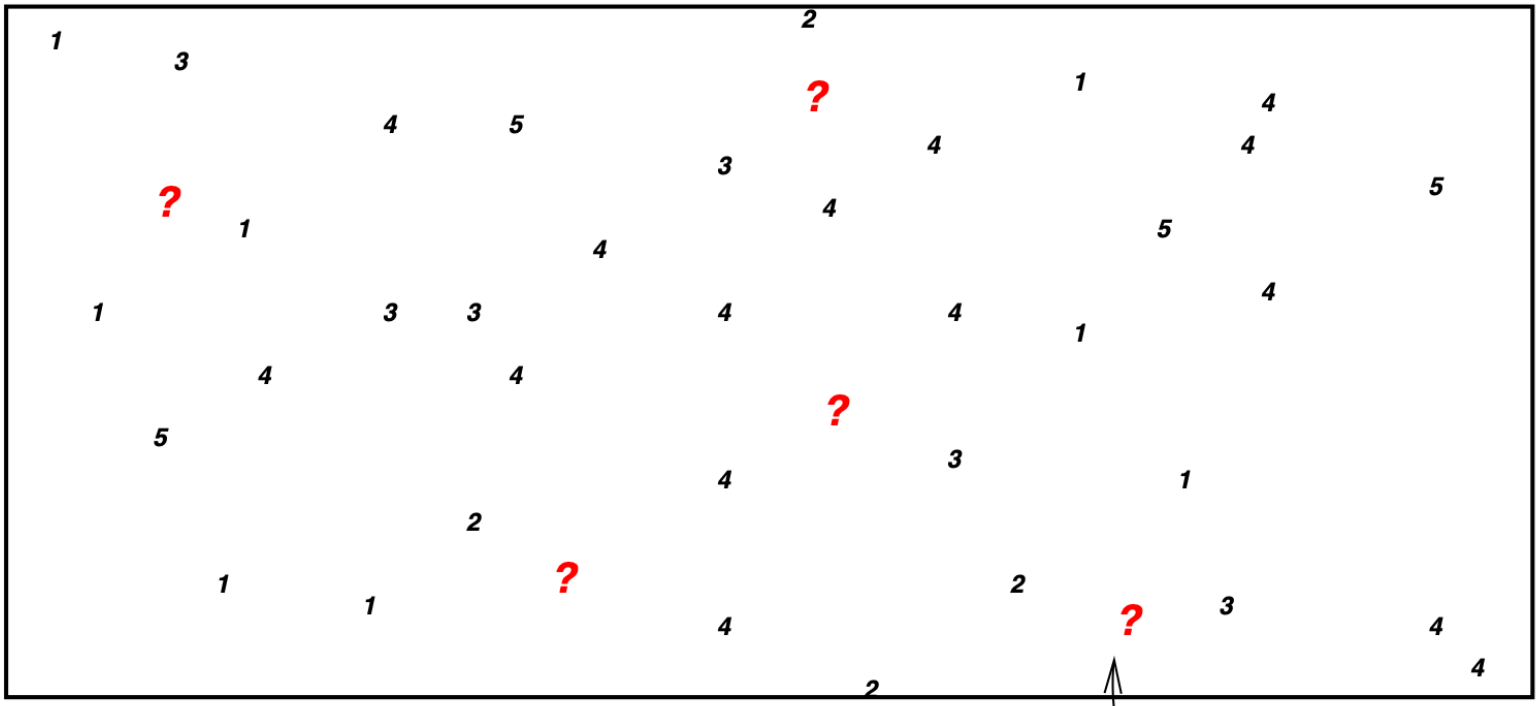
Matrix Completion

Matrix completion for recommendation systems

Netflix challenge dataset



- users provide ratings on a few movies, and we want to predict the missing entries in this ratings matrix, so that we can make recommendations
- without any assumptions, the missing entries can be anything, and no prediction is possible

Matrix completion problem

$r << \min\{d, n\}$

$X = U A^T$ low-rank (r)

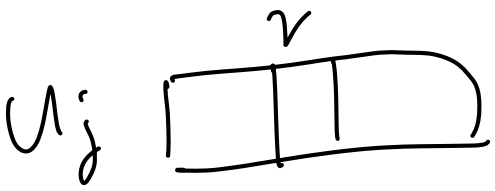
U : $d \times r$ movie features

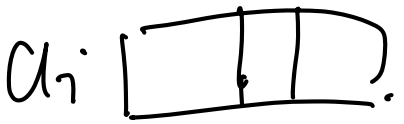
A : $n \times r$ user features

$X_{ji} = \langle u_j, a_i \rangle$: user i's rating on movie j.

u_j : r-dim feature for movie j.

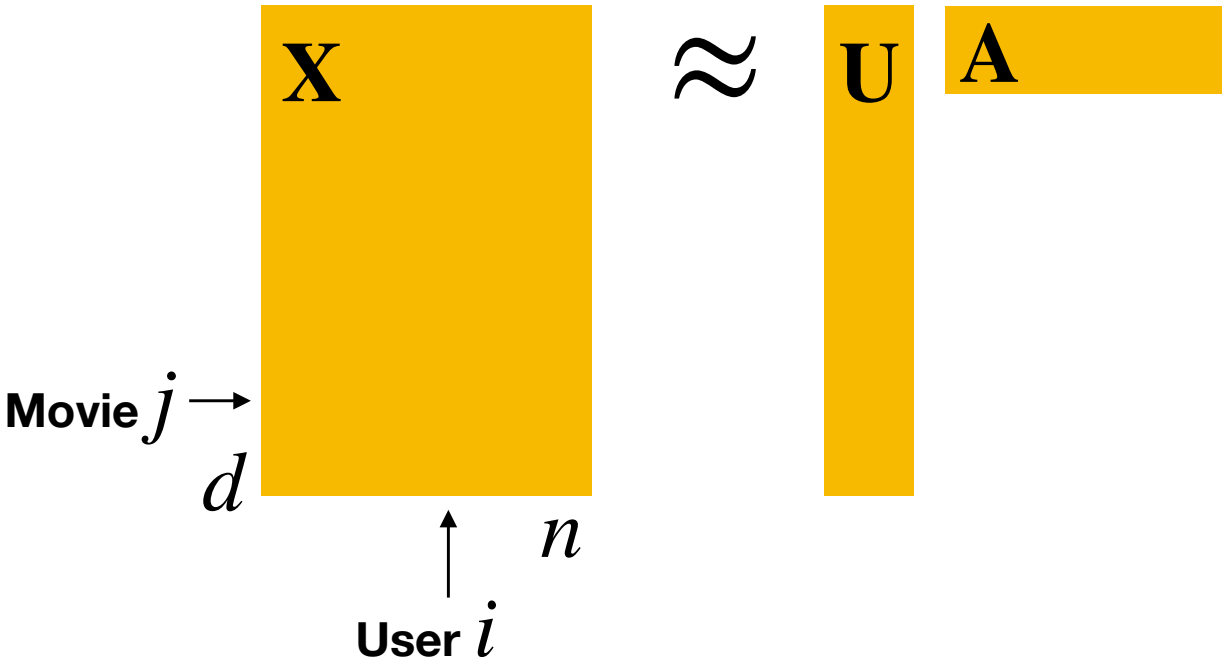
a_i : r-dim feature for user i.





Matrix completion

- let $\mathbf{X} = [x_1 \quad x_2 \quad \cdots \quad x_n] \in \mathbb{R}^{d \times n}$ be the ratings matrix, and assume it is fully observed, i.e. we know all the entries
- then we want to find $\mathbf{U} \in \mathbb{R}^{d \times r}$ and $\mathbf{A} = [a_1 \quad a_2 \quad \cdots \quad a_n] \in \mathbb{R}^{r \times n}$ that approximates $\mathbf{X}$



- if we **observe all entries** of $\mathbf{X}$, then we can find the best rank-r approximation with SVD

Optimization for Matrix Completion

- a natural approach to fit v_j's and $a_i's$ to given training data is to solve

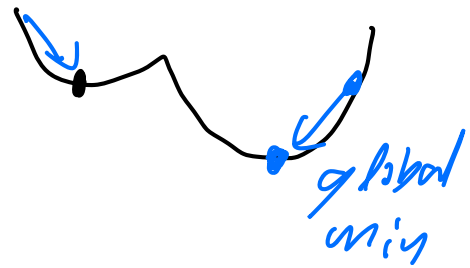
$$\text{minimize}_{\mathbf{U},\mathbf{A}} \sum_{(i,j)\in S_{\text{train}}} (\mathbf{X}_{ji} - v_j^T a_i)^2$$

non-convex

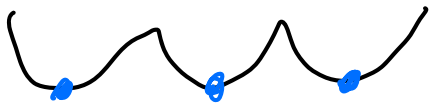
S_{train}: observed entry indices

- this can be solved, for example via gradient descent or alternating minimization

1) Find a good initialization then apply gradient descent (locally convex)



2) run randomly initialized gradient descent



Matrix Recovery

$d \times n$

- a natural approach to fit v_j's and $a_i's$ to given training data is to solve

$$\text{minimize}_{\mathbf{U},\mathbf{A}} \sum_{(i,j)\in S_{\text{train}}} (\mathbf{X}_{ji} - v_j^T a_i)^2$$

- If X is low rank, why only observe a few (<< dn) entries, we can recover X?

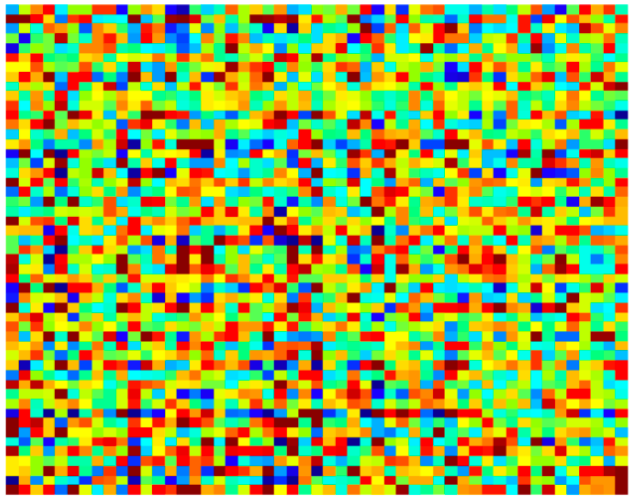
If X is low-rank

$X = U A^T$

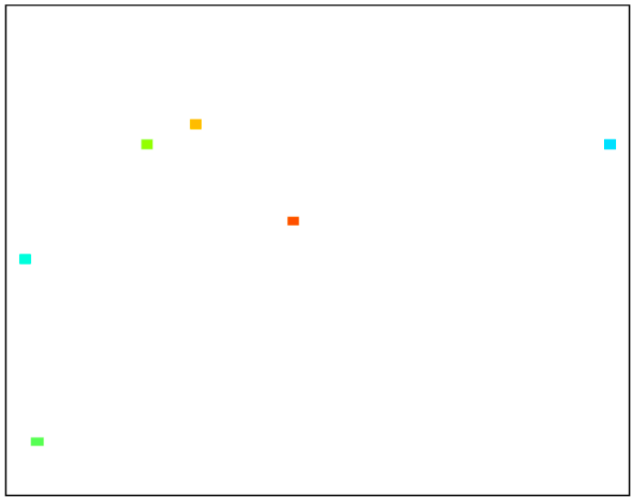
$d \times r \quad r \times n \rightarrow r(n+d)$

Example: 2000×2000 rank-8 random matrix

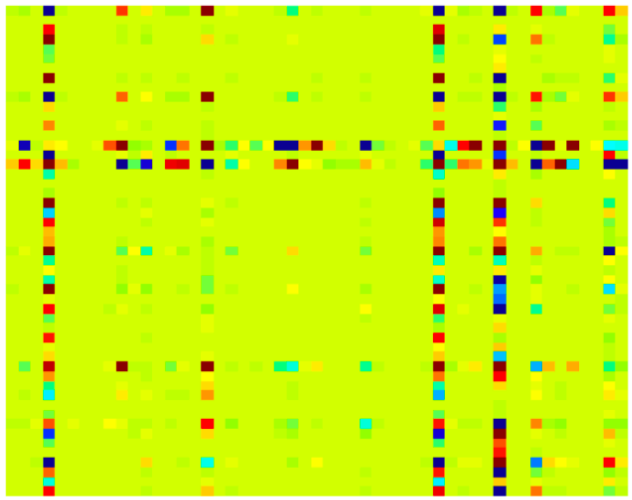
low-rank matrix $\mathbf{X}$



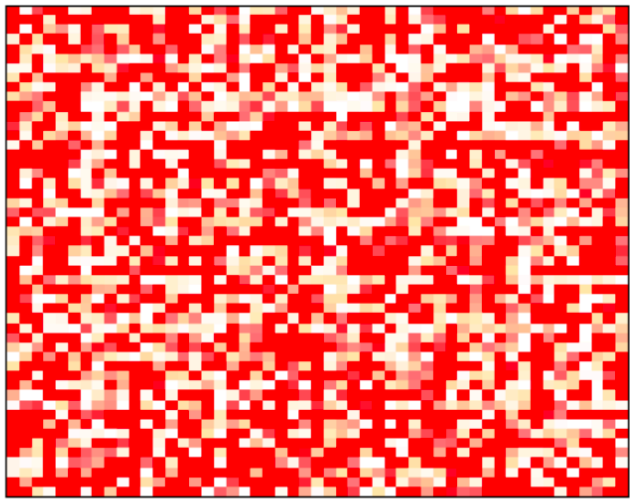
sampled matrix



Gradient descent output $\mathbf{UA}$



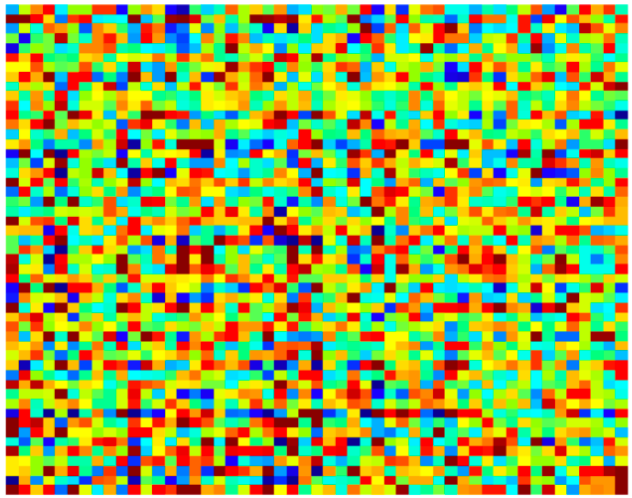
squared error $(\mathbf{X}_{ji} - (\mathbf{UA})_{ji})^2$



