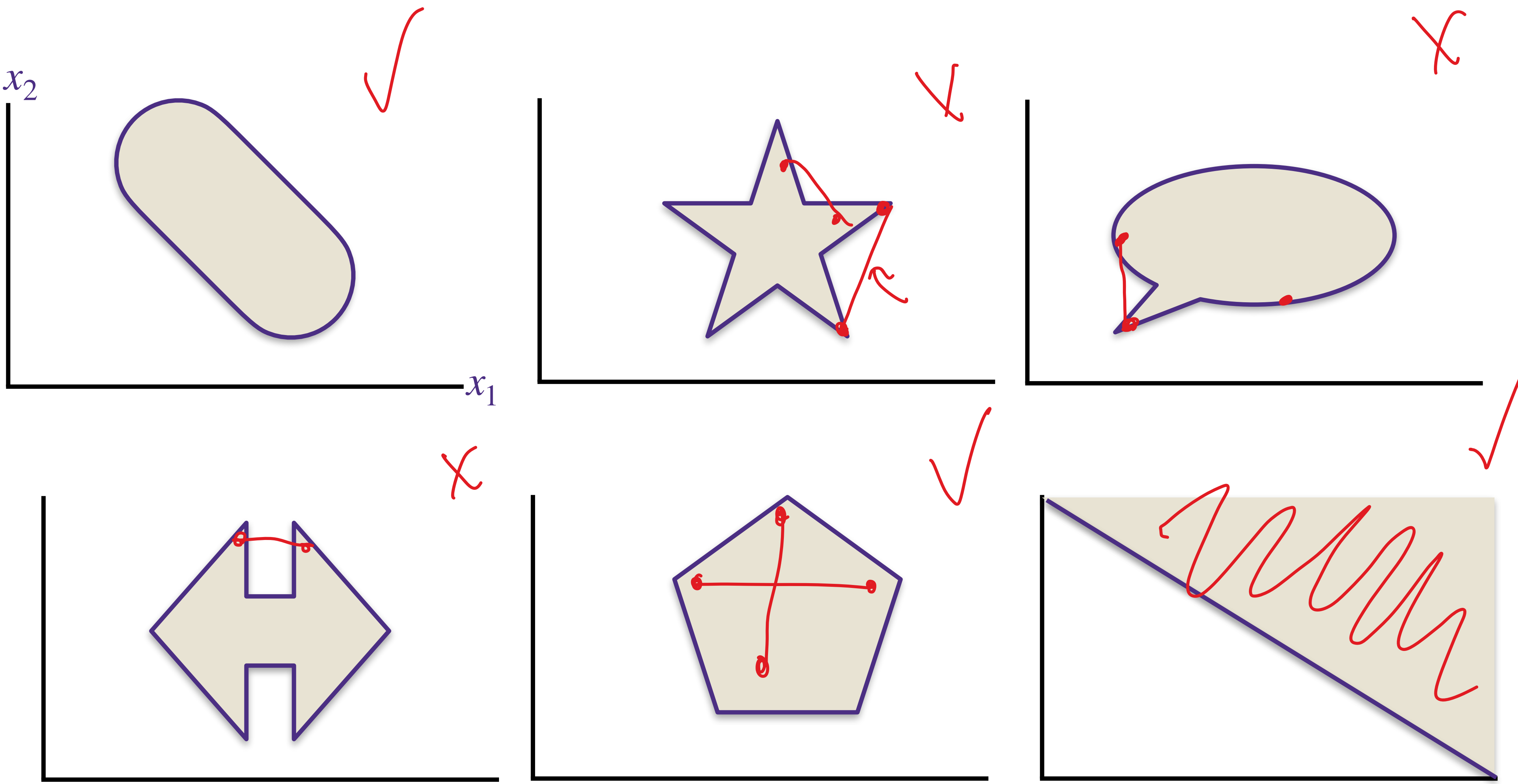
