

Convolutional Neural Network



Multi-layer Neural Network

$$a^{(1)} = x$$

$$z^{(2)} = \Theta^{(1)} a^{(1)}$$

$$a^{(2)} = g(z^{(2)})$$

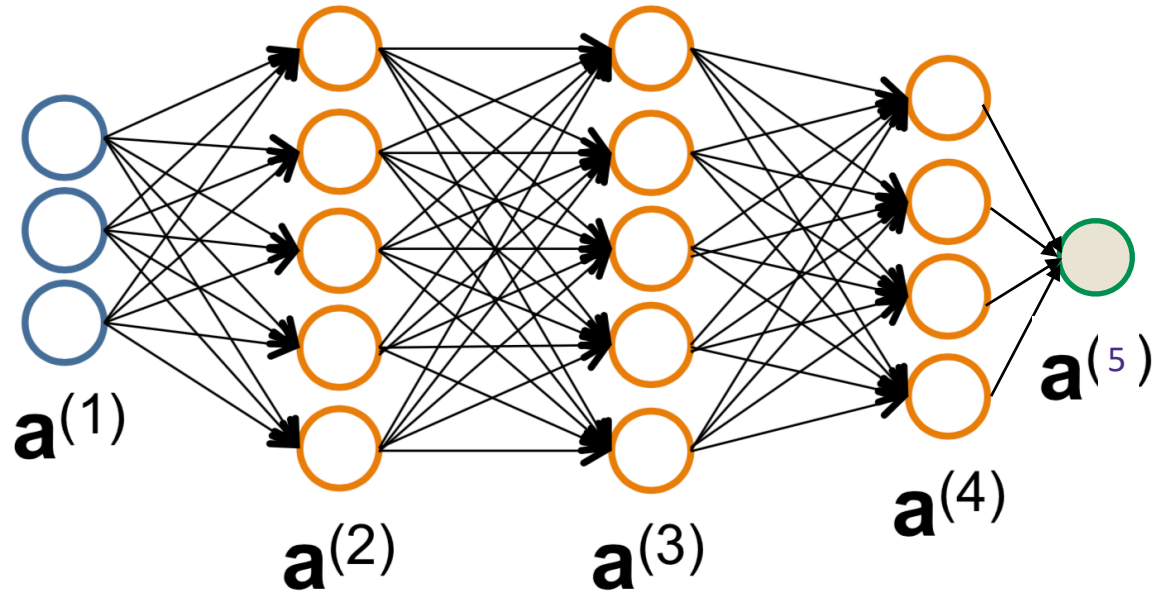
$$\vdots$$

$$z^{(l+1)} = \Theta^{(l)} a^{(l)}$$

$$a^{(l+1)} = g(z^{(l+1)})$$

$$\vdots$$

$$\hat{y} = a^{(L+1)}$$



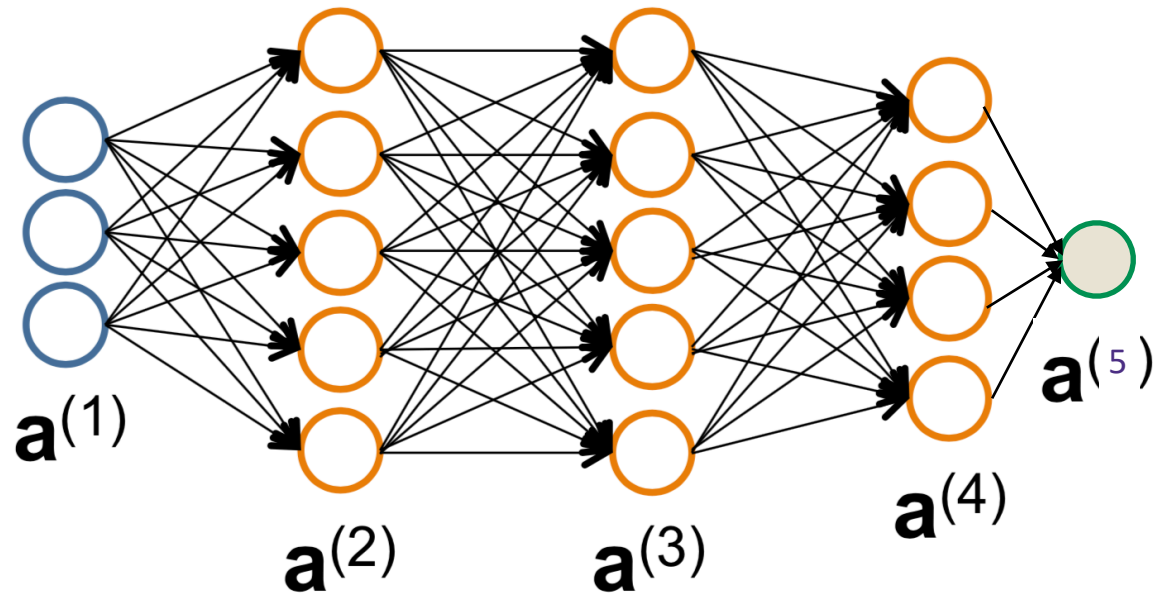
$$L(y, \hat{y}) = y \log(\hat{y}) + (1 - y) \log(1 - \hat{y})$$