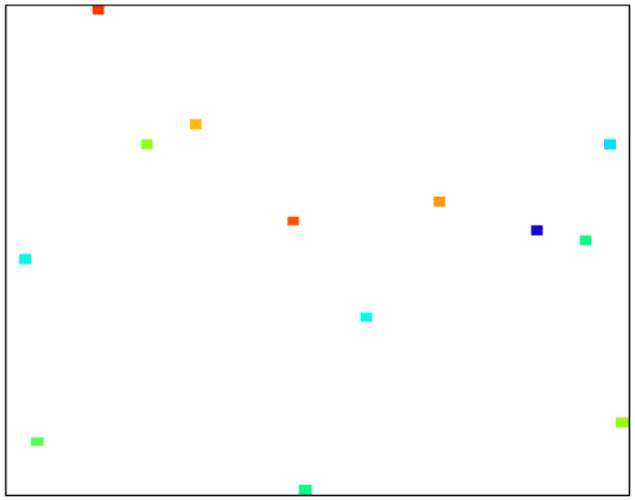
0.25% sampled

Example: 2000×2000 rank-8 random matrix

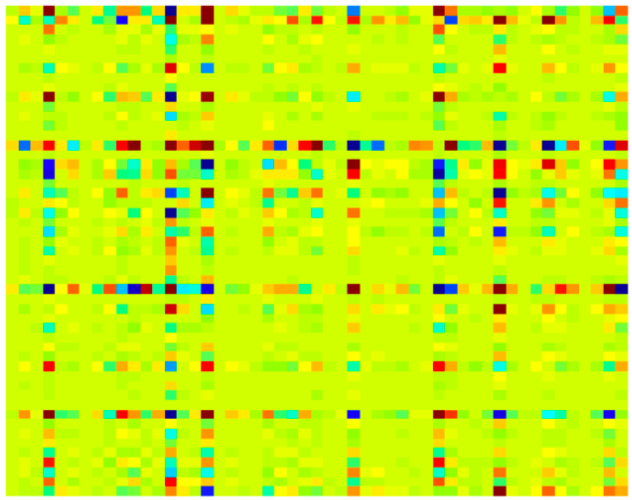
low-rank matrix $\mathbf{X}$



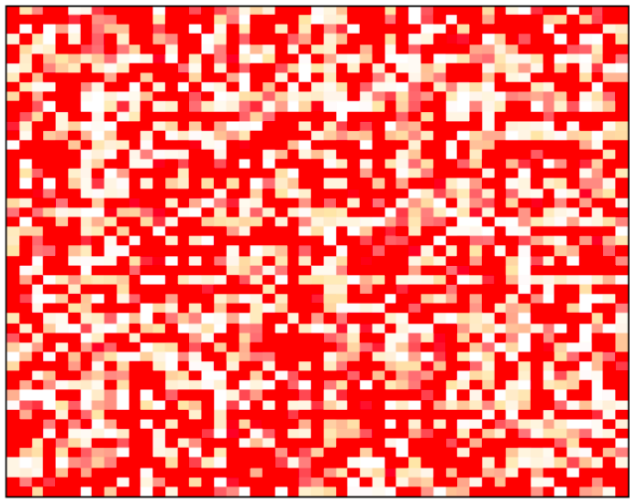
sampled matrix



Gradient descent output $\mathbf{UA}$



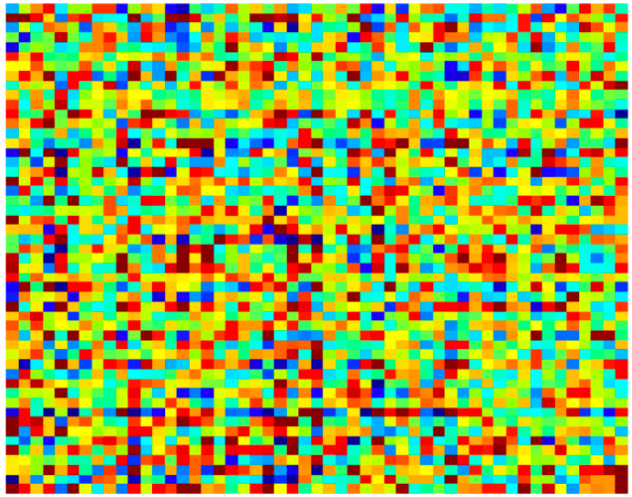
squared error $(\mathbf{X}_{ji} - (\mathbf{UA})_{ji})^2$



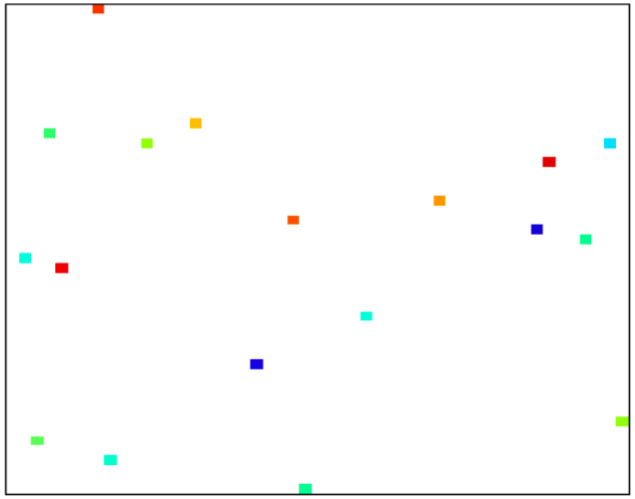
0.50% sampled

Example: 2000×2000 rank-8 random matrix

low-rank matrix $\mathbf{X}$



sampled matrix



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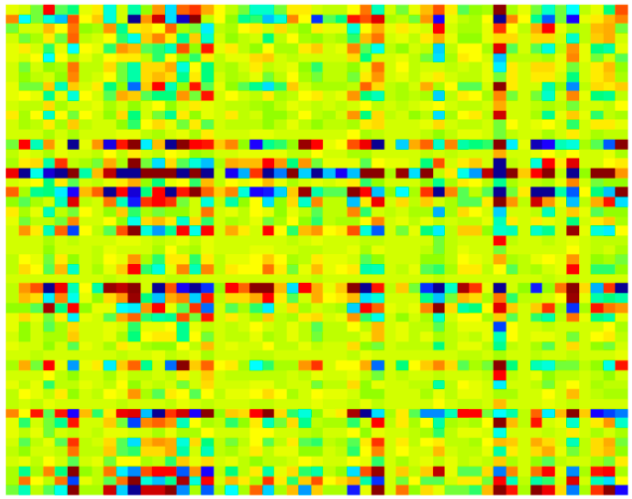
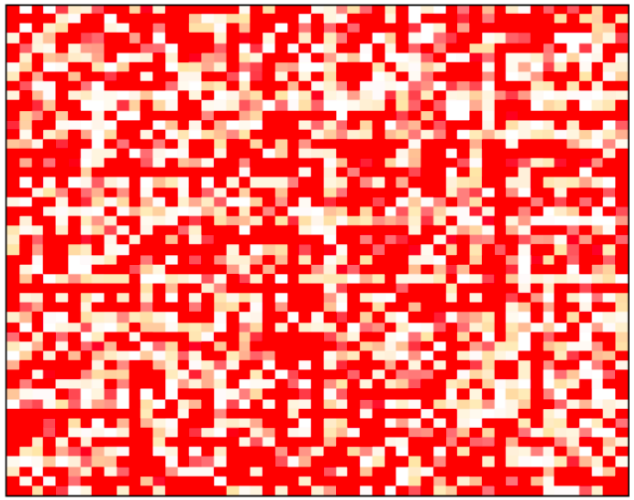


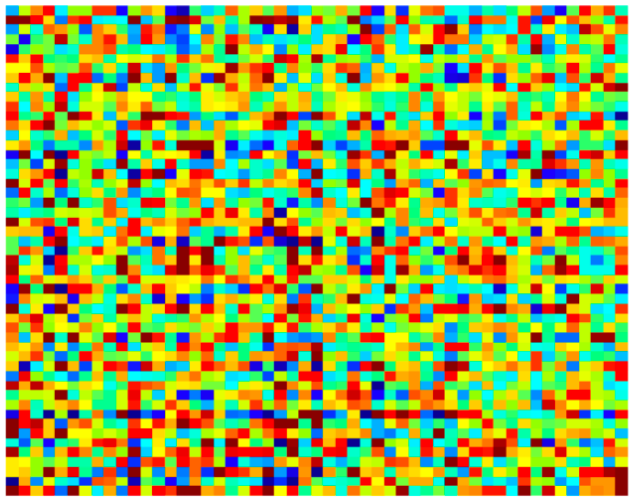
squared error $(\mathbf{X}_{ji} - (\mathbf{UA})_{ji})^2$



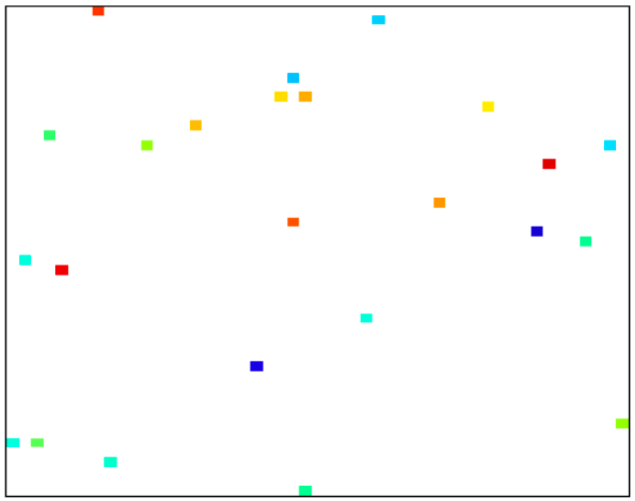
0.75% sampled

Example: 2000×2000 rank-8 random matrix

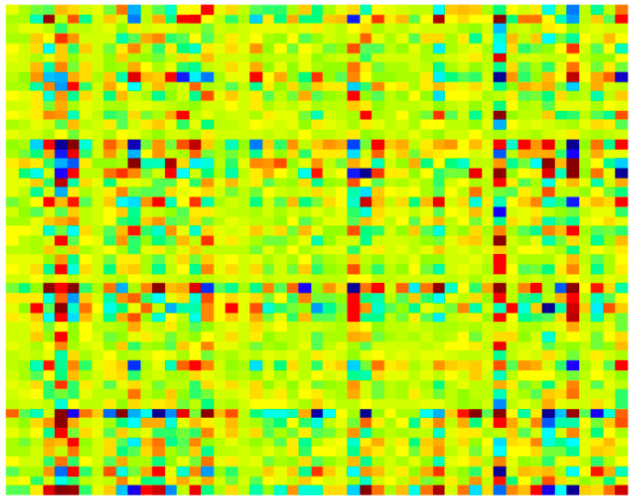
low-rank matrix $\mathbf{X}$



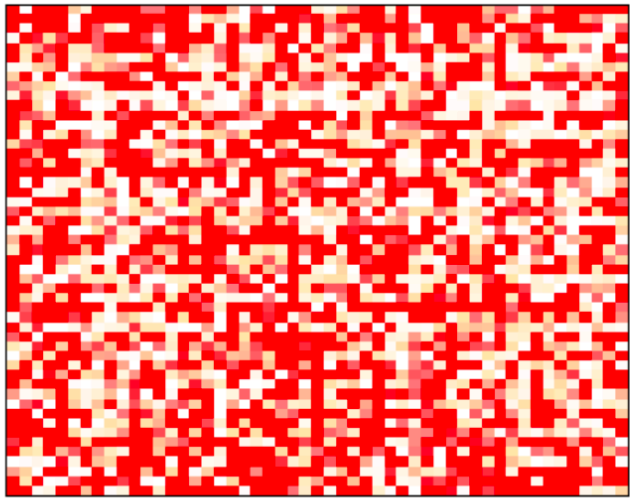
sampled matrix



Gradient descent output $\mathbf{UA}$



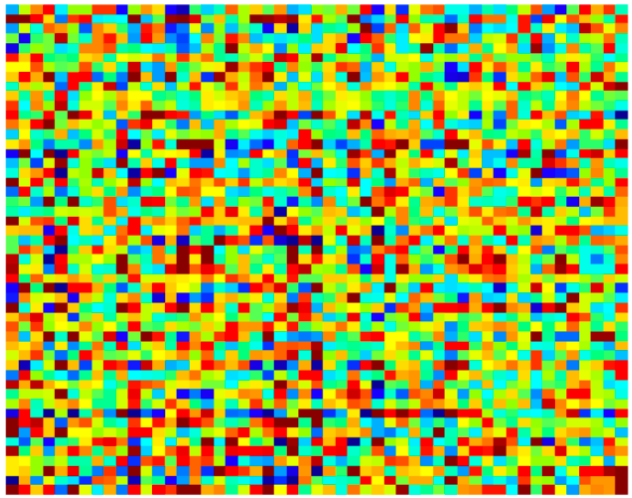
squared error $(\mathbf{X}_{ji} - (\mathbf{UA})_{ji})^2$



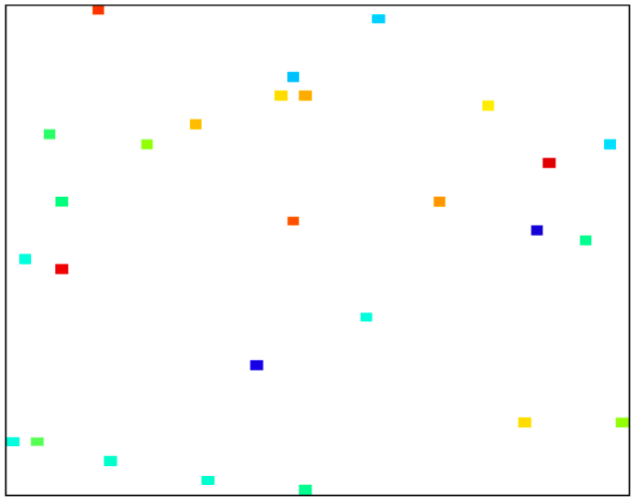
1.00% sampled

Example: 2000×2000 rank-8 random matrix

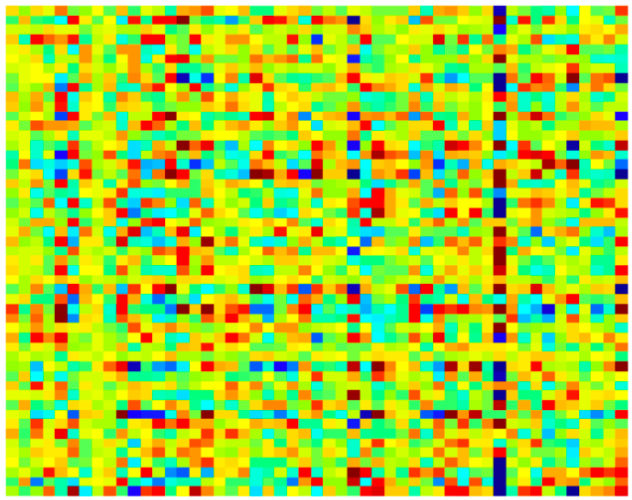
low-rank matrix $\mathbf{X}$



sampled matrix



Gradient descent output $\mathbf{UA}$



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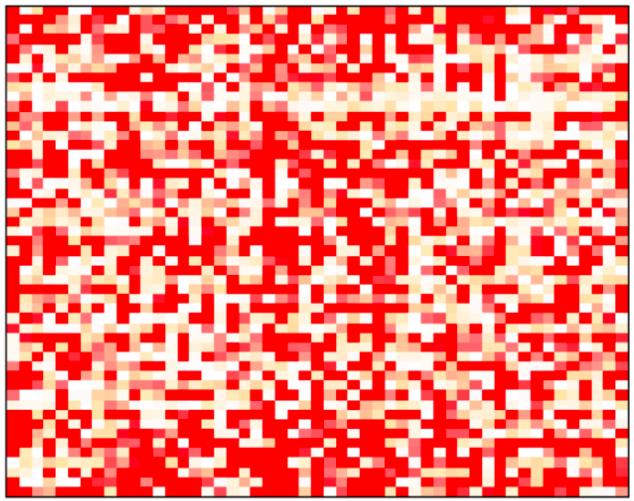
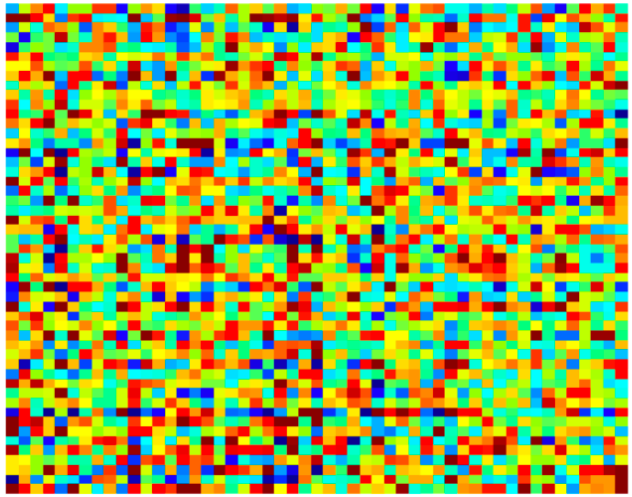