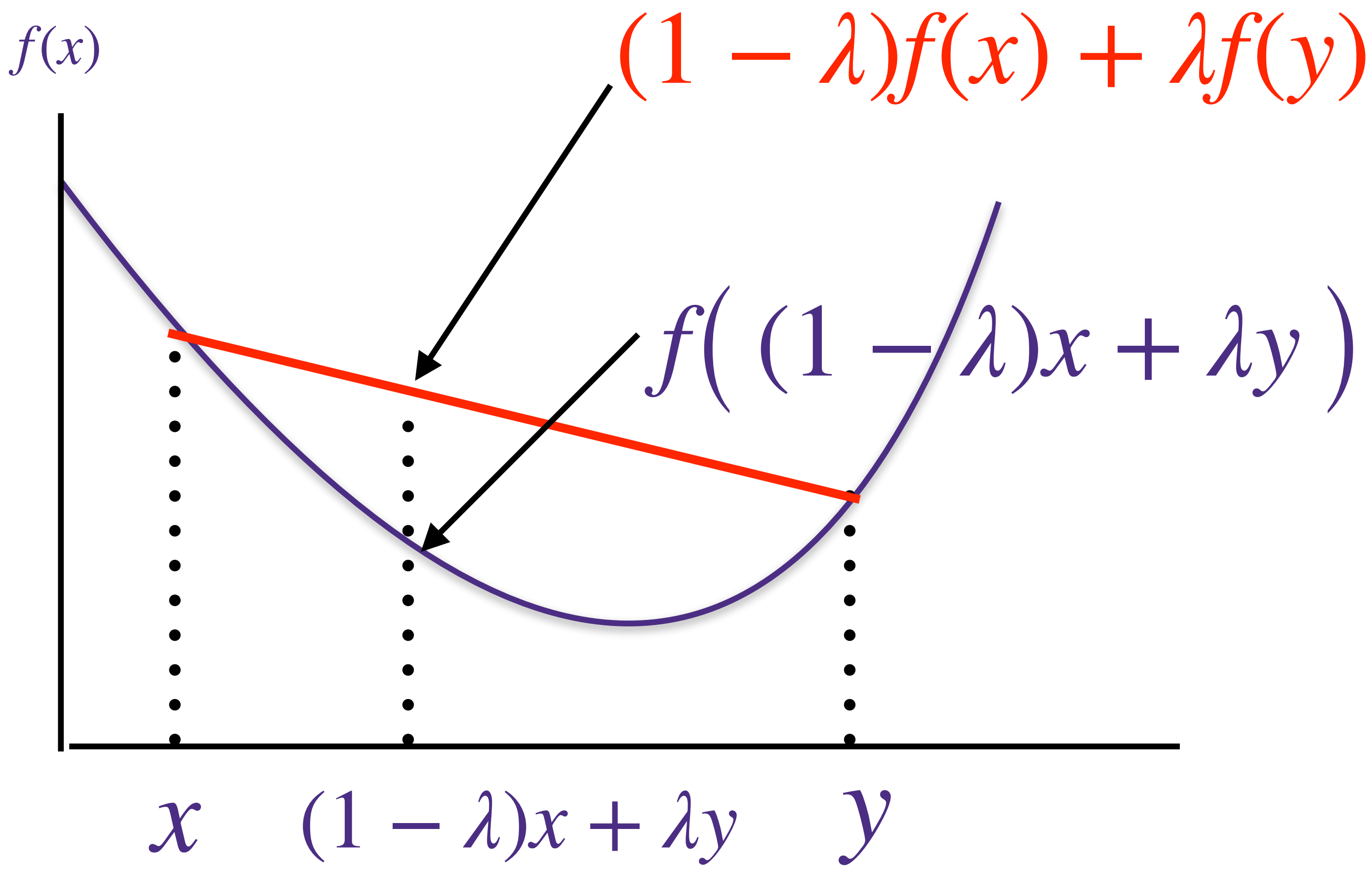