$$g(z) = \frac{1}{1 + e^{-z}}$$

Binary
Logistic
Regression

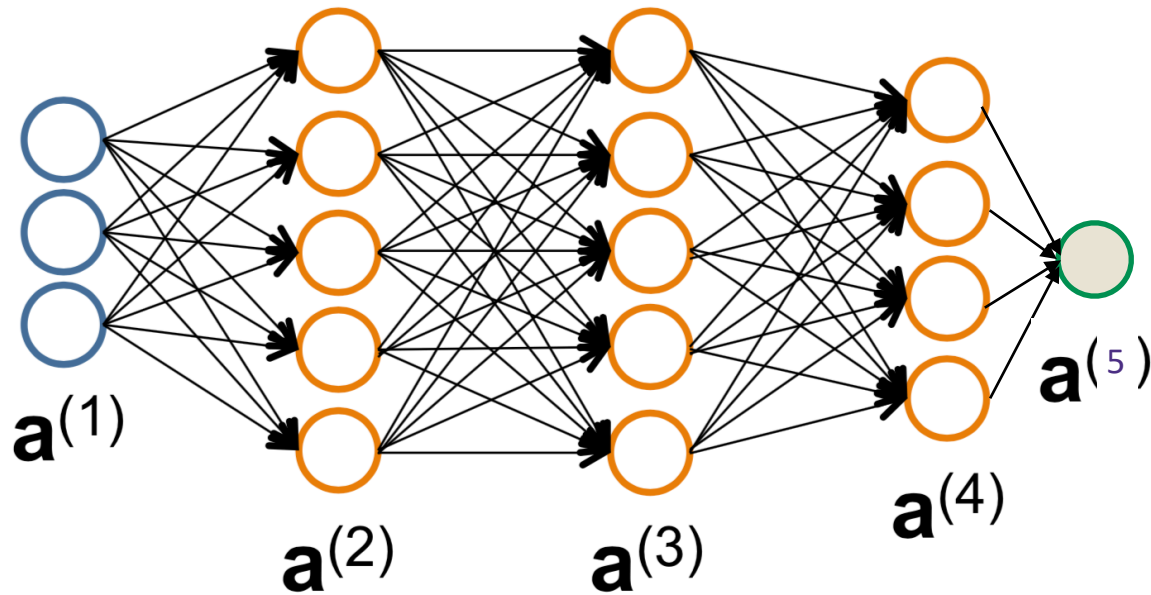
Neural Network Architecture

The neural network architecture is defined by the number of layers, and the number of nodes in each layer, but also by **allowable edges**.



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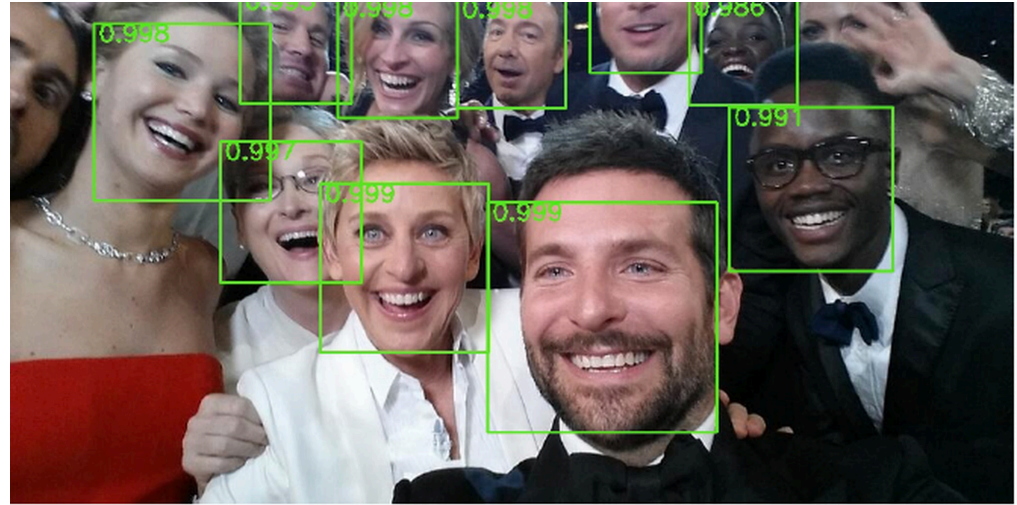
We say a layer is **Fully Connected (FC)** if all linear mappings from the current layer to the next layer are permissible.

$$\mathbf{a}^{(k+1)} = g(\Theta \mathbf{a}^{(k)}) \quad \text{for any } \Theta \in \mathbb{R}^{n_{k+1} \times n_k}$$