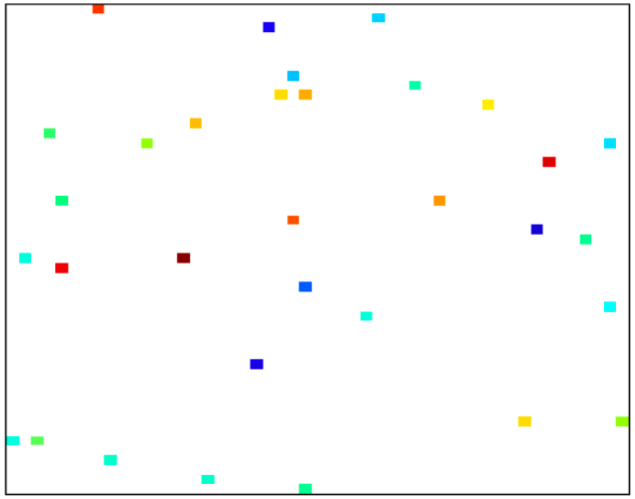
1.25% sampled

Example: 2000×2000 rank-8 random matrix

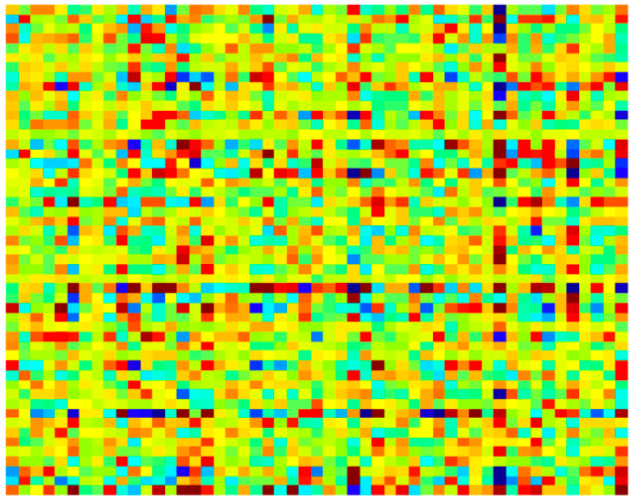
low-rank matrix $\mathbf{X}$



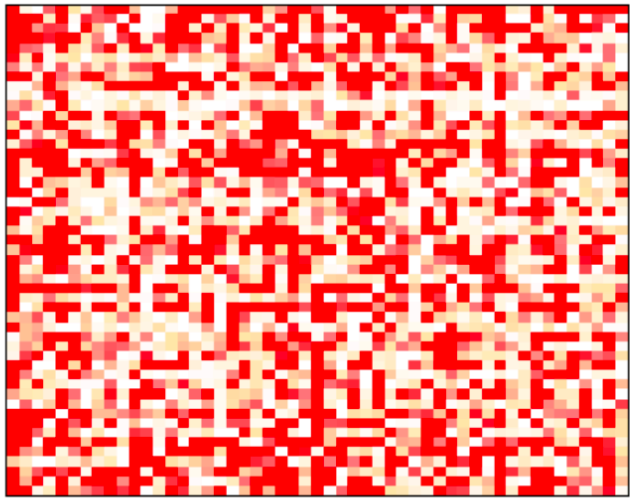
sampled matrix



Gradient descent output $\mathbf{UA}$



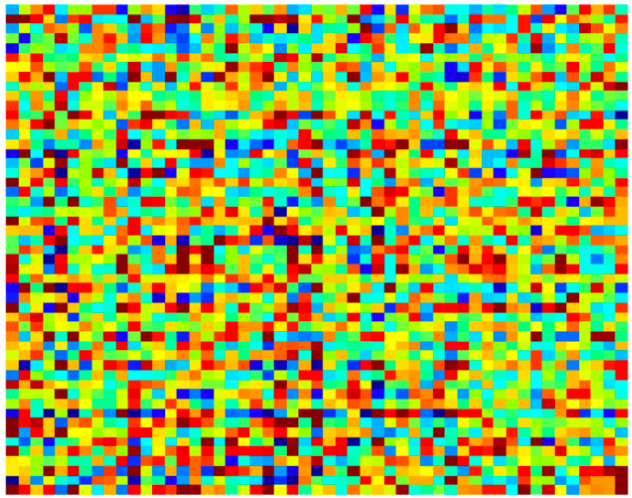
squared error $(\mathbf{X}_{ji} - (\mathbf{UA})_{ji})^2$



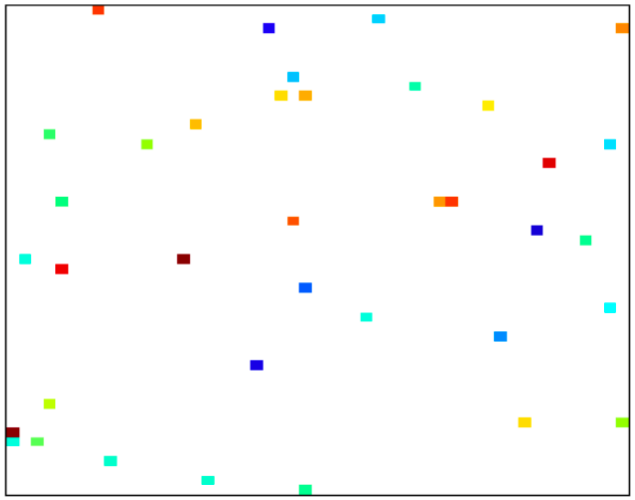
1.50% sampled

Example: 2000×2000 rank-8 random matrix

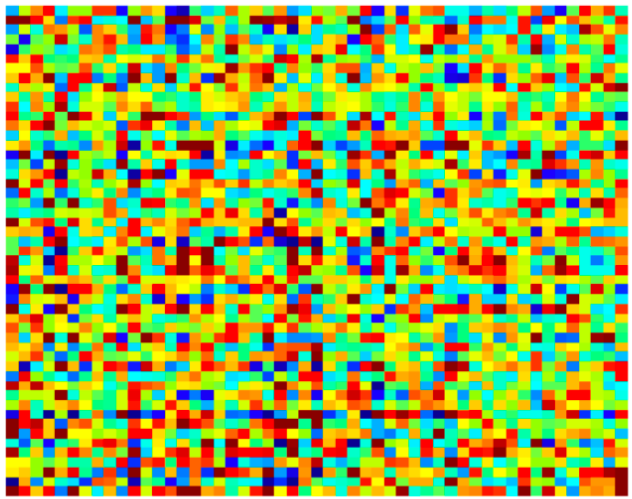
low-rank matrix $\mathbf{X}$



sampled matrix



Gradient descent output $\mathbf{UA}$



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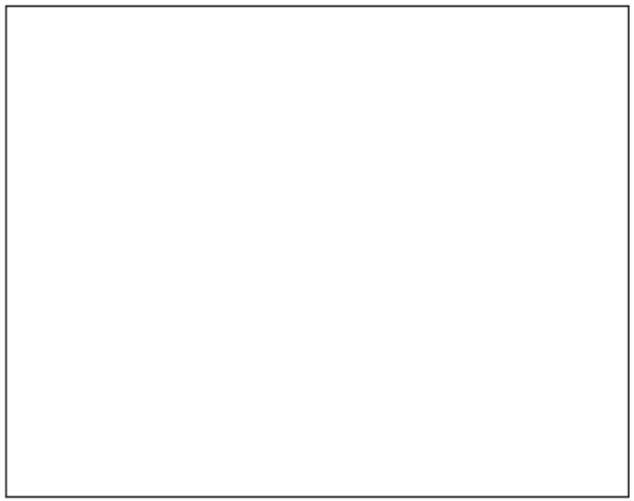


1.75% sampled

Convolutional Neural Networks

Multi-layer Neural Network

$$a^{(1)} = x$$

$$z^{(2)} = \Theta^{(1)} a^{(1)}$$

$$a^{(2)} = g\left(z^{(2)}\right)$$

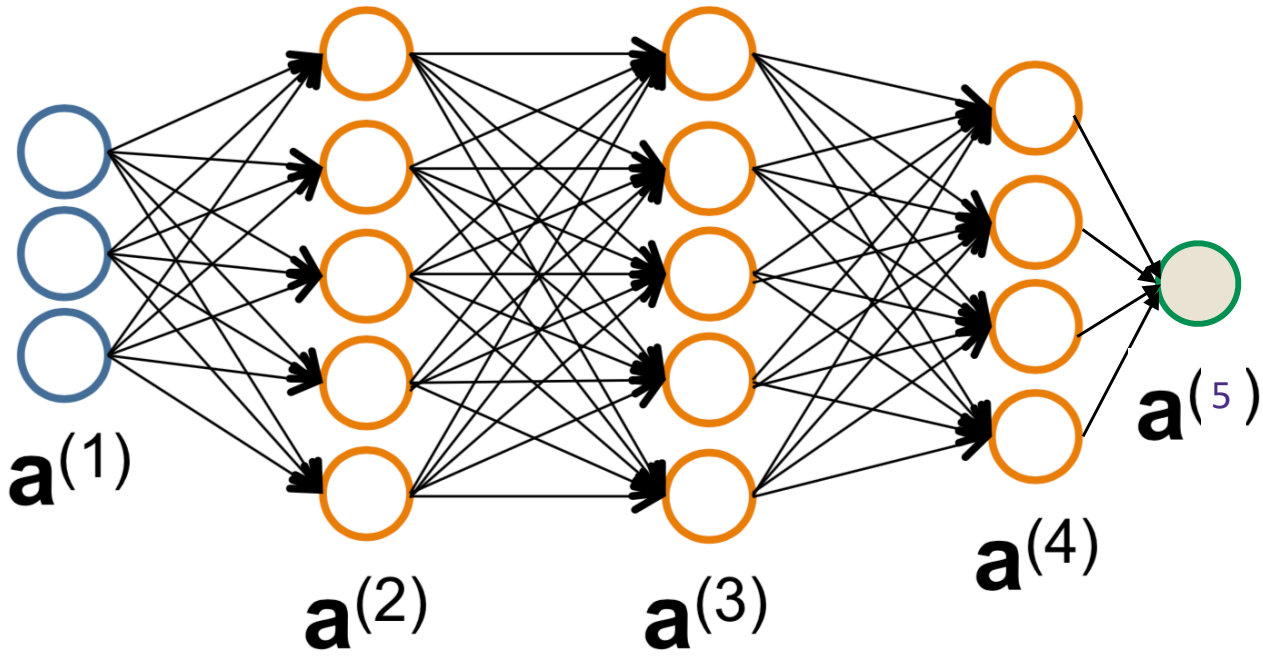
$$\vdots$$

$$z^{(l+1)} = \Theta^{(l)} a^{(l)}$$

$$a^{(l+1)} = g\left(z^{(l+1)}\right)$$

$$\vdots$$

$$\hat{y} = a^{(L+1)}$$



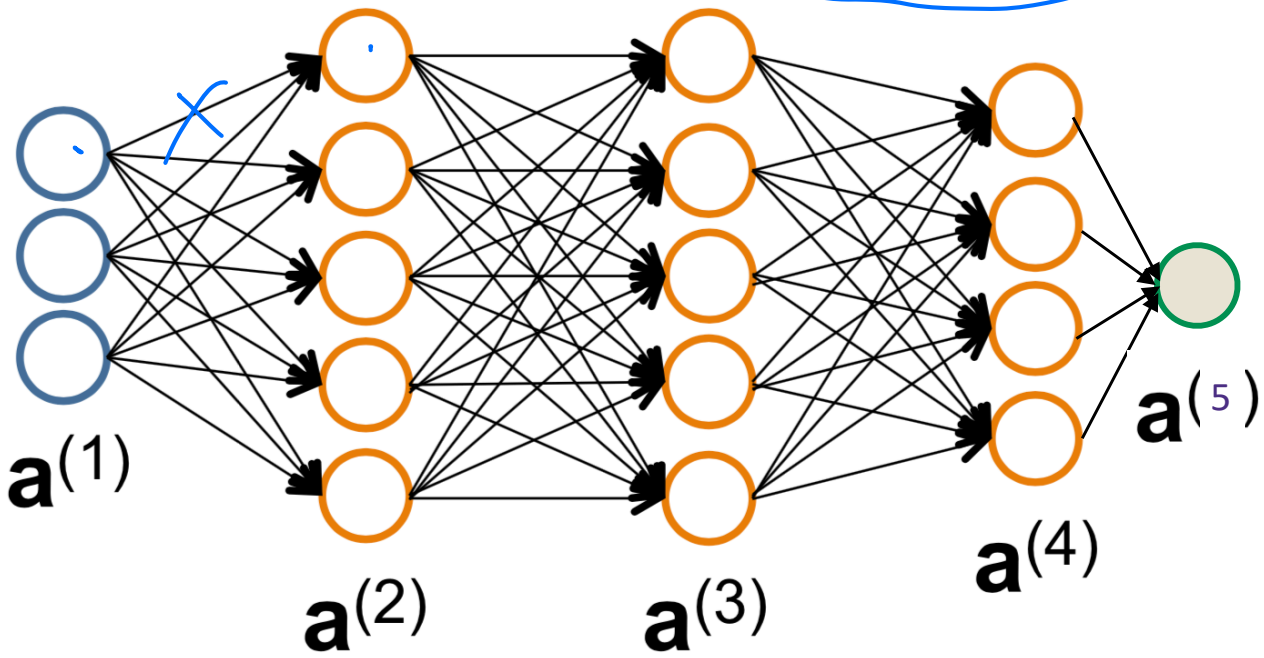
$$L(y, \hat{y}) = y \log(\hat{y}) + (1 - y)\log(1 - \hat{y})$$

$$g(z) = \frac{1}{1 + e^{-z}}$$

Binary Logistic Regression

Neural Network Architecture

The neural network architecture is defined by the number of layers, and the number of nodes in each layer, but also by **allowable edges**.



Neural Network Architecture

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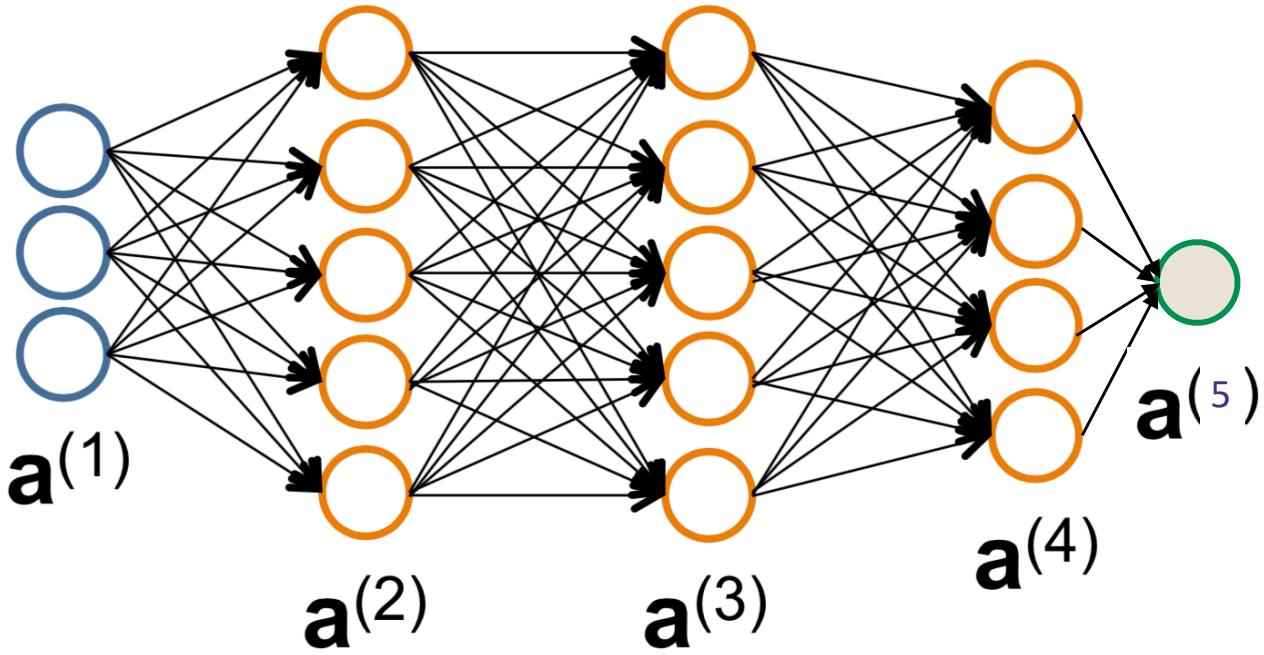