What is a convex function?

A function $f : \mathbb{R}^d \to \mathbb{R}$ is convex if $f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$ for all $x, y \in \mathbb{R}^d$ and $\lambda \in [0, 1]$



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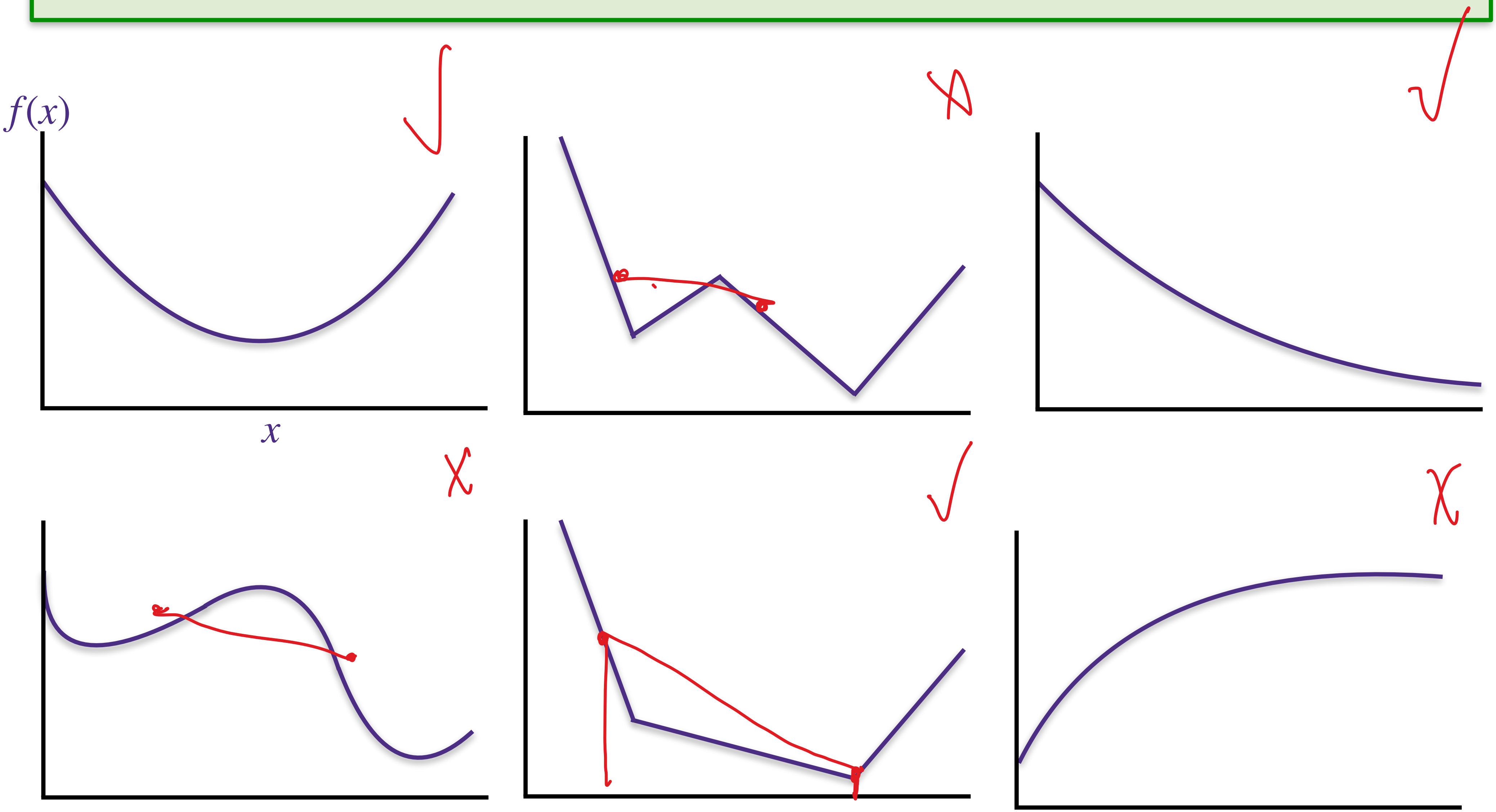