A lot of parameters!! $n_1 n_2 + n_2 n_3 + \cdots + n_L n_{L+1}$

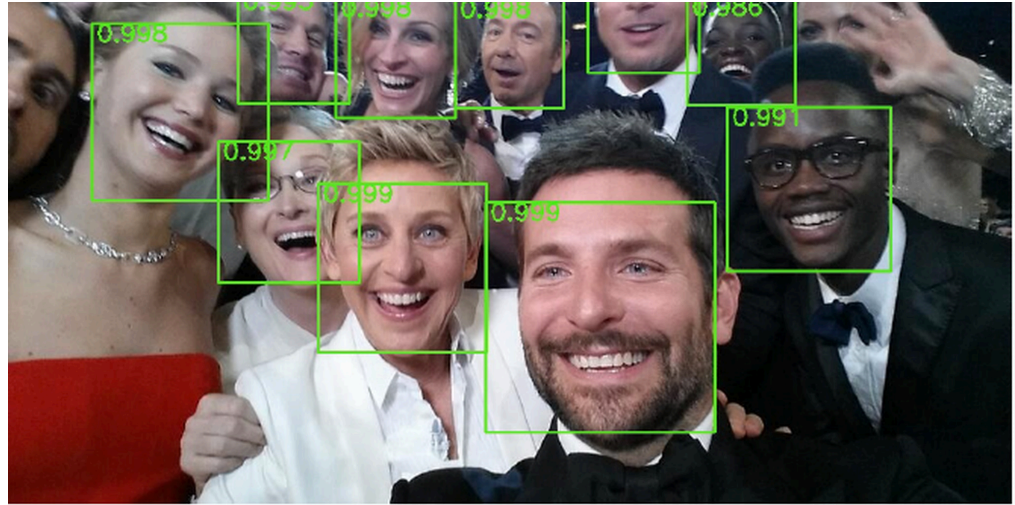
Neural Network Architecture

Objects are often **localized in space** so to find the faces in an image, not every pixel is important for classification—makes sense to drag a window across an image.

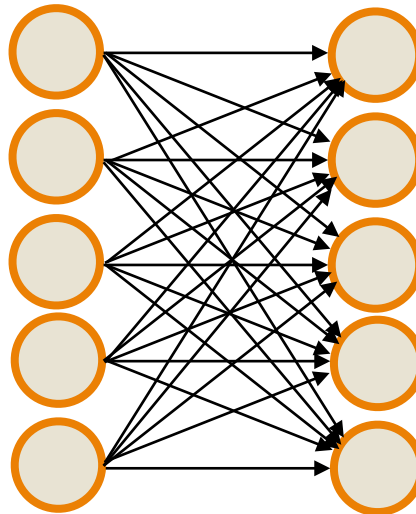


Neural Network Architecture

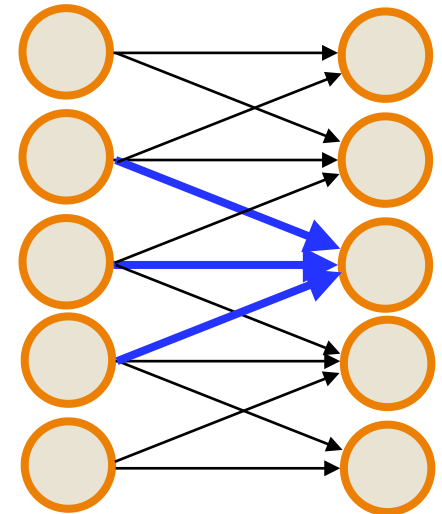
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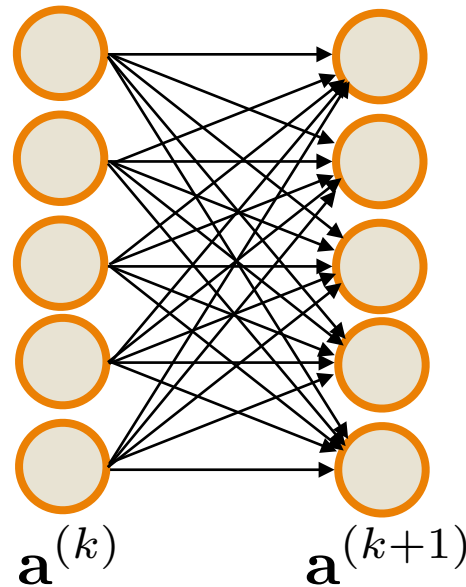
Similarly, to identify edges or other local structure, it makes sense to only look at **local information**



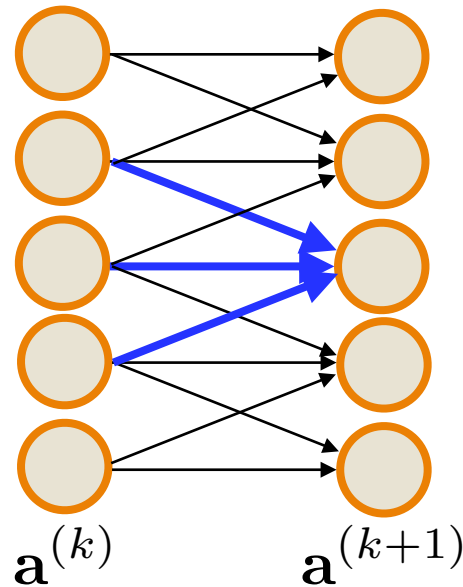
vs.