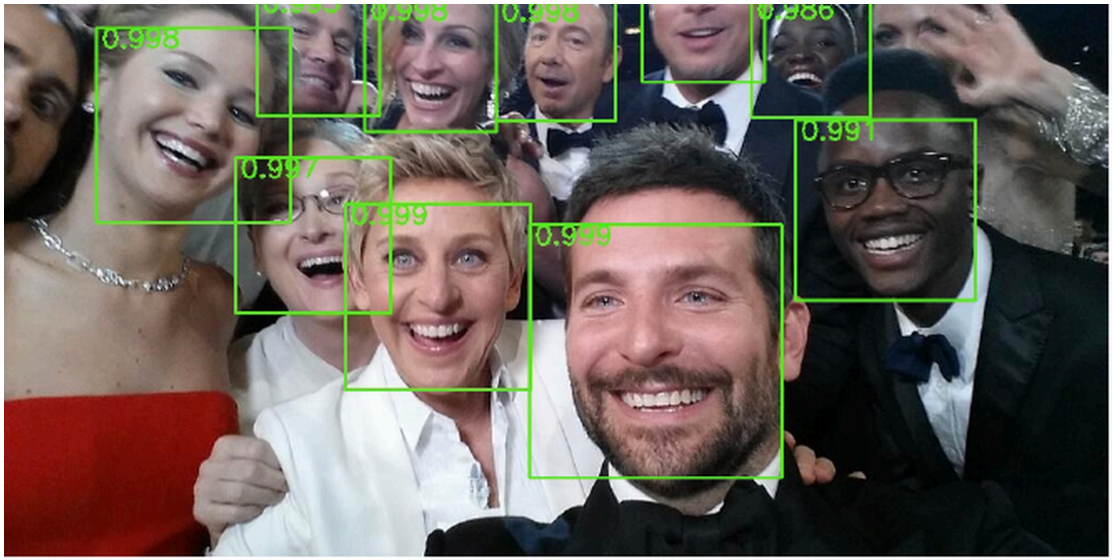
We say a layer is **Fully Connected (FC)** if all linear mappings from the current layer to the next layer are permissible.

$$\mathbf{a}^{(k+1)} = g(\Theta \mathbf{a}^{(k)}) \quad \text{for any } \Theta \in \mathbb{R}^{n_{k+1} \times n_k}$$

A lot of parameters!! $n_1 n_2 + n_2 n_3 + \cdots + n_L n_{L+1}$

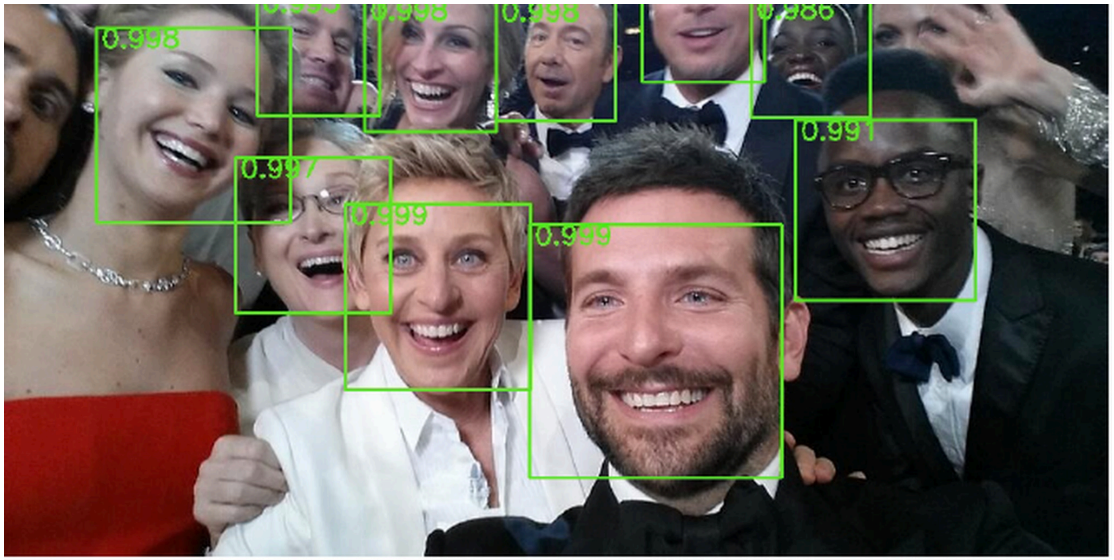
Neural Network Architecture

Objects are often **localized in space** so to find the faces in an image, not every pixel is important for classification—makes sense to drag a window across an image.

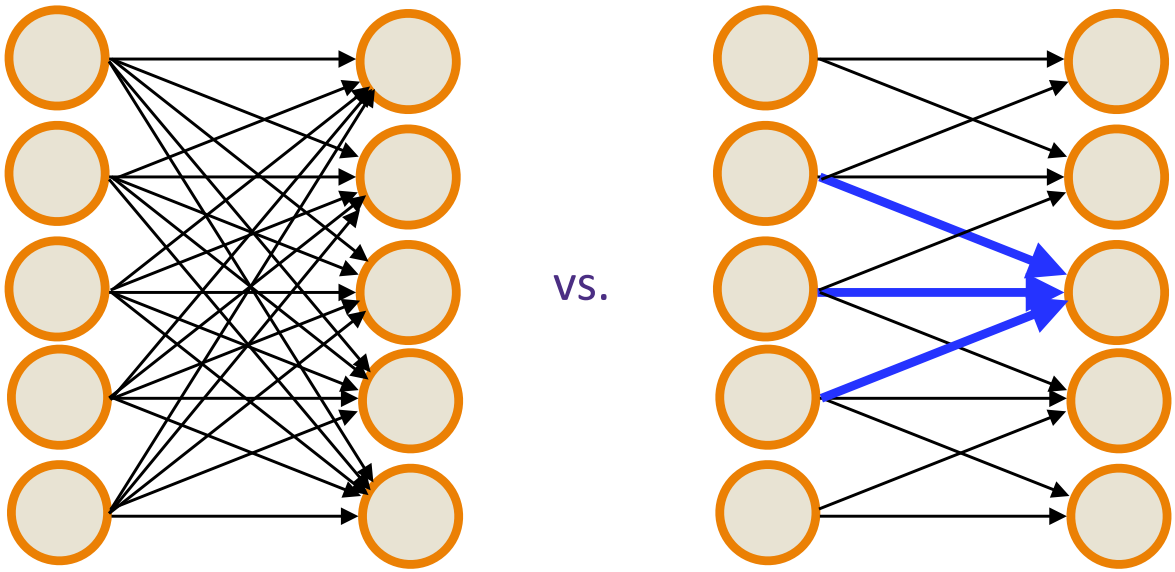


Neural Network Architecture

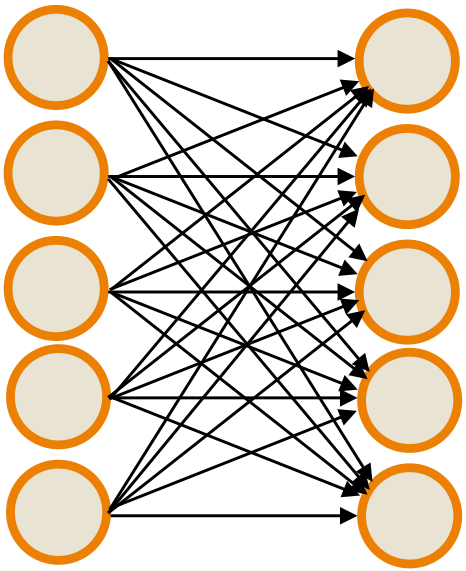
Objects are often **localized in space** so to find the faces in an image, not every pixel is important for classification—makes sense to drag a window across an image.



Similarly, to identify edges or other local structure, it makes sense to only look at **local information**

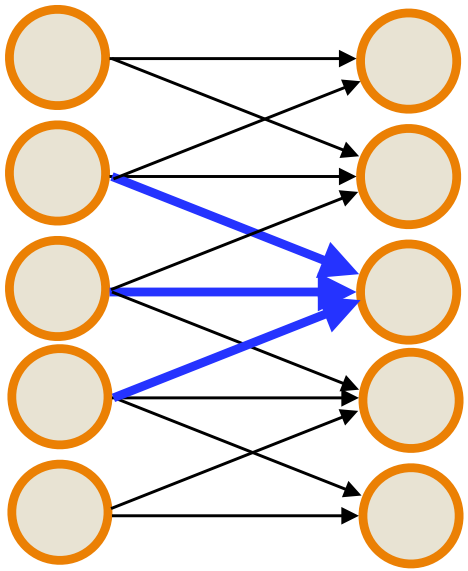


Neural Network Architecture



$\mathbf{a}^{(k)}$ $\mathbf{a}^{(k+1)}$

vs.



$\mathbf{a}^{(k)}$ $\mathbf{a}^{(k+1)}$

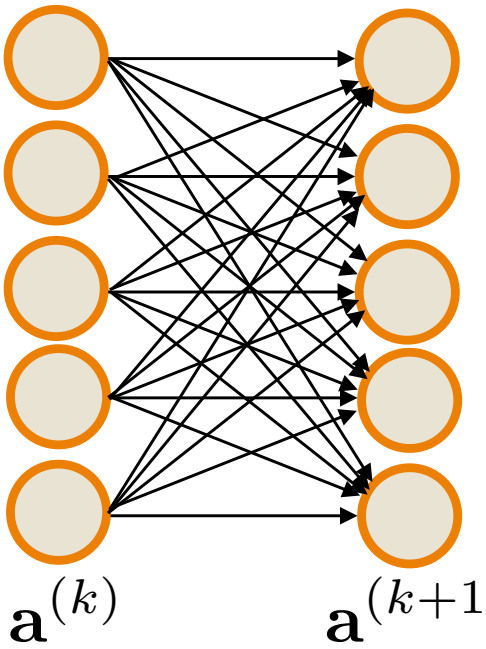
hand coded

$$\begin{bmatrix} \Theta_{0,0} & \Theta_{0,1} & \Theta_{0,2} & \Theta_{0,3} & \Theta_{0,4} \\ \Theta_{1,0} & \Theta_{1,1} & \Theta_{1,2} & \Theta_{1,3} & \Theta_{1,4} \\ \Theta_{2,0} & \Theta_{2,1} & \Theta_{2,2} & \Theta_{2,3} & \Theta_{2,4} \\ \Theta_{3,0} & \Theta_{3,1} & \Theta_{3,2} & \Theta_{3,3} & \Theta_{3,4} \\ \Theta_{4,0} & \Theta_{4,1} & \Theta_{4,2} & \Theta_{4,3} & \Theta_{4,4} \end{bmatrix} \qquad \begin{bmatrix} \Theta_{0,0} & \Theta_{0,1} & 0 & 0 & 0 \\ \Theta_{1,0} & \Theta_{1,1} & \Theta_{1,2} & 0 & 0 \\ 0 & \Theta_{2,1} & \Theta_{2,2} & \Theta_{2,3} & 0 \\ 0 & 0 & \Theta_{3,2} & \Theta_{3,3} & \Theta_{3,4} \\ 0 & 0 & 0 & \Theta_{4,3} & \Theta_{4,4} \end{bmatrix}$$

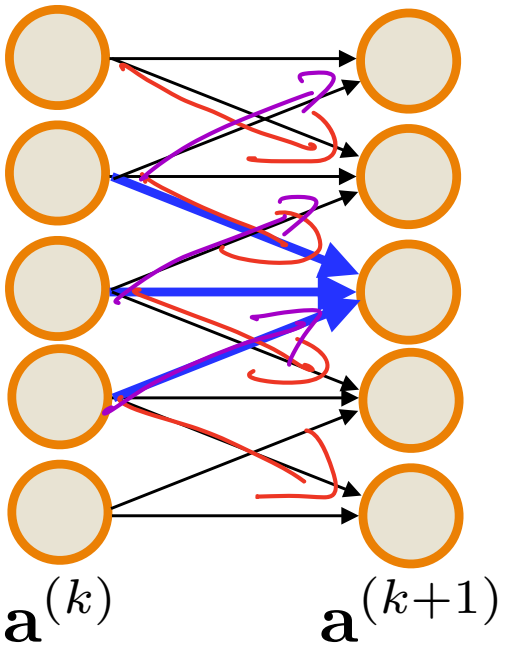
Parameters: n^2 $3n-2$

$$\mathbf{a}_i^{(k+1)} = g\left(\sum_{j=0}^{n-1} \Theta_{i,j}\mathbf{a}_j^{(k)}\right)$$

Neural Network Architecture



vs.



Mirror/share local weights everywhere (e.g., structure equally likely to be anywhere in image)

$$\begin{bmatrix} \Theta_{0,0} & \Theta_{0,1} & \Theta_{0,2} & \Theta_{0,3} & \Theta_{0,4} \\ \Theta_{1,0} & \Theta_{1,1} & \Theta_{1,2} & \Theta_{1,3} & \Theta_{1,4} \\ \Theta_{2,0} & \Theta_{2,1} & \Theta_{2,2} & \Theta_{2,3} & \Theta_{2,4} \\ \Theta_{3,0} & \Theta_{3,1} & \Theta_{3,2} & \Theta_{3,3} & \Theta_{3,4} \\ \Theta_{4,0} & \Theta_{4,1} & \Theta_{4,2} & \Theta_{4,3} & \Theta_{4,4} \end{bmatrix} \qquad \begin{bmatrix} \Theta_{0,0} & \Theta_{0,1} & 0 & 0 & 0 \\ \Theta_{1,0} & \Theta_{1,1} & \Theta_{1,2} & 0 & 0 \\ 0 & \Theta_{2,1} & \Theta_{2,2} & \Theta_{2,3} & 0 \\ 0 & 0 & \Theta_{3,2} & \Theta_{3,3} & \Theta_{3,4} \\ 0 & 0 & 0 & \Theta_{4,3} & \Theta_{4,4} \end{bmatrix} \qquad \begin{bmatrix} \theta_1 & \theta_2 & 0 & 0 & 0 \\ \theta_0 & \theta_1 & \theta_2 & 0 & 0 \\ 0 & \theta_0 & \theta_1 & \theta_2 & 0 \\ 0 & 0 & \theta_0 & \theta_1 & \theta_2 \\ 0 & 0 & 0 & \theta_0 & \theta_1 \end{bmatrix}$$

Parameters: n^2 $3n-2$ 3

$$\mathbf{a}_i^{(k+1)} = g\left(\sum_{j=0}^{n-1} \Theta_{i,j}\mathbf{a}_j^{(k)}\right) \qquad \mathbf{a}_i^{(k+1)} = g\left(\sum_{j=0}^{m-1} \theta_j\mathbf{a}_{i+j}^{(k)}\right)$$

Neural Network Architecture

Fully Connected (FC) Layer

$$\begin{bmatrix} \Theta_{0,0} & \Theta_{0,1} & \Theta_{0,2} & \Theta_{0,3} & \Theta_{0,4} \\ \Theta_{1,0} & \Theta_{1,1} & \Theta_{1,2} & \Theta_{1,3} & \Theta_{1,4} \\ \Theta_{2,0} & \Theta_{2,1} & \Theta_{2,2} & \Theta_{2,3} & \Theta_{2,4} \\ \Theta_{3,0} & \Theta_{3,1} & \Theta_{3,2} & \Theta_{3,3} & \Theta_{3,4} \\ \Theta_{4,0} & \Theta_{4,1} & \Theta_{4,2} & \Theta_{4,3} & \Theta_{4,4} \end{bmatrix}$$

$$\mathbf{a}_i^{(k+1)} = g\left(\sum_{j=0}^{n-1} \Theta_{i,j}\mathbf{a}_j^{(k)}\right)$$

Convolutional (CONV) Layer (1 filter)

$$\begin{bmatrix} \theta_1 & \theta_2 & 0 & 0 & 0 \\ \theta_0 & \theta_1 & \theta_2 & 0 & 0 \\ 0 & \theta_0 & \theta_1 & \theta_2 & 0 \\ 0 & 0 & \theta_0 & \theta_1 & \theta_2 \\ 0 & 0 & 0 & \theta_0 & \theta_1 \end{bmatrix}$$

m=3

$$\mathbf{a}_i^{(k+1)} = g\left(\sum_{j=0}^{m-1} \theta_j \mathbf{a}_{i+j}^{(k)}\right) = g([\theta * \mathbf{a}^{(k)}]_i)$$

Convolution*

$\theta = (\theta_0, \ldots, \theta_{m-1}) \in \mathbb{R}^m$ is referred to as a "filter"