Convex functions and convex sets?

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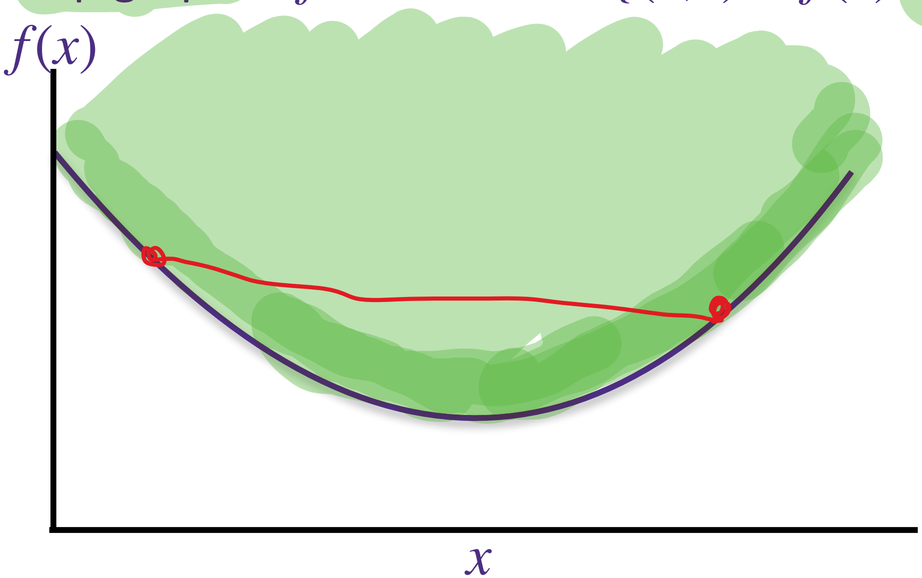
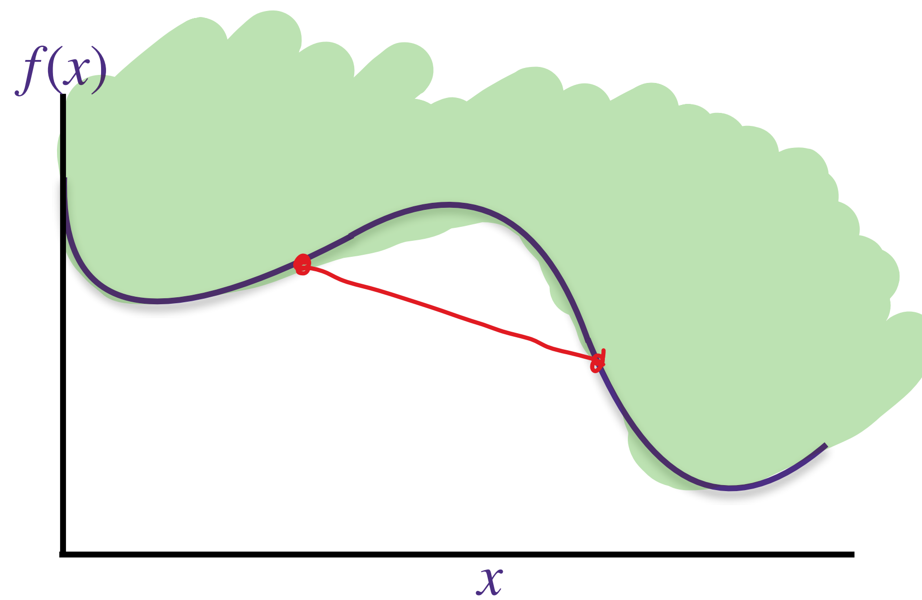
A function $f : \mathbb{R}^d \to \mathbb{R}$ is convex if the set $\{(x,t) \in \mathbb{R}^{d+1} : f(x) \leq t\}$ is convex