Neural Network Architecture



vs.



$$\begin{bmatrix} \Theta_{0,0} & \Theta_{0,1} & \Theta_{0,2} & \Theta_{0,3} & \Theta_{0,4} \\ \Theta_{1,0} & \Theta_{1,1} & \Theta_{1,2} & \Theta_{1,3} & \Theta_{1,4} \\ \Theta_{2,0} & \Theta_{2,1} & \Theta_{2,2} & \Theta_{2,3} & \Theta_{2,4} \\ \Theta_{3,0} & \Theta_{3,1} & \Theta_{3,2} & \Theta_{3,3} & \Theta_{3,4} \\ \Theta_{4,0} & \Theta_{4,1} & \Theta_{4,2} & \Theta_{4,3} & \Theta_{4,4} \end{bmatrix}$$

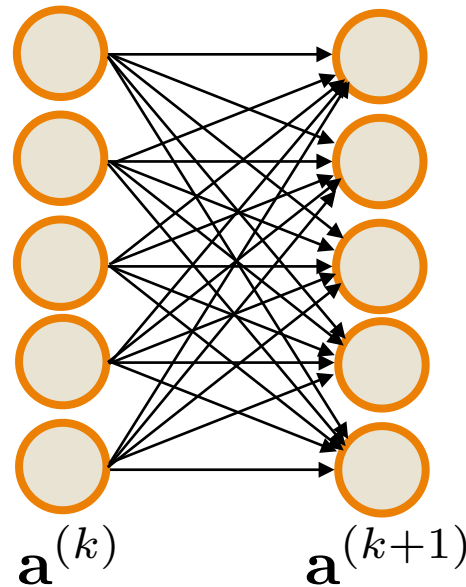
$$\begin{bmatrix} \Theta_{0,0} & \Theta_{0,1} & 0 & 0 & 0 \\ \Theta_{1,0} & \Theta_{1,1} & \Theta_{1,2} & 0 & 0 \\ 0 & \Theta_{2,1} & \Theta_{2,2} & \Theta_{2,3} & 0 \\ 0 & 0 & \Theta_{3,2} & \Theta_{3,3} & \Theta_{3,4} \\ 0 & 0 & 0 & \Theta_{4,3} & \Theta_{4,4} \end{bmatrix}$$

Parameters: n^2

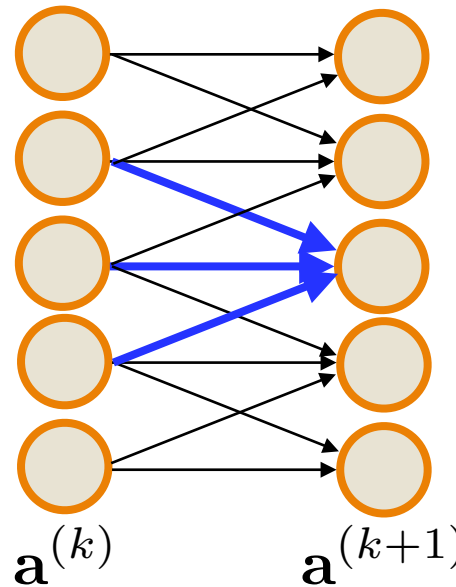
$3n - 2$

$$\mathbf{a}_i^{(k+1)} = g \left(\sum_{j=0}^{n-1} \Theta_{i,j} \mathbf{a}_j^{(k)} \right)$$

Neural Network Architecture



vs.



Mirror/share local weights everywhere
(e.g., structure equally likely to be anywhere in image)

$$\begin{bmatrix} \Theta_{0,0} & \Theta_{0,1} & \Theta_{0,2} & \Theta_{0,3} & \Theta_{0,4} \\ \Theta_{1,0} & \Theta_{1,1} & \Theta_{1,2} & \Theta_{1,3} & \Theta_{1,4} \\ \Theta_{2,0} & \Theta_{2,1} & \Theta_{2,2} & \Theta_{2,3} & \Theta_{2,4} \\ \Theta_{3,0} & \Theta_{3,1} & \Theta_{3,2} & \Theta_{3,3} & \Theta_{3,4} \\ \Theta_{4,0} & \Theta_{4,1} & \Theta_{4,2} & \Theta_{4,3} & \Theta_{4,4} \end{bmatrix}$$