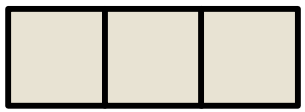
Example (1d convolution)

$$(\theta * x)_i = \sum_{j=0}^{m-1} \theta_j x_{i+j}$$

stride = 1

Input $x \in \mathbb{R}^n$

Filter $\theta \in \mathbb{R}^m$



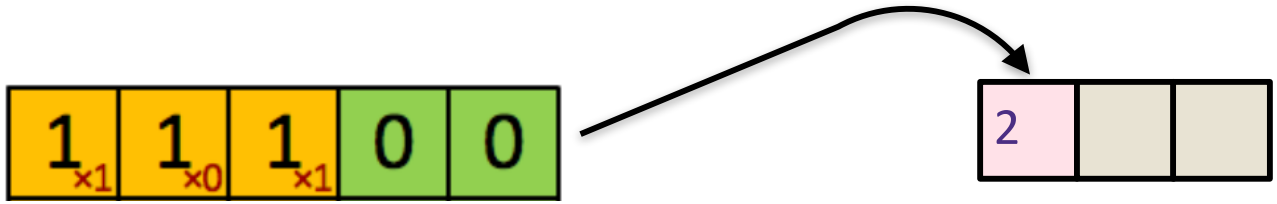
Output $\theta * x$

Example (1d convolution)

$$(\theta * x)_i = \sum_{j=0}^{m-1} \theta_j x_{i+j}$$

Input $x \in \mathbb{R}^n$

Filter $\theta \in \mathbb{R}^m$



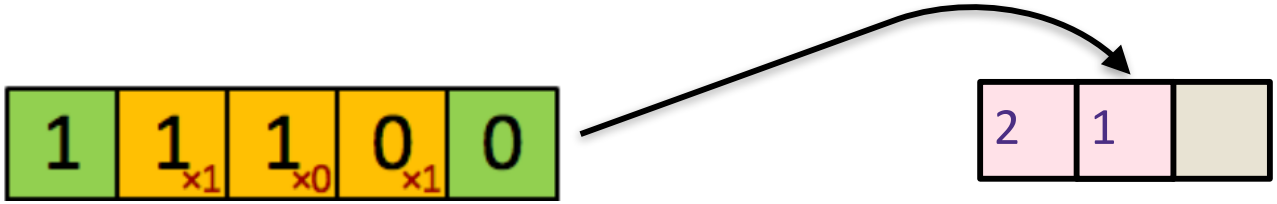
Output $\theta * x$

Example (1d convolution)

$$(\theta * x)_i = \sum_{j=0}^{m-1} \theta_j x_{i+j}$$

Input $x \in \mathbb{R}^n$

Filter $\theta \in \mathbb{R}^m$

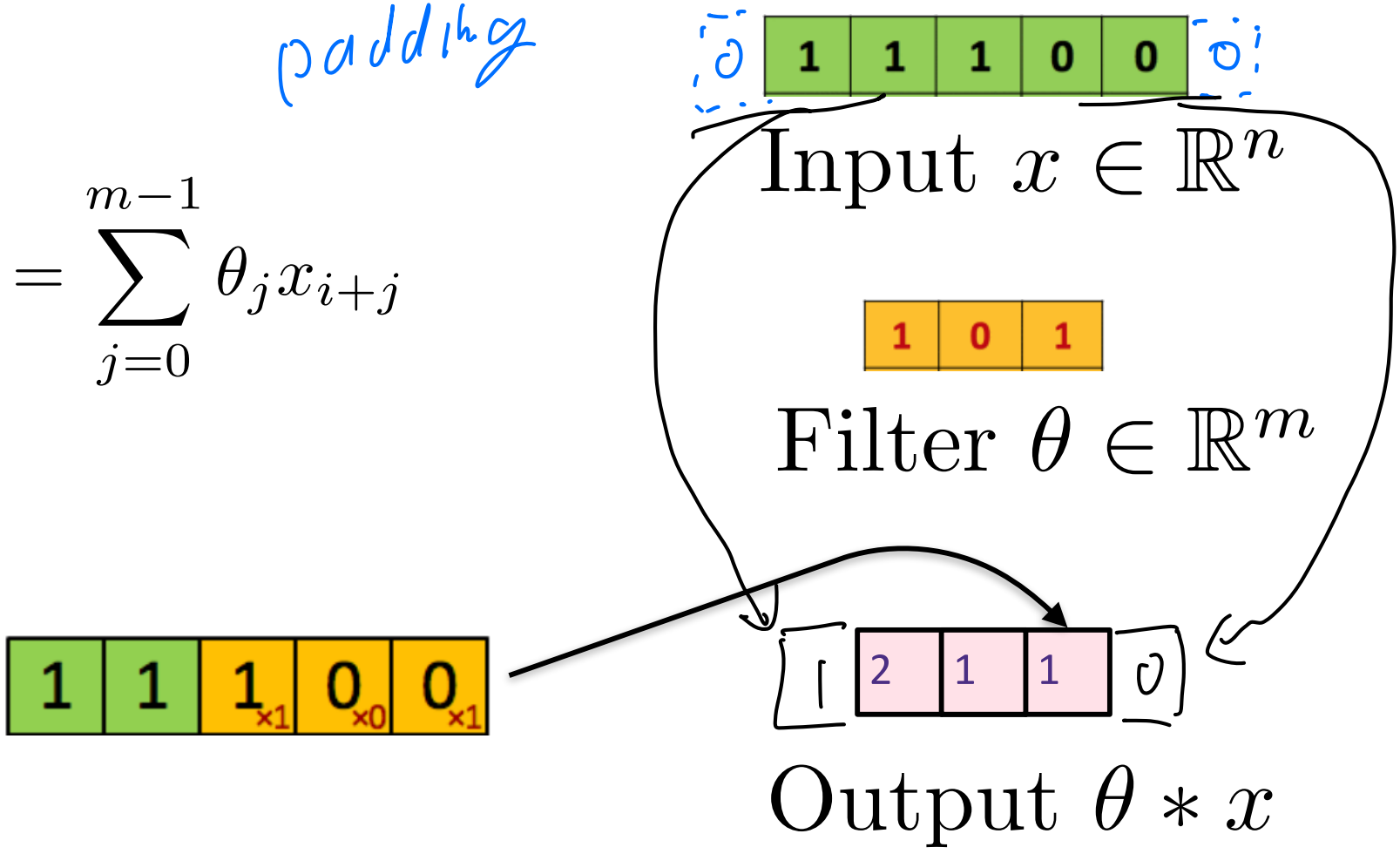


Output $\theta * x$

Example (1d convolution)

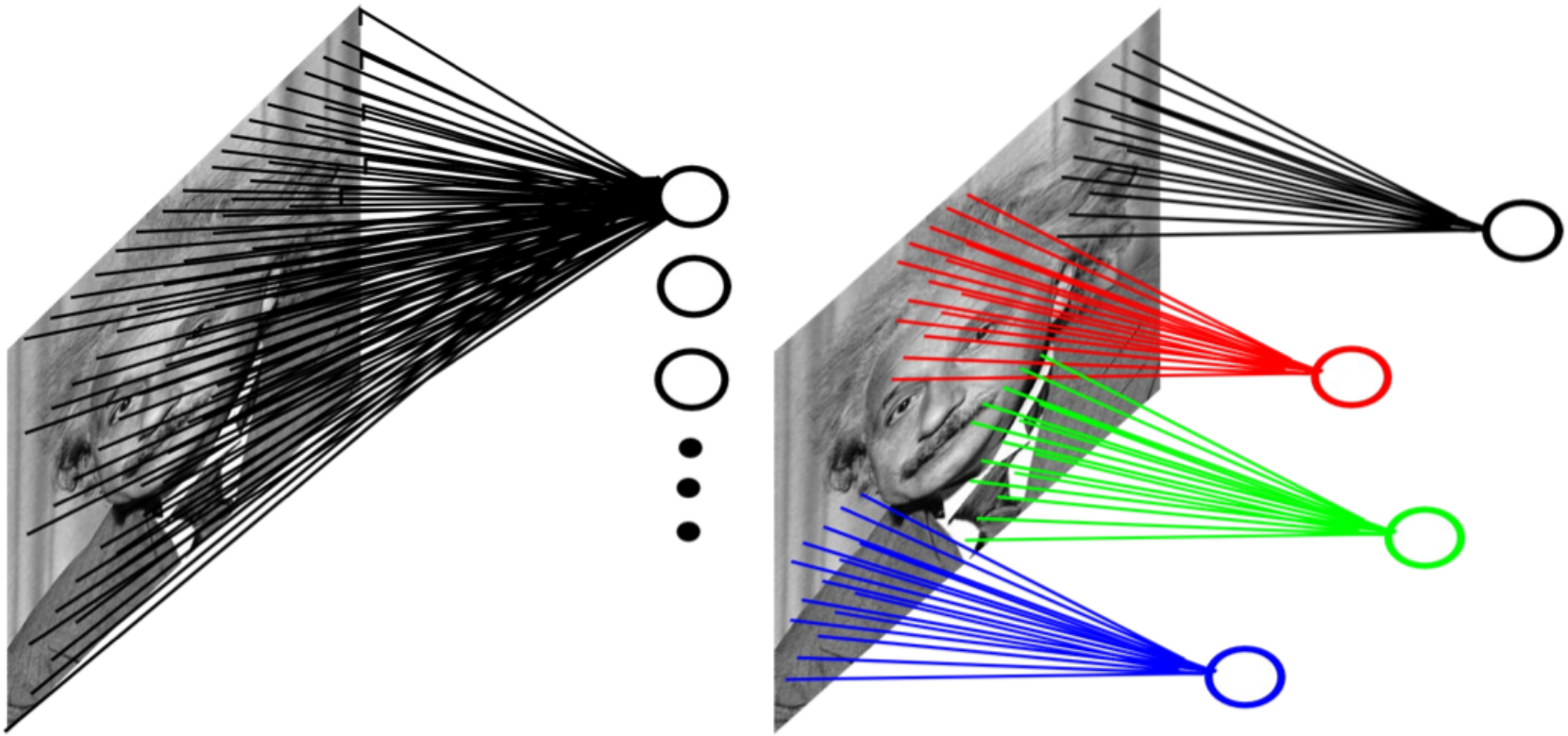
padding

$$(\theta * x)_i = \sum_{j=0}^{m-1} \theta_j x_{i+j}$$



2d Convolution Layer

- **Example: 200x200 image**
  - Fully-connected, 400,000 hidden units = 16 billion parameters
  - Locally-connected, 400,000 hidden units 10x10 fields = 40 million params
  - Local connections capture local dependencies



Convolution of images (2d convolution)

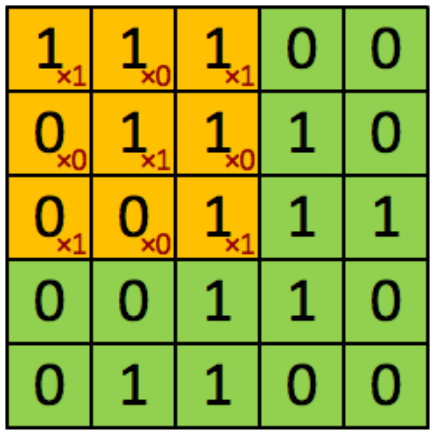
$$(I * K)(i,j) = \sum_m \sum_n I(i+m, j+n) K(m,n)$$

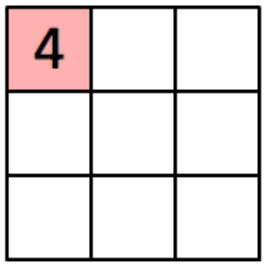
1	1	1	0	0
0	1	1	1	0
0	0	1	1	1
0	0	1	1	0
0	1	1	0	0

Image I

1	0	1
0	1	0
1	0	1

Filter K



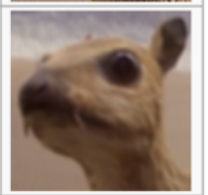
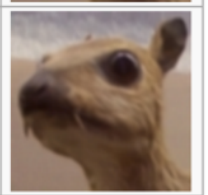


$I * K$

Convolution of images

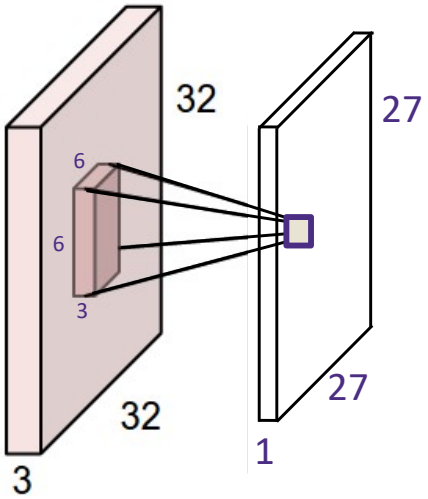
$$(I * K)(i,j) = \sum_m \sum_n I(i+m, j+n) K(m,n)$$

Image I

Operation	Filter K	Convolved Image $I * K$
Edge detection	$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$	
	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	
	$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$	
Sharpen	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix}$	
Box blur (normalized)	$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	
Gaussian blur (approximation)	$\frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$	

Stacking convolved images

$K : k \times k \times r$

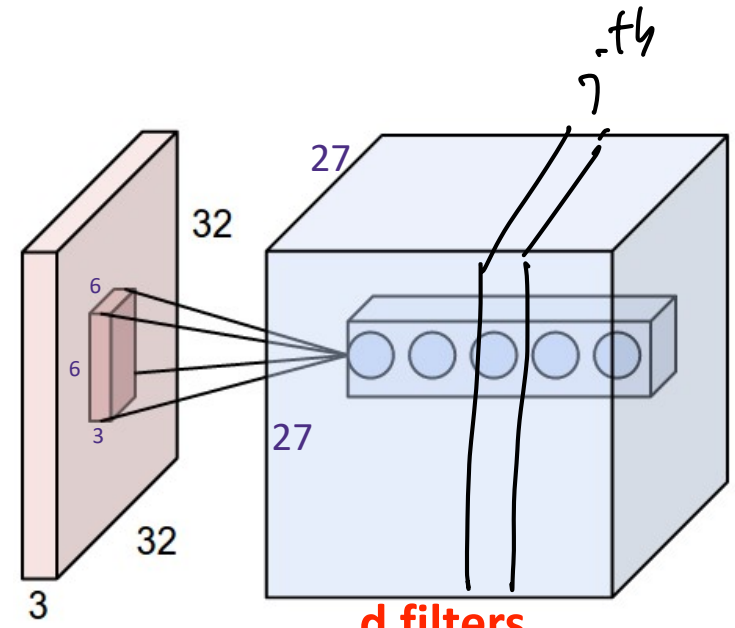


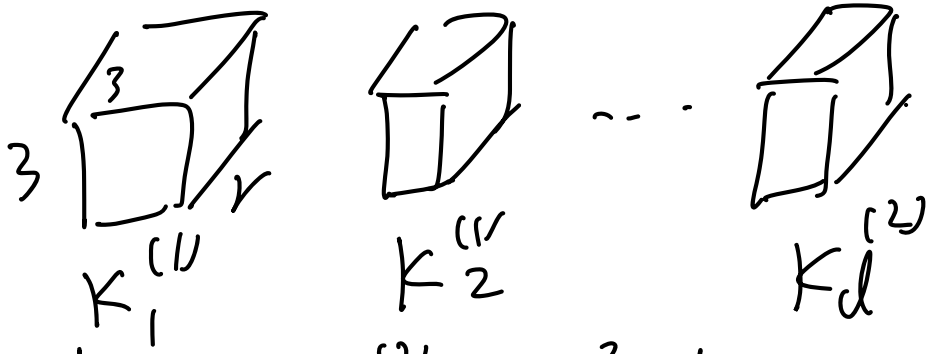
$x \in \mathbb{R}^{n \times n \times r}$

R G B

output: $Z = \sum_{\alpha=1}^{r} X[:,:,\alpha] * K[:,:,\alpha]$

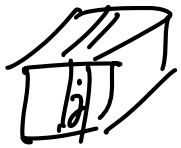
Stacking convolved images





$\{K_j^{(2)}\}_{j=1}^{d}$, $K_j^{(2)} : 3 \times 3 \times d$

Repeat with d filters!