Graph of f is defined as $\{(x,t) \ : \ f(x) = t\}$

Epigraph of f is defined as $\{(x,t) \ : \ f(x) \leq t\}$





More definitions of convexity

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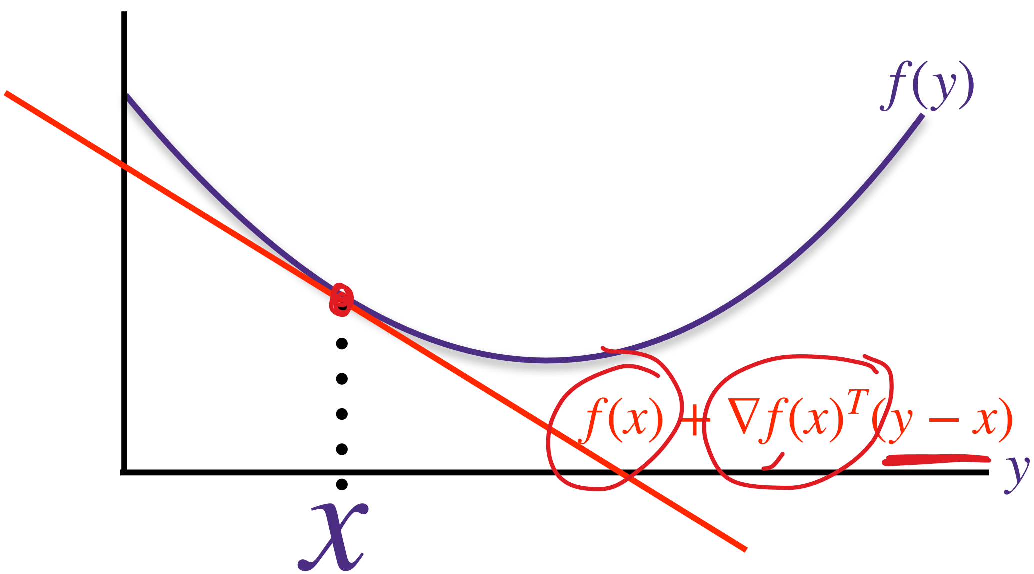
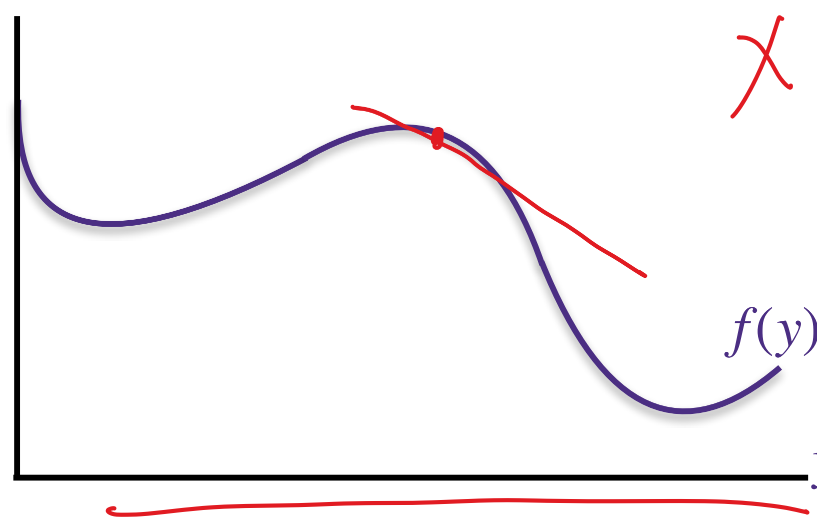
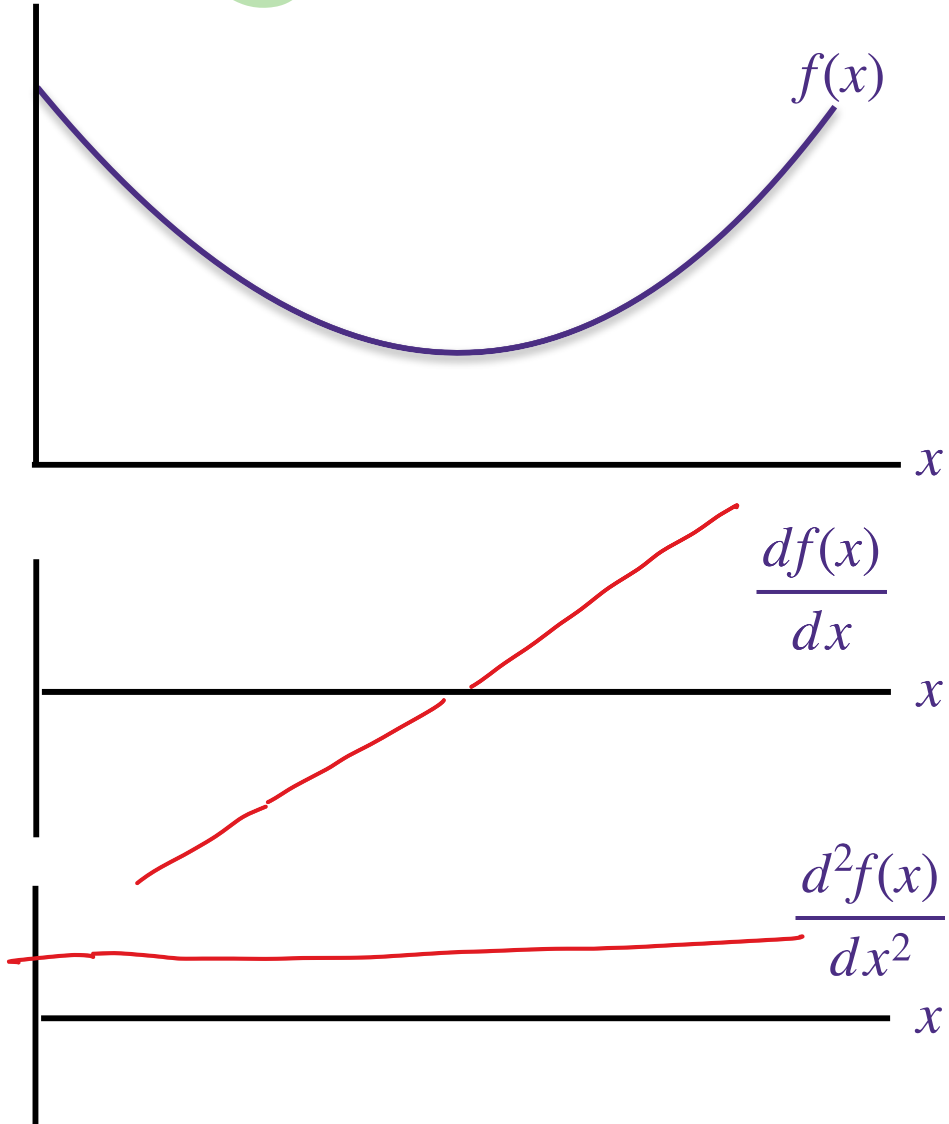
A function $f : \mathbb{R}^d \to \mathbb{R}$ that is differentiable everywhere is convex if $f(y) \geq f(x) + \nabla f(x)^\top (y - x)$ for all $x, y \in dom(f)$