Parameters: n^2

$$\begin{bmatrix} \Theta_{0,0} & \Theta_{0,1} & 0 & 0 & 0 \\ \Theta_{1,0} & \Theta_{1,1} & \Theta_{1,2} & 0 & 0 \\ 0 & \Theta_{2,1} & \Theta_{2,2} & \Theta_{2,3} & 0 \\ 0 & 0 & \Theta_{3,2} & \Theta_{3,3} & \Theta_{3,4} \\ 0 & 0 & 0 & \Theta_{4,3} & \Theta_{4,4} \end{bmatrix}$$

$3n - 2$

$$\begin{bmatrix} \theta_1 & \theta_2 & 0 & 0 & 0 \\ \theta_0 & \theta_1 & \theta_2 & 0 & 0 \\ 0 & \theta_0 & \theta_1 & \theta_2 & 0 \\ 0 & 0 & \theta_0 & \theta_1 & \theta_2 \\ 0 & 0 & 0 & \theta_0 & \theta_1 \end{bmatrix}$$

3

$$\mathbf{a}_i^{(k+1)} = g \left(\sum_{j=0}^{n-1} \Theta_{i,j} \mathbf{a}_j^{(k)} \right)$$

$$\mathbf{a}_i^{(k+1)} = g \left(\sum_{j=0}^{m-1} \theta_j \mathbf{a}_{i+j}^{(k)} \right)$$

Neural Network Architecture

Fully Connected (FC) Layer

$$\begin{bmatrix} \Theta_{0,0} & \Theta_{0,1} & \Theta_{0,2} & \Theta_{0,3} & \Theta_{0,4} \\ \Theta_{1,0} & \Theta_{1,1} & \Theta_{1,2} & \Theta_{1,3} & \Theta_{1,4} \\ \Theta_{2,0} & \Theta_{2,1} & \Theta_{2,2} & \Theta_{2,3} & \Theta_{2,4} \\ \Theta_{3,0} & \Theta_{3,1} & \Theta_{3,2} & \Theta_{3,3} & \Theta_{3,4} \\ \Theta_{4,0} & \Theta_{4,1} & \Theta_{4,2} & \Theta_{4,3} & \Theta_{4,4} \end{bmatrix}$$

Convolutional (CONV) Layer (1 filter)

$$\begin{bmatrix} \theta_1 & \theta_2 & 0 & 0 & 0 \\ \theta_0 & \theta_1 & \theta_2 & 0 & 0 \\ 0 & \theta_0 & \theta_1 & \theta_2 & 0 \\ 0 & 0 & \theta_0 & \theta_1 & \theta_2 \\ 0 & 0 & 0 & \theta_0 & \theta_1 \end{bmatrix} \quad m=3$$

$$\mathbf{a}_i^{(k+1)} = g \left(\sum_{j=0}^{n-1} \Theta_{i,j} \mathbf{a}_j^{(k)} \right)$$

$$\mathbf{a}_i^{(k+1)} = g \left(\sum_{j=0}^{m-1} \theta_j \mathbf{a}_{i+j}^{(k)} \right) = g([\theta * \mathbf{a}^{(k)}]_i)$$

Convolution*

$\theta = (\theta_0, \dots, \theta_{m-1}) \in \mathbb{R}^m$ is referred to as a “filter”

Example (1d convolution)

$$(\theta * x)_i = \sum_{j=0}^{m-1} \theta_j x_{i+j}$$

1	1	1	0	0
---	---	---	---	---

Input $x \in \mathbb{R}^n$

1	0	1
---	---	---

Filter $\theta \in \mathbb{R}^m$

--	--	--

Output $\theta * x$

Example (1d convolution)

$$(\theta * x)_i = \sum_{j=0}^{m-1} \theta_j x_{i+j}$$

1	1	1	0	0
---	---	---	---	---

Input $x \in \mathbb{R}^n$

1	0	1
---	---	---

Filter $\theta \in \mathbb{R}^m$

1 _{x1}	1 _{x0}	1 _{x1}	0	0
-----------------	-----------------	-----------------	---	---

2		
---	--	--

Output $\theta * x$

Example (1d convolution)

$$(\theta * x)_i = \sum_{j=0}^{m-1} \theta_j x_{i+j}$$

1	1	1	0	0
---	---	---	---	---

Input $x \in \mathbb{R}^n$

1	0	1
---	---	---

Filter $\theta \in \mathbb{R}^m$

1	1 _{x1}	1 _{x0}	0 _{x1}	0
---	-----------------	-----------------	-----------------	---

2	1	
---	---	--

Output $\theta * x$

Example (1d convolution)

$$(\theta * x)_i = \sum_{j=0}^{m-1} \theta_j x_{i+j}$$

1	1	1	0	0
---	---	---	---	---

Input $x \in \mathbb{R}^n$

1	0	1
---	---	---

Filter $\theta \in \mathbb{R}^m$

1	1	1 _{x1}	0 _{x0}	0 _{x1}
---	---	-----------------	-----------------	-----------------