$a^{(2)}$

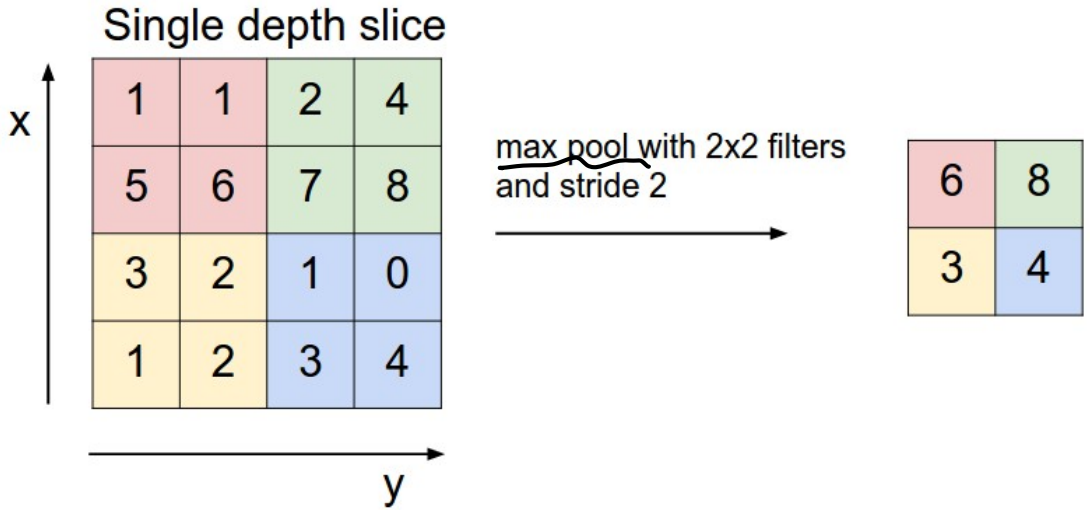
X $a_7^{(1)}$

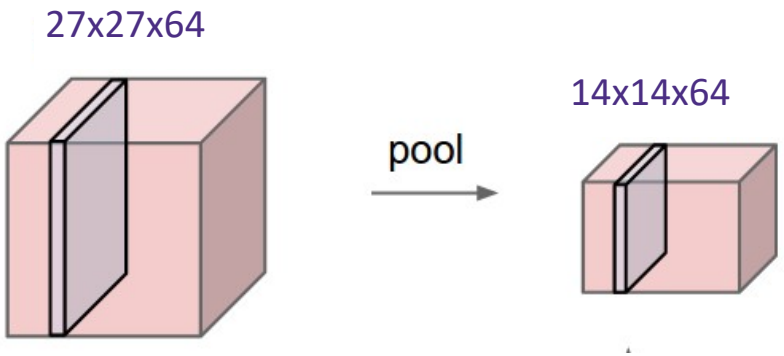
$$a_7^{(1)} = g\left(\sum_{\alpha=1}^{r} X[:,:,\alpha] * K_7^{(1)}[:,:,\alpha]\right)$$

$$a_j^{(2)} = g\left(\sum_{7=1}^{d} a^{(1)}[:,:,7] * K_j^{(2)}[:,:,7]\right)$$

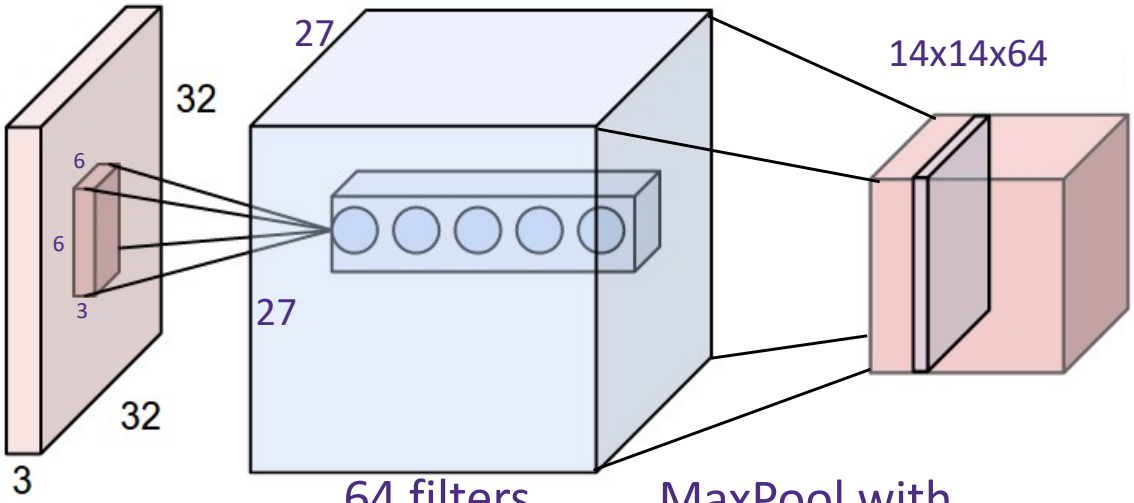
Pooling

Pooling reduces the dimension and can be interpreted as “This filter had a high response in this general region”





Pooling Convolution layer

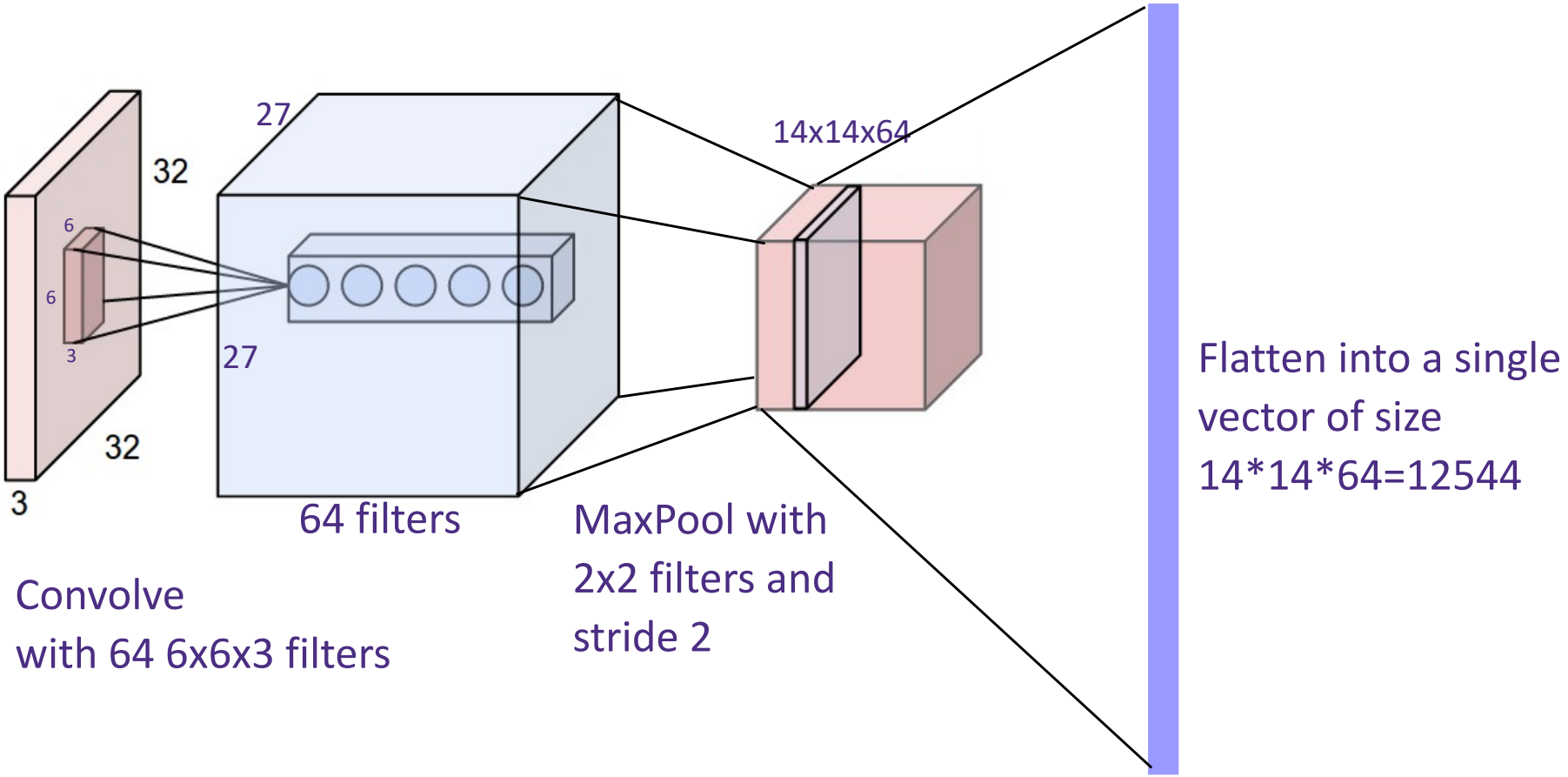


64 filters

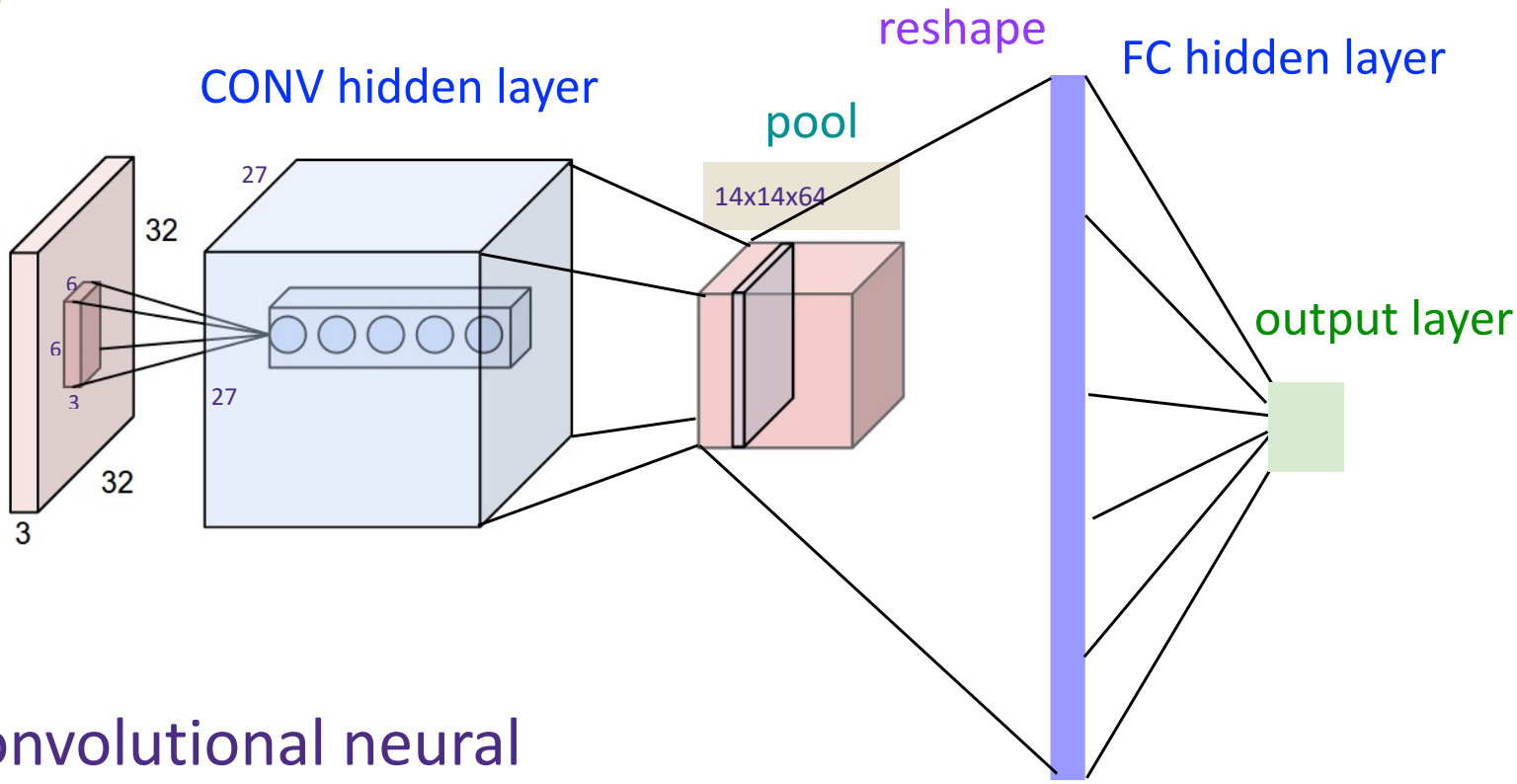
MaxPool with
2x2 filters and
stride 2

Convolve
with 64 6x6x3 filters

Flattening

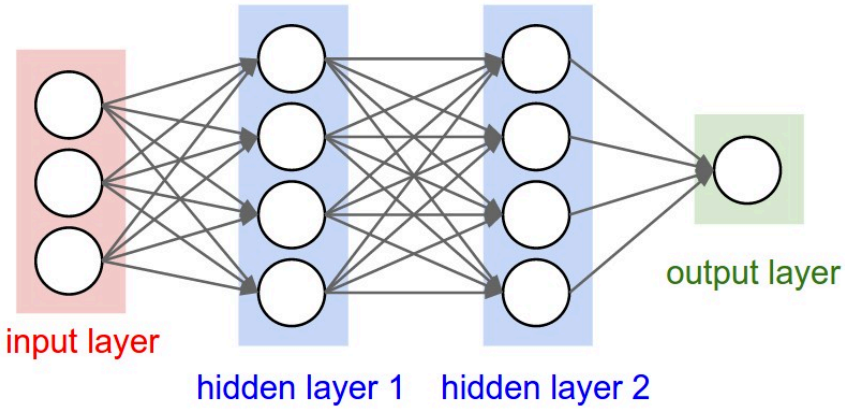

32
6
6
3
32
3
27
27
64 filters
14x14x64
Convolve
with 64 6x6x3 filters
MaxPool with
2x2 filters and
stride 2
Flatten into a single
vector of size
14*14*64=12544

Training Convolutional Networks

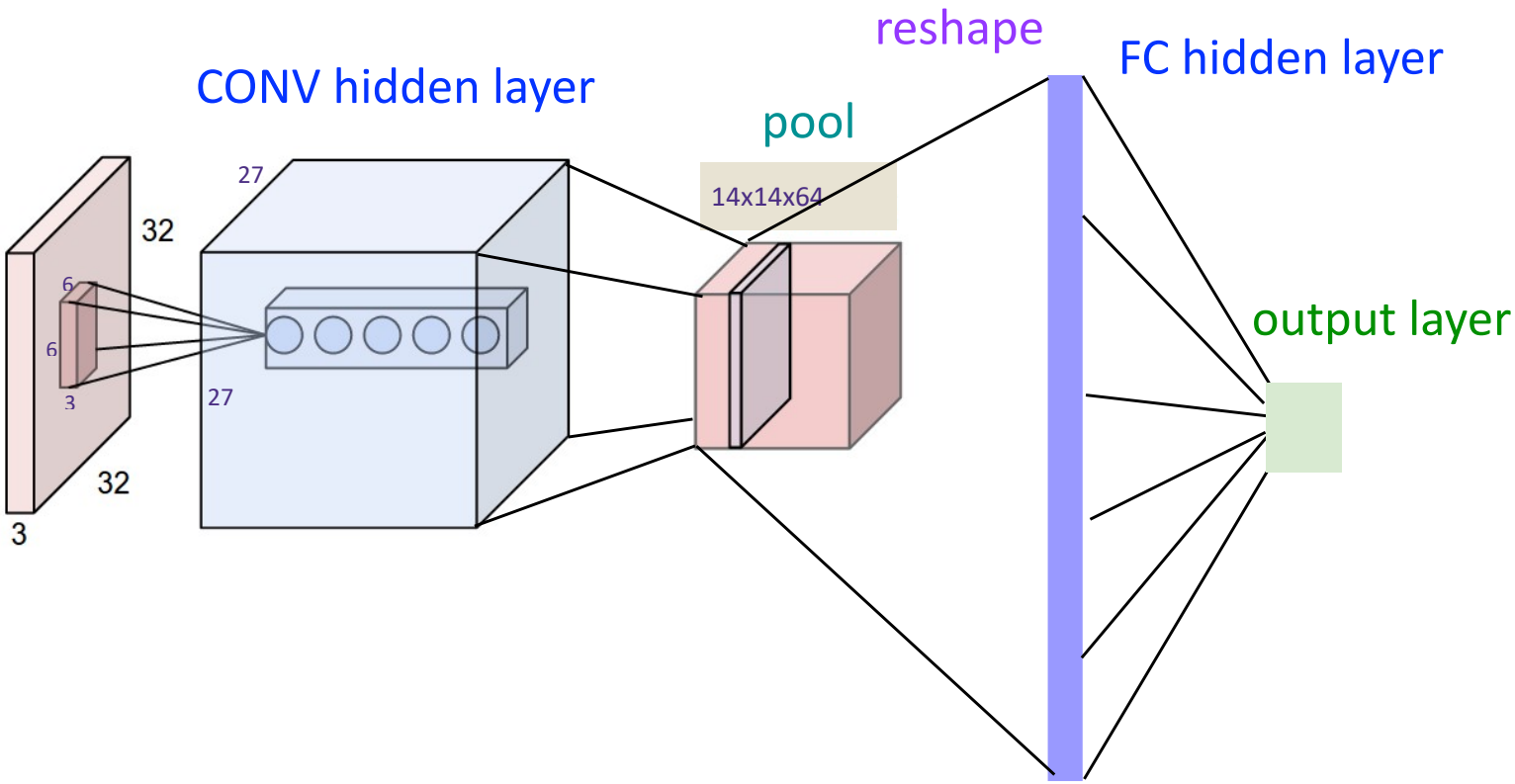


Recall: Convolutional neural networks (CNN) are just regular fully connected (FC) neural networks with some connections removed.