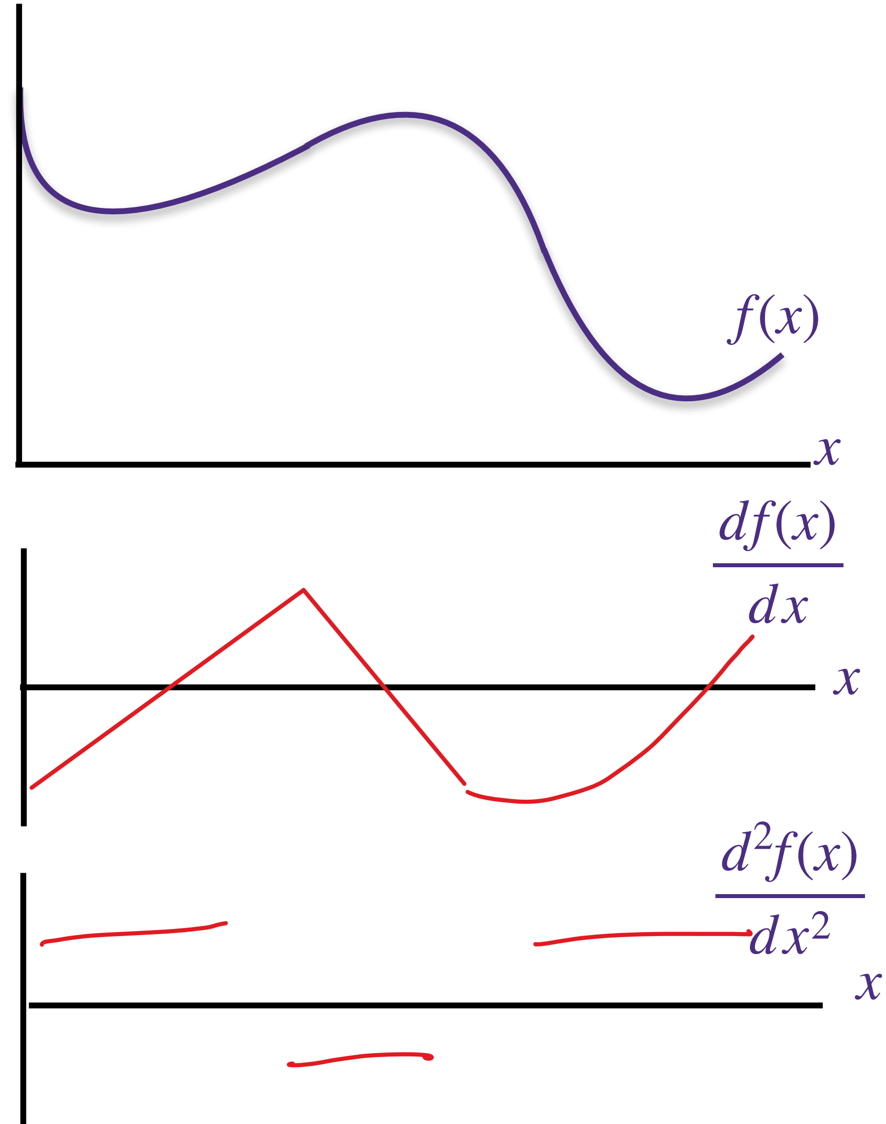




More definitions of convexity

A function $f : \mathbb{R}^d \to \mathbb{R}$ that is twice-differentiable everywhere is convex if $\nabla^2 f(x) \succeq 0$ for all $x \in dom(f)$





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A function $f : \mathbb{R}^d \to \mathbb{R}$ that is twice-differentiable everywhere is convex if $\nabla^2 f(x) \succeq 0$ for all $x \in dom(f)$

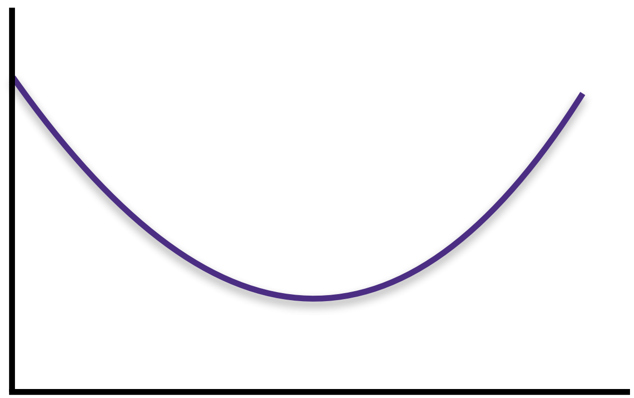
$y^T \nabla^2 f(x)\, y \geq 0 \quad \forall y$

Why do we care about convexity?

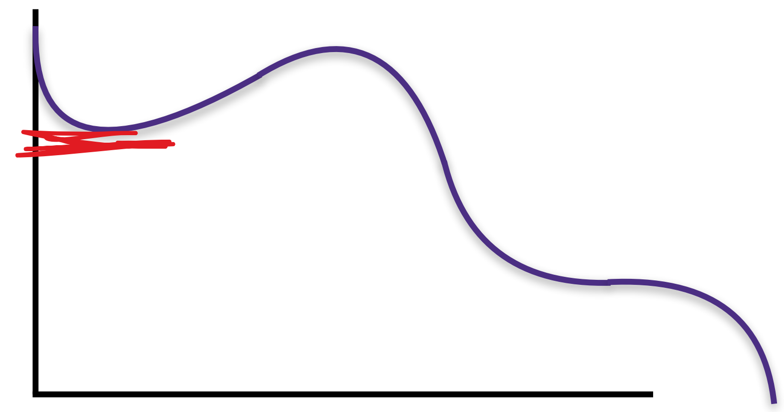
Convex functions

- All local minima are global minima
- Efficient to optimize (e.g., gradient descent)