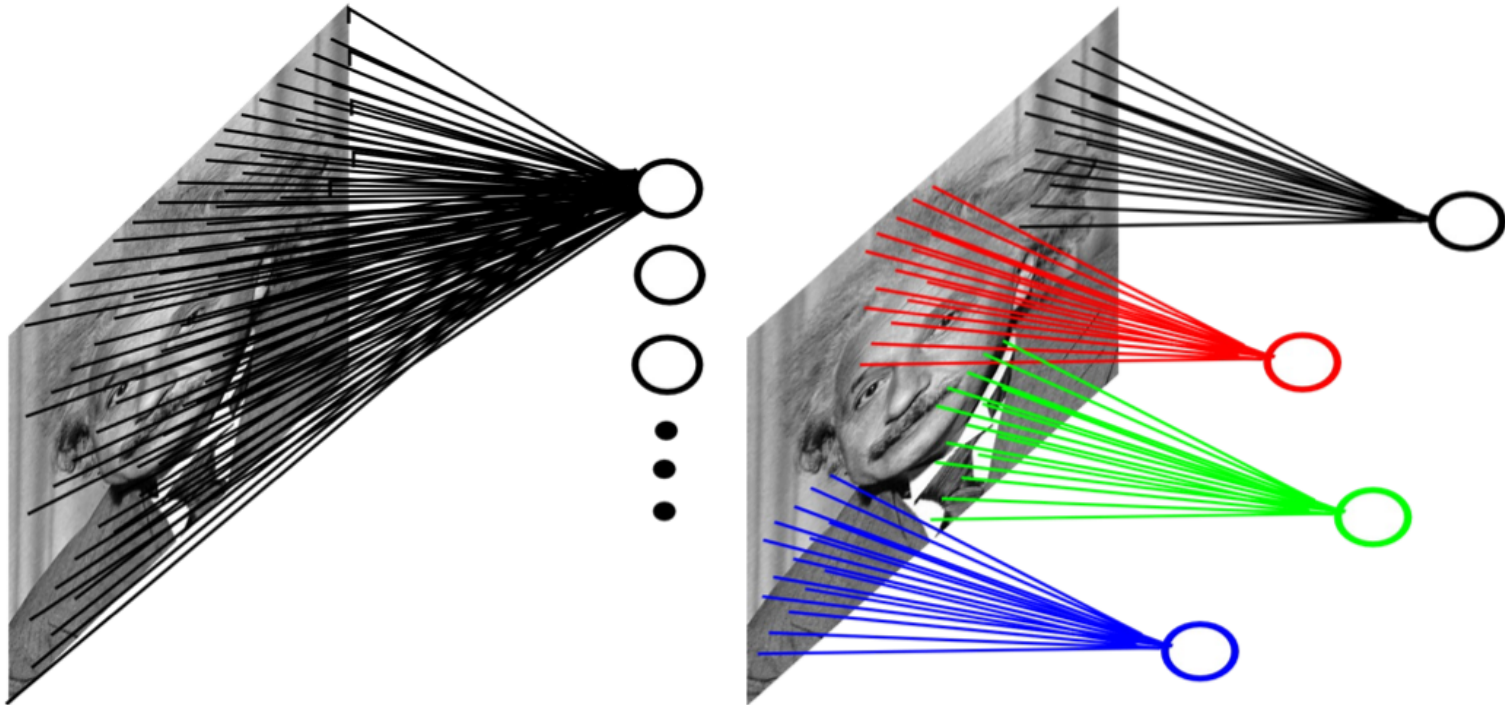
2	1	1
---	---	---

Output $\theta * x$

2d Convolution Layer

■ Example: 200x200 image

- ▶ Fully-connected, 400,000 hidden units = 16 billion parameters
- ▶ Locally-connected, 400,000 hidden units 10x10 fields = 40 million params
- ▶ Local connections capture local dependencies



Convolution of images (2d convolution)

$$(I * K)(i, j) = \sum_m \sum_n I(i + m, j + n) K(m, n)$$

1	1	1	0	0
0	1	1	1	0
0	0	1	1	1
0	0	1	1	0
0	1	1	0	0

Image I

1	0	1
0	1	0
1	0	1

Filter K

1 _{x1}	1 _{x0}	1 _{x1}	0	0
0 _{x0}	1 _{x1}	1 _{x0}	1	0
0 _{x1}	0 _{x0}	1 _{x1}	1	1
0	0	1	1	0
0	1	1	0	0

Image

4		

Convolved
Feature

$$I * K$$

Convolution of images

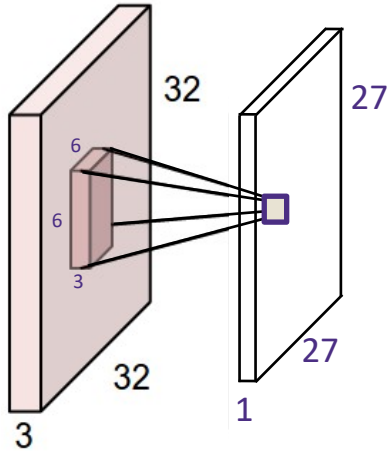
$$(I * K)(i, j) = \sum_m \sum_n I(i + m, j + n) K(m, n)$$