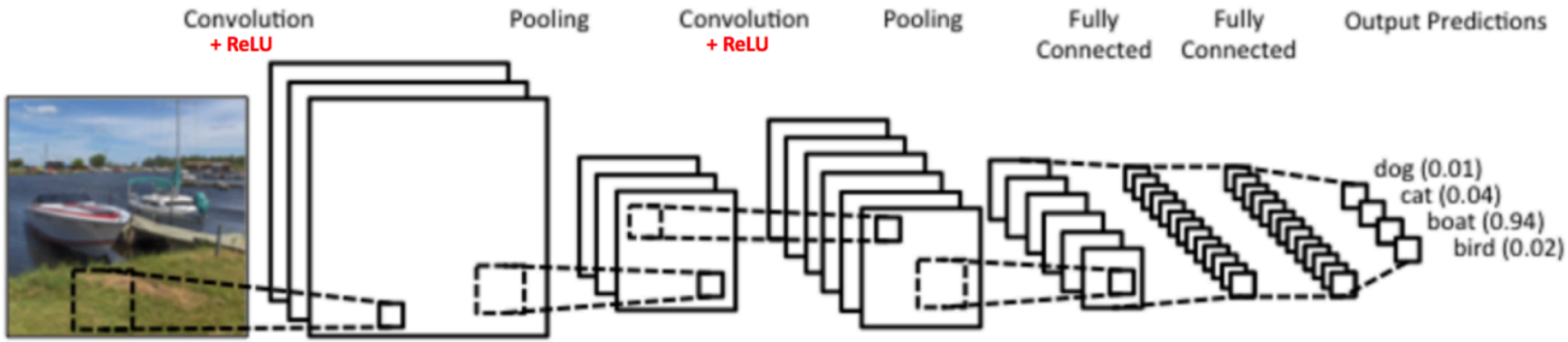
Train with SGD!

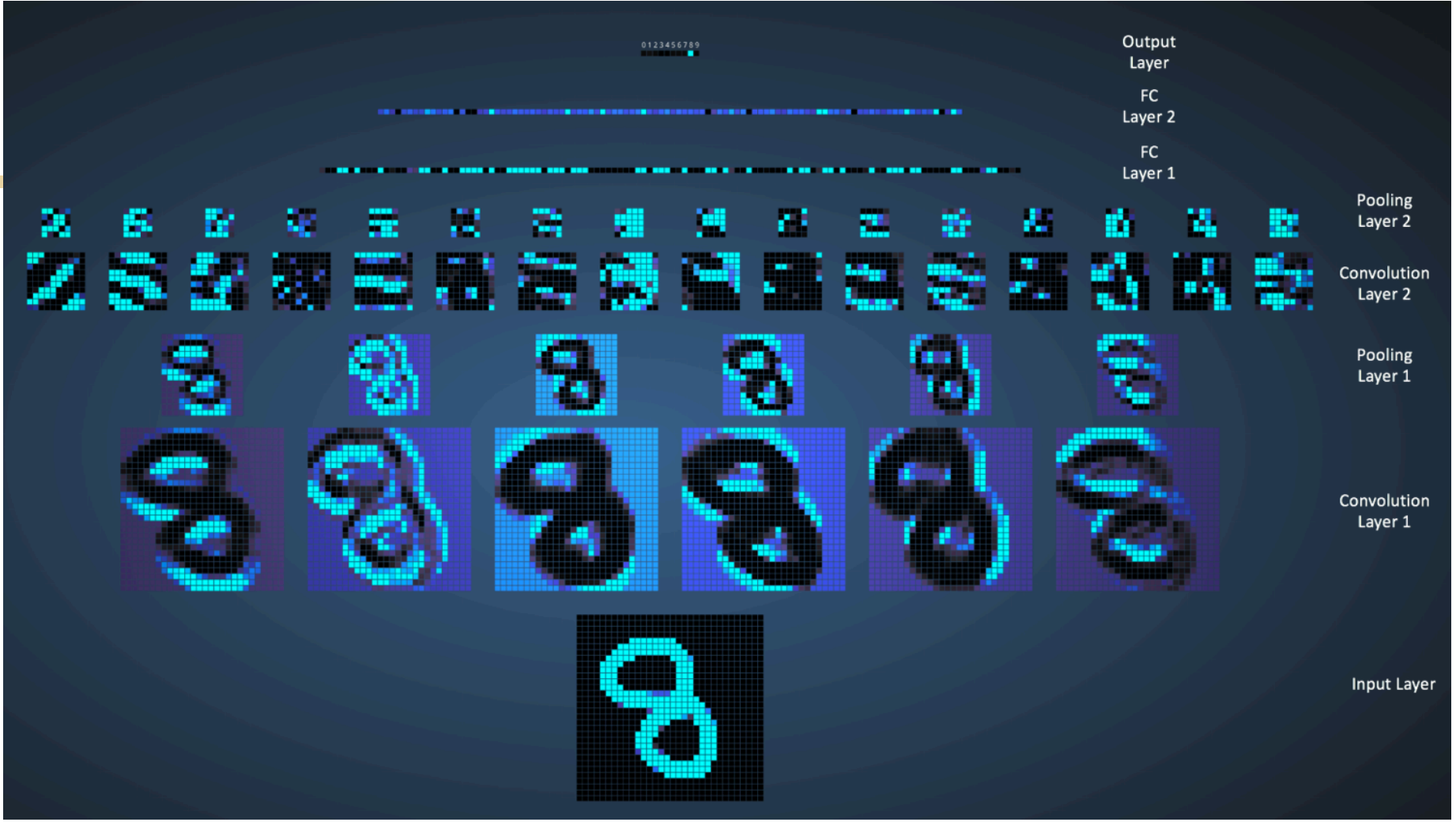


Training Convolutional Networks

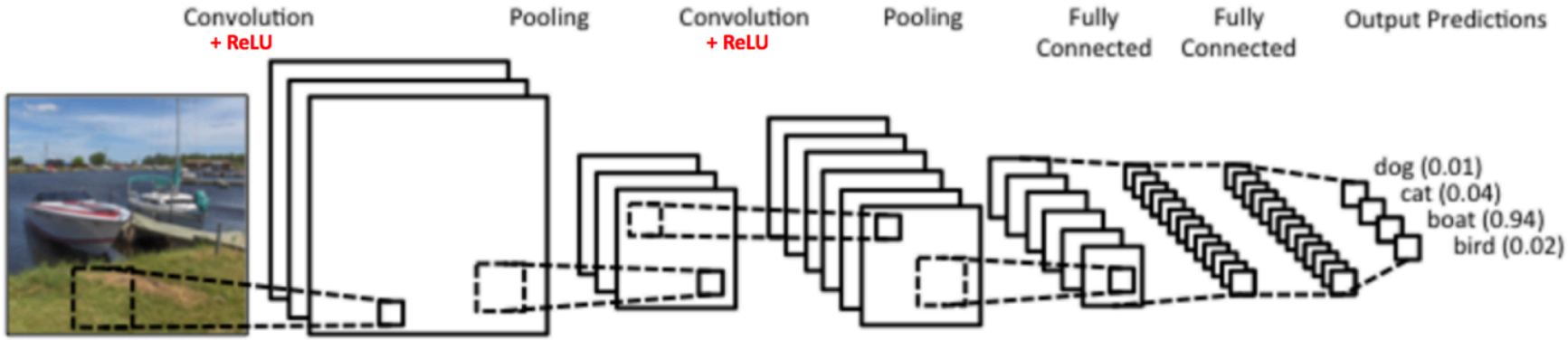


Real example network: LeNet





Real example network: LeNet



Famous CNNs

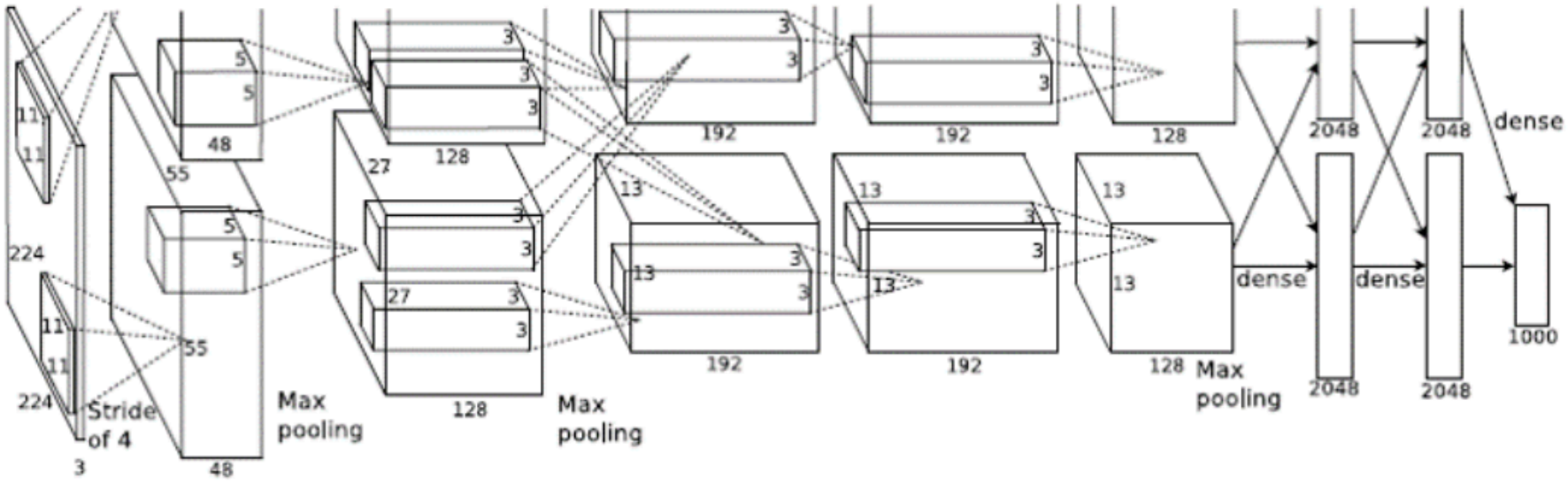
ImageNet Dataset

~14 million images, 20k classes

Deng et al. “Imagenet: a large scale hierarchical image database” ‘09

AlexNet

Breakthrough on ImageNet: ~the beginning of deep learning era



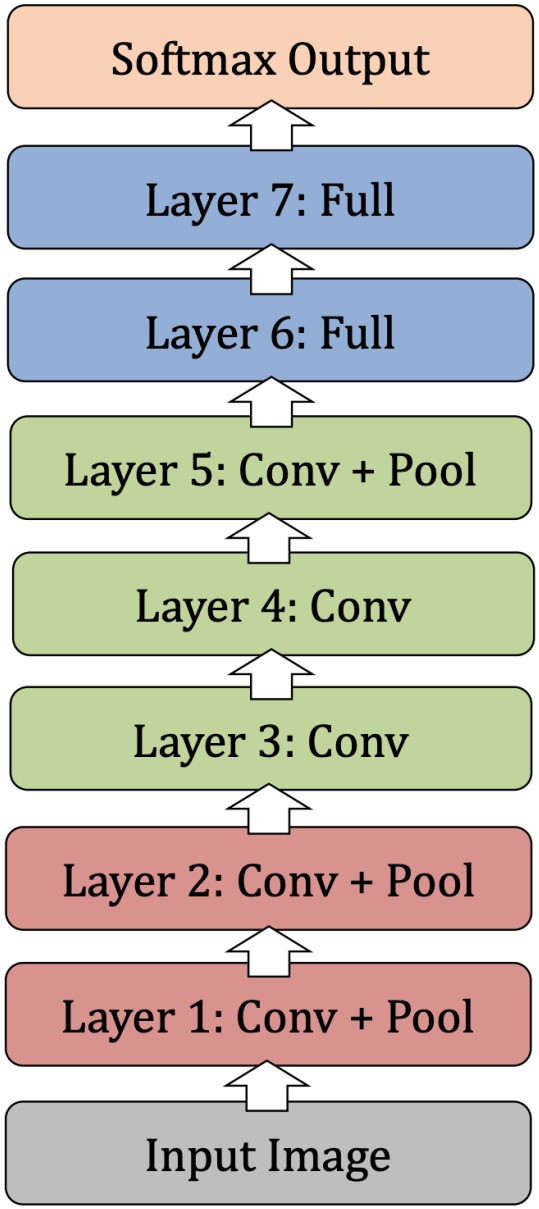
Krizhevsky, Sutskever, Hinton “ImageNet Claasification with Deep Convolutional Neural Networks”, NIPS 2012.

AlexNet

8 layers, ~60M parameters

Top5 error: 18.2%

Techniques used:
ReLU activation, overlapping pooling, dropout, ensemble (create 10 patches by cropping and average the predictions), data-augmentation (intensity of RGB channels)



[From Rob Fergus' CIFAR 2016 tutorial]

GoogLeNet

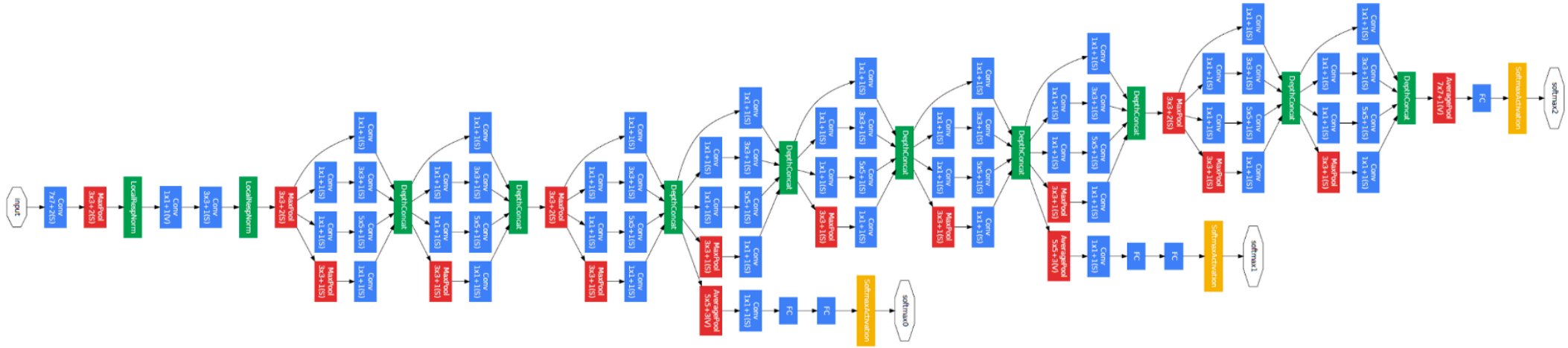
Motivation: multiscale nature of images

Large kernel for global features, and **smaller kernel** for local features.