Convex Function



Non-convex Function



We only need to find a point with $\nabla f(x) = 0$, which for convex functions implies that it is a local minima and a global minima

For non-convex functions, a stationary point with $\nabla f(x) = 0$ could be a local minima, a local maxima, or a saddle point

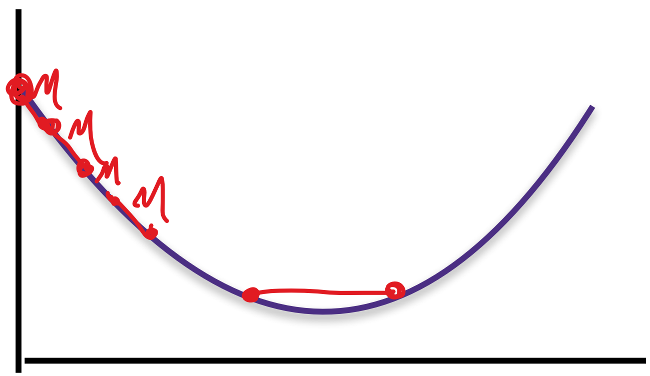
Gradient Descent on $\min_{w} f(w)$

$$\text{Initialize: } w_0 = 0$$

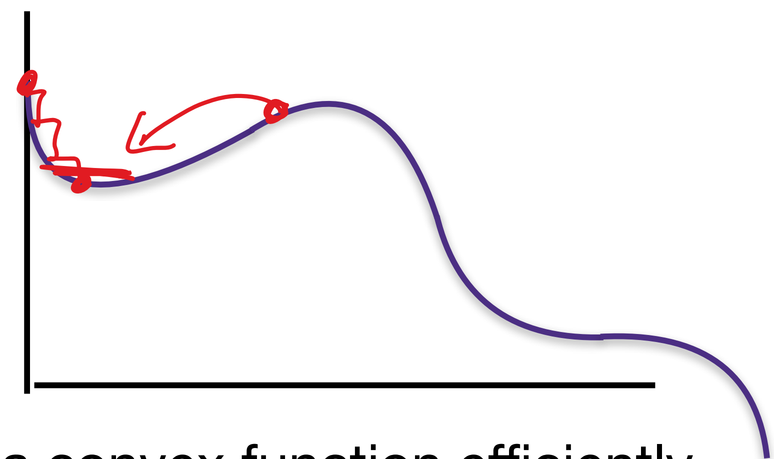
$$\text{for } t = 1, 2, \ldots$$

$$w_{t+1} = w_t - \eta \nabla f(w_t)$$

Convex Function

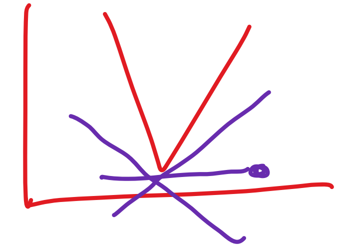


Non-convex Function



- Strength: Can find global minima of a convex function efficiently
- Weakness: Can only be applied to smooth functions
  - i.e., functions that is differentiable everywhere,
  - otherwise $\nabla f(x)$ is not defined and gradient descent cannot be applied