Image I



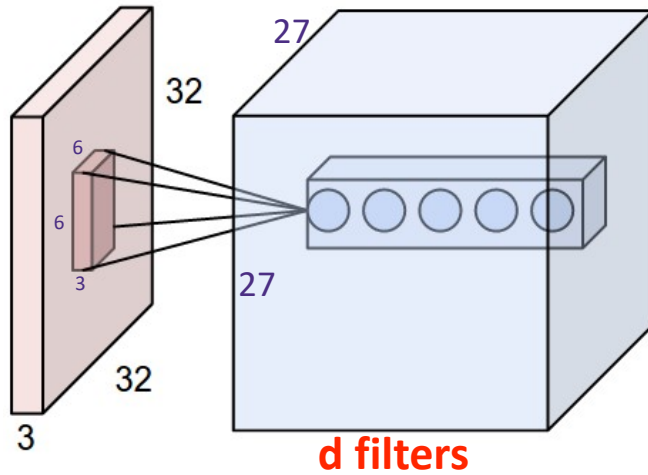
Operation	Filter K	Convolved Image $I * K$
Edge detection	$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$	
	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	
	$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$	
Sharpen	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix}$	
Box blur (normalized)	$\frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	
Gaussian blur (approximation)	$\frac{1}{16} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$	

Stacking convolved images



$$x \in \mathbb{R}^{n \times n \times r}$$

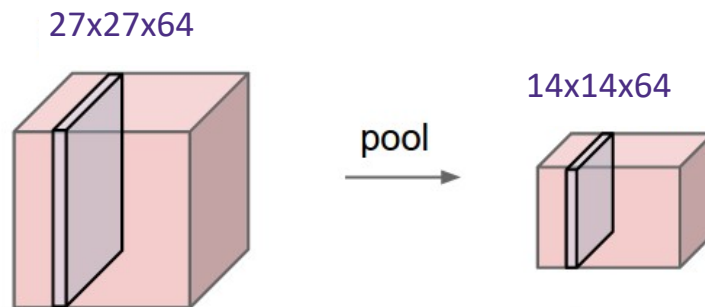
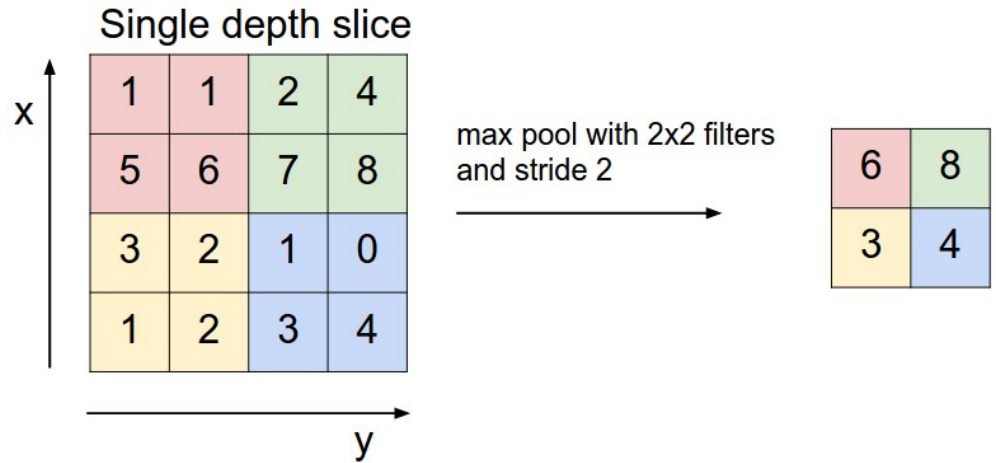
Stacking convolved images



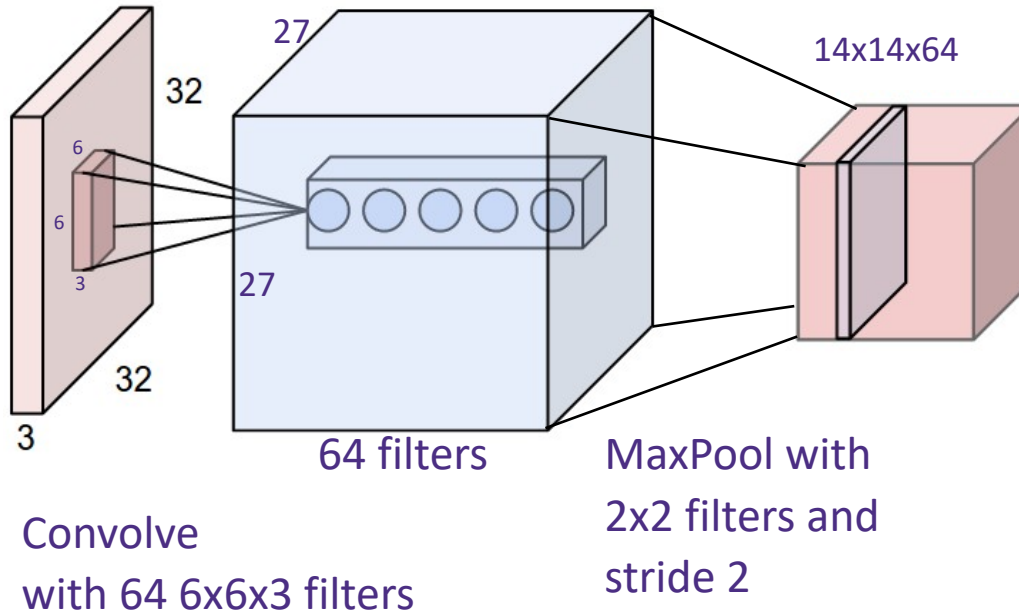
Repeat with d filters!