Idea: have multiple different-size kernels at any layer.

[Going Deep with Convolutions, Szegedy et al. ’14]

GoogLeNet

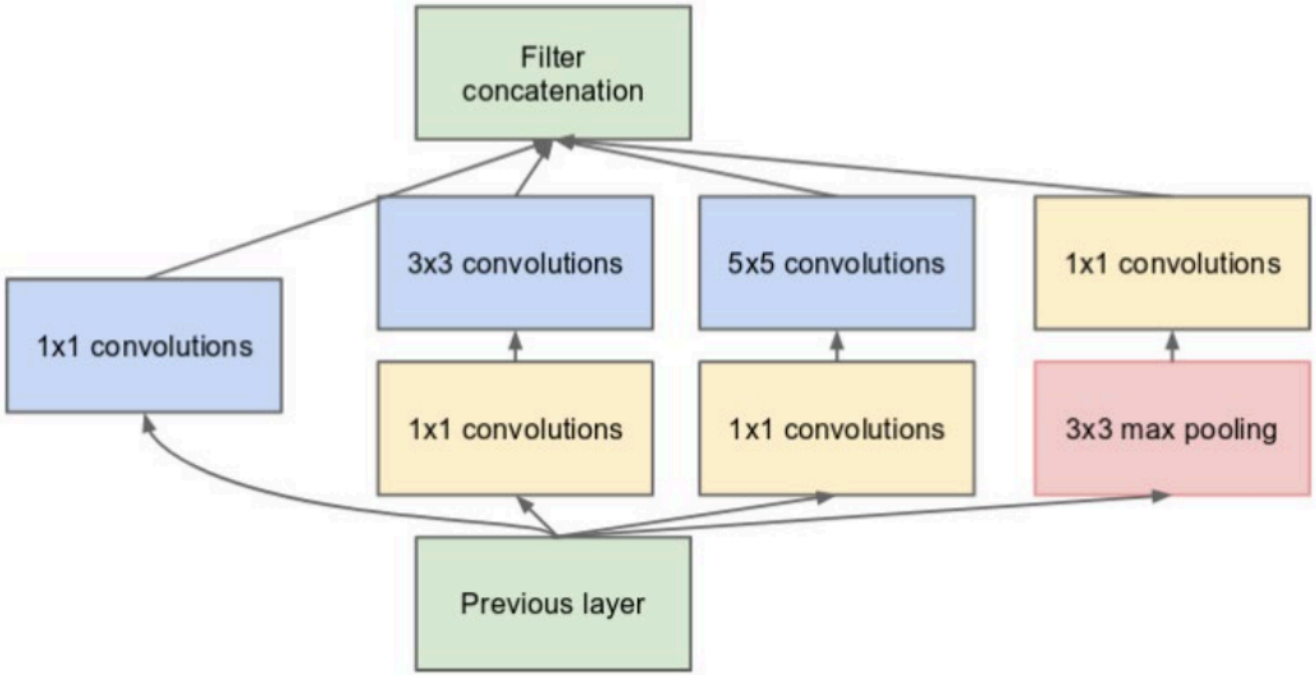


Large kernel for global features, and **smaller kernel** for local features.

Idea: have multiple different-size kernels at any layer.

[Going Deep with Convolutions, Szegedy et al. ’14]

Inception Module



Multiple filter scales at each layer

Dimensionality reduction to keep computational requirements down

[Going Deep with Convolutions, Szegedy et al. ’14]

Residual Networks

Motivation: extremely deep nets are hard to train (gradient explosion/ vanishing)

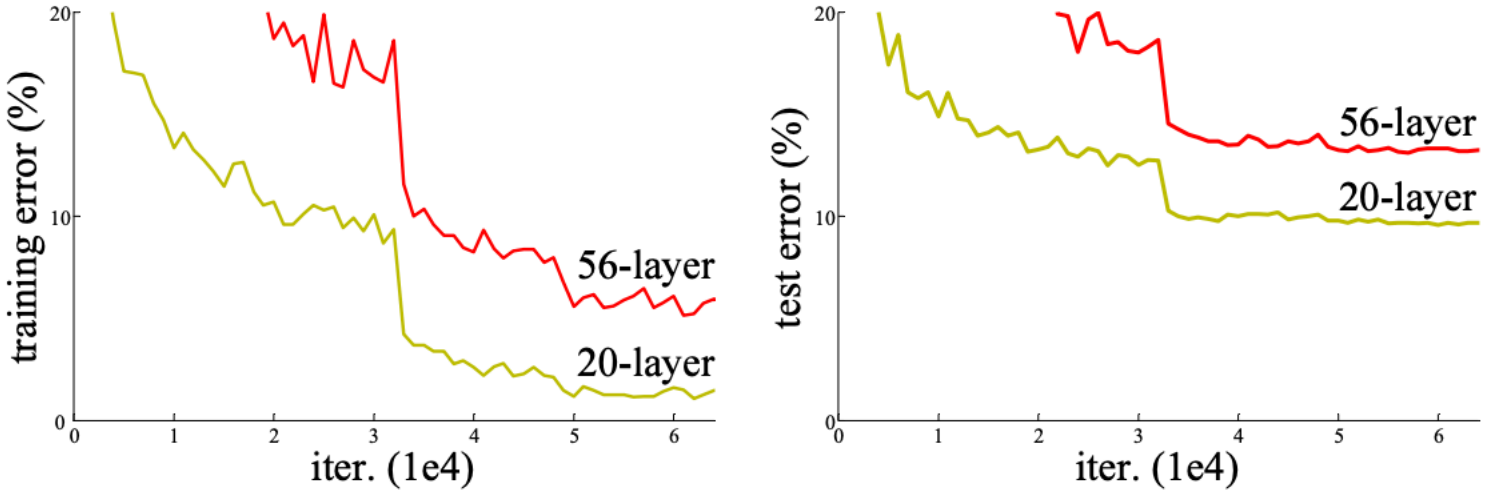


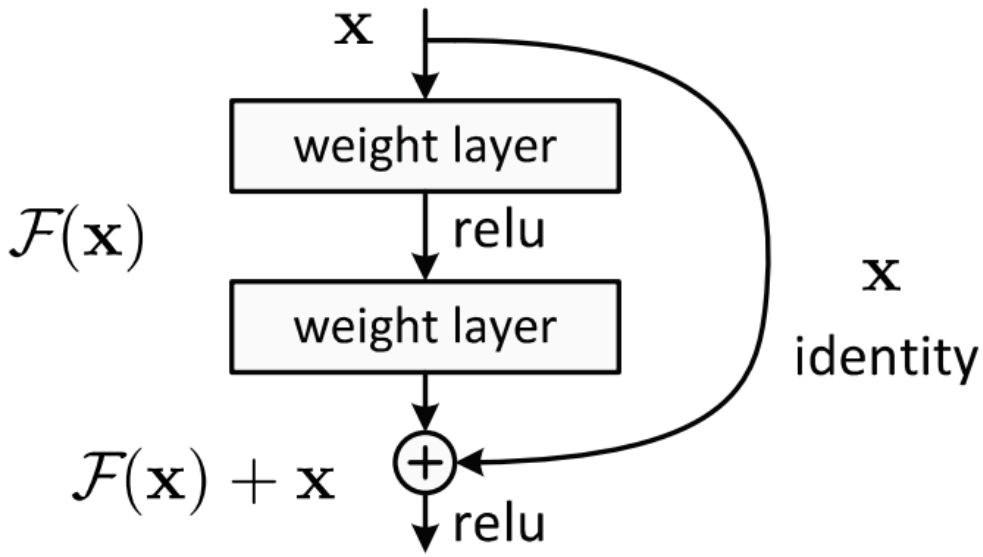
Figure 1. Training error (left) and test error (right) on CIFAR-10 with 20-layer and 56-layer "plain" networks. The deeper network has higher training error, and thus test error. Similar phenomena on ImageNet is presented in Fig. 4.

[He, Zhang, Ren, Sun, '16]

Residual Networks

Idea: identity shortcut, skip one or more layers.

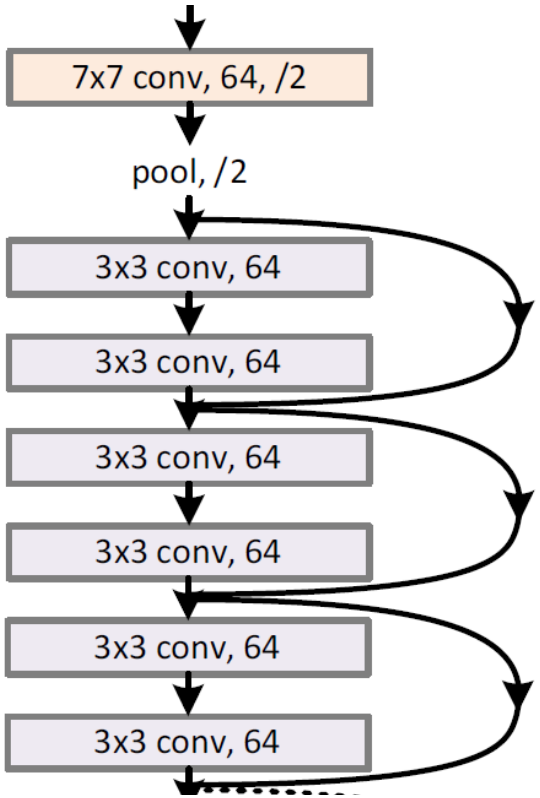
Justification: network can easily simulate shallow network ($F \approx 0$), so performance should not degrade by going deeper.

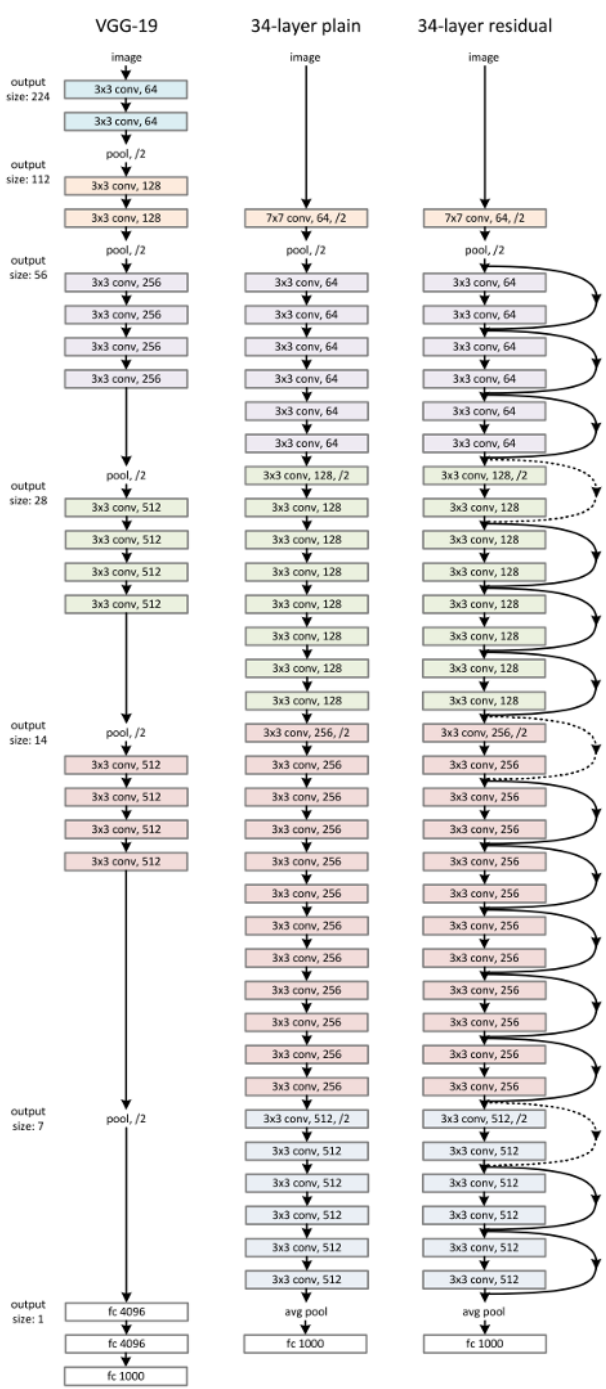


[He, Zhang, Ren, Sun, '16]

Residual Networks

- 3.57% top-5 error on ImageNet
- First deep network with > 100 layers.
- Widely used in many domains (AlphaGo)





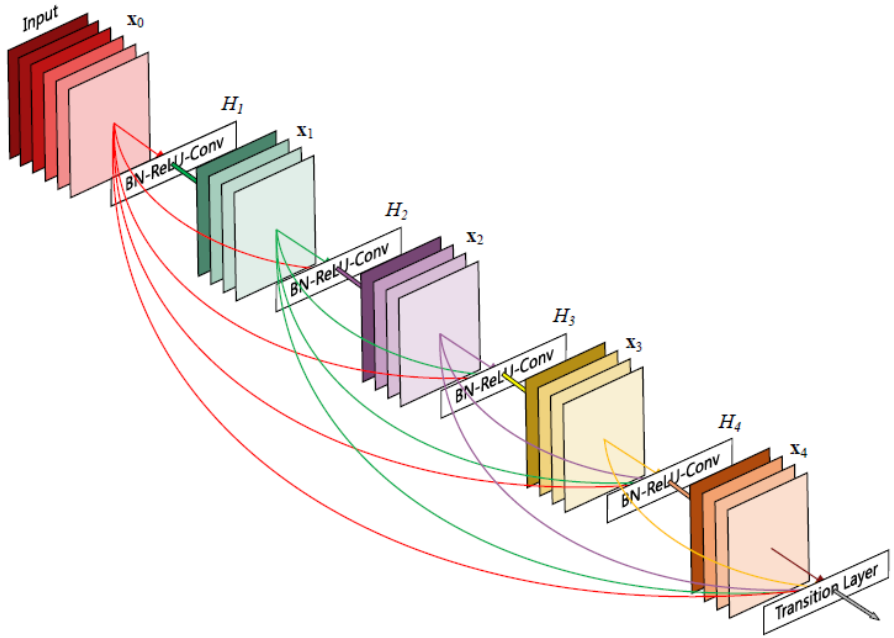
[He, Zhang, Ren, Sun, '16]

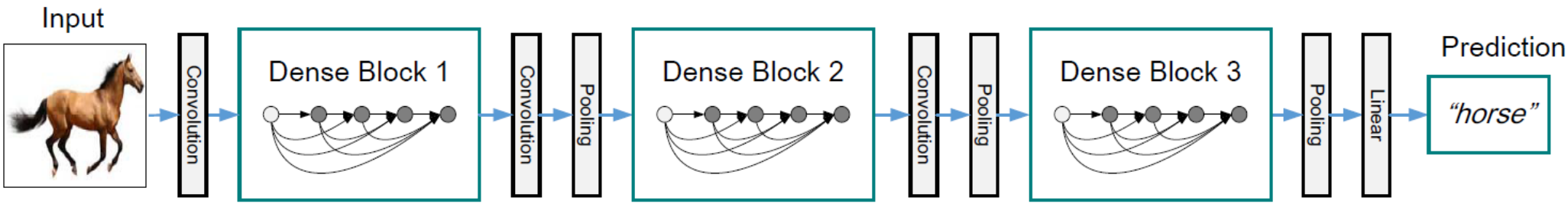
Densely Connected Network

Idea: explicit forward output of layer to all future layers (by concatenation)

Intuition: helps vanishing gradients, encourage reuse features (reduce parameter count)

Issues: network maybe too wide, need to be careful about memory consumption





[He, Zhang, Ren, Sun, '16]

Neural Architecture / Hyper-Parameter Search

Many design choices:

- Number of layers, width, kernel size, pooling, connections, etc.
- Normalization, learning rate, batch size, etc.

Strategies:

- Grid search
- Random search [Bergestra & Bengio ’12]
- Bandit-based [Li et al. ’16]
- Gradient-based (DARTS) [Liu et al. ’19]
- Neural tangent kernel [Xu et al. ’21]
- …