Sub-Gradient

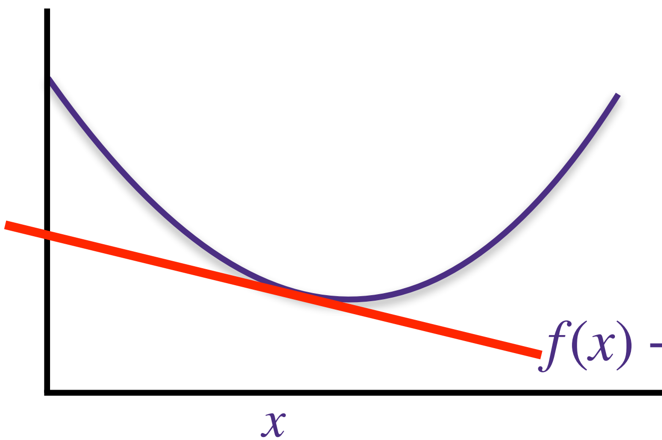


Definition: a function is **non-smooth** if it is not differentiable everywhere

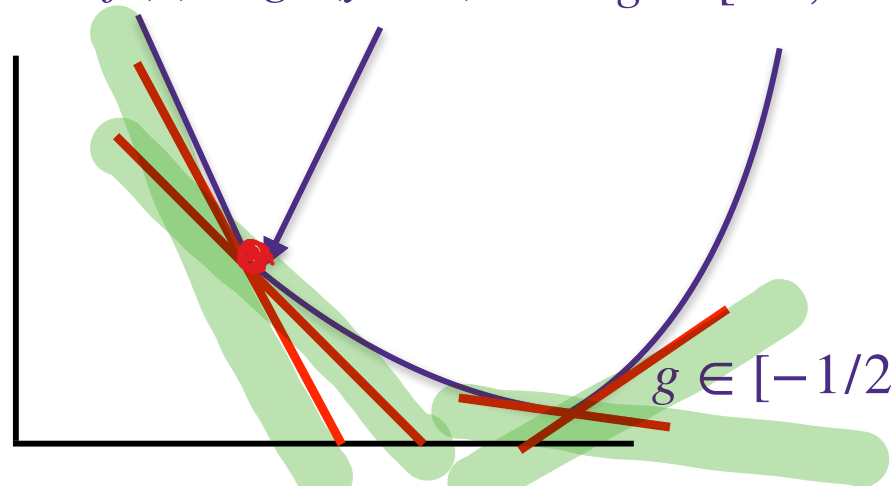
Definition: a vector $g \in \mathbb{R}^d$ is a **sub-gradient** at x if it satisfies

$$f(y) \geq f(x) + g^T(y - x) \text{ for all } y \in \mathbb{R}^d$$

Smooth Convex Function



Non-smooth Convex Function



- for smooth convex functions,
  - gradient is the unique sub-gradient, and
  - the global minimum is achieved at points where gradient is zero

- for non-smooth convex functions,
  - the minimum is achieved at points where sub-gradient set includes the zero vector

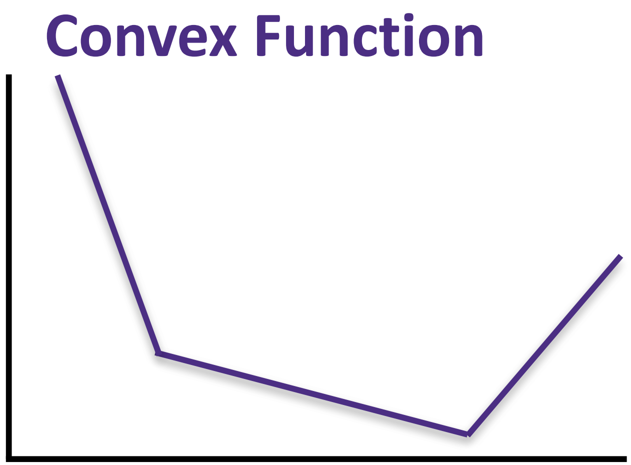
Sub-Gradient Descent for non-smooth functions

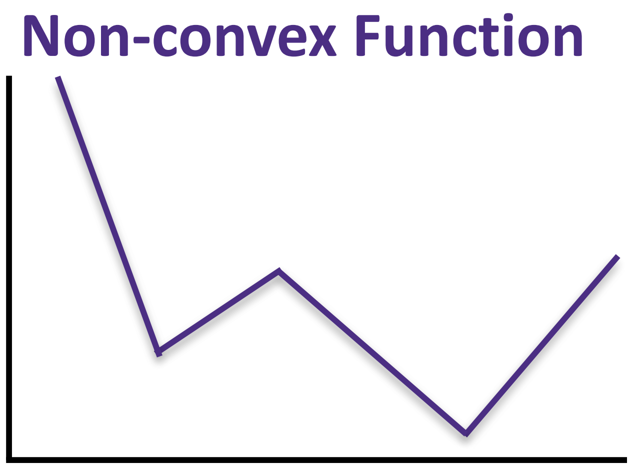
Initialize: $w_0 = 0$

for $t = 1, 2, \ldots$

Find any g_t such that $f(y) \geq f(w_t) + g_t^\top (y - w_t)$

$w_{t+1} \leftarrow w_t - \eta_t g_t$





- Strength: finds global minima for **non-smooth convex functions**
- Weakness: it is slower than gradient descent on convex smooth functions, because the gradient do not get smaller near the global minima
  - Instead of last iterate w_t, we use the best one we saw in all iterates
  - The stepsize needs to decrease with t

Coordinate descent

Initialize: $w_0 = 0$

for $t = 1, 2, \ldots$

Let $i_t = t \ \% \ d$

$$w_{t+1}[i_t] \leftarrow w_t[i_t] - \eta_t \frac{\partial f(w_t)}{\partial w[i_t]}$$

Optimization

- You can always run gradient descent whether f is convex or not. But you only have guarantees if f is convex
- Many bells and whistles can be added onto gradient descent such as momentum and dimension-specific step-sizes (Nesterov, Adagrad, ADAM, etc.)

Questions?