Pooling

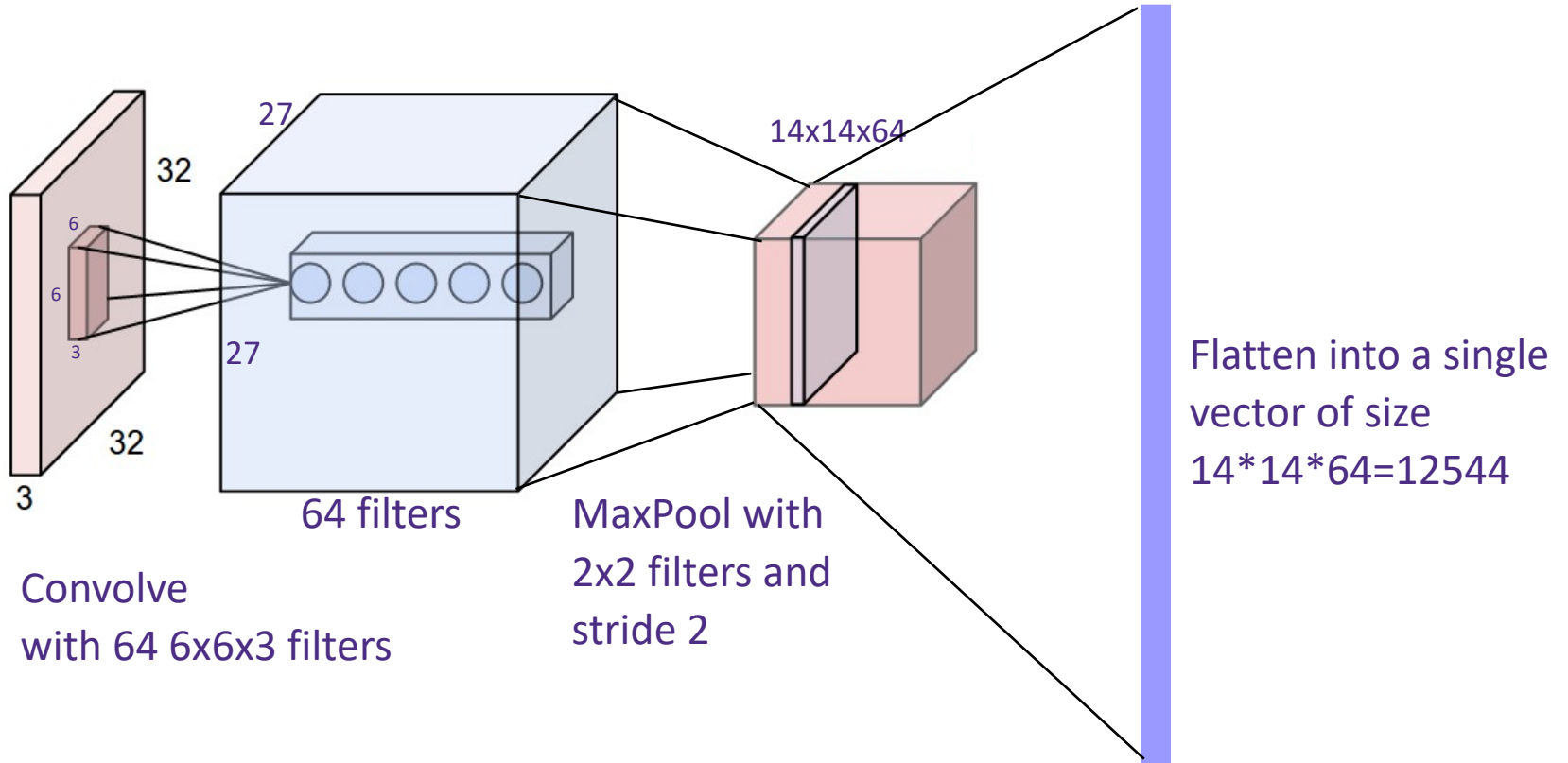
Pooling reduces the dimension and can be interpreted as “This filter had a high response in this general region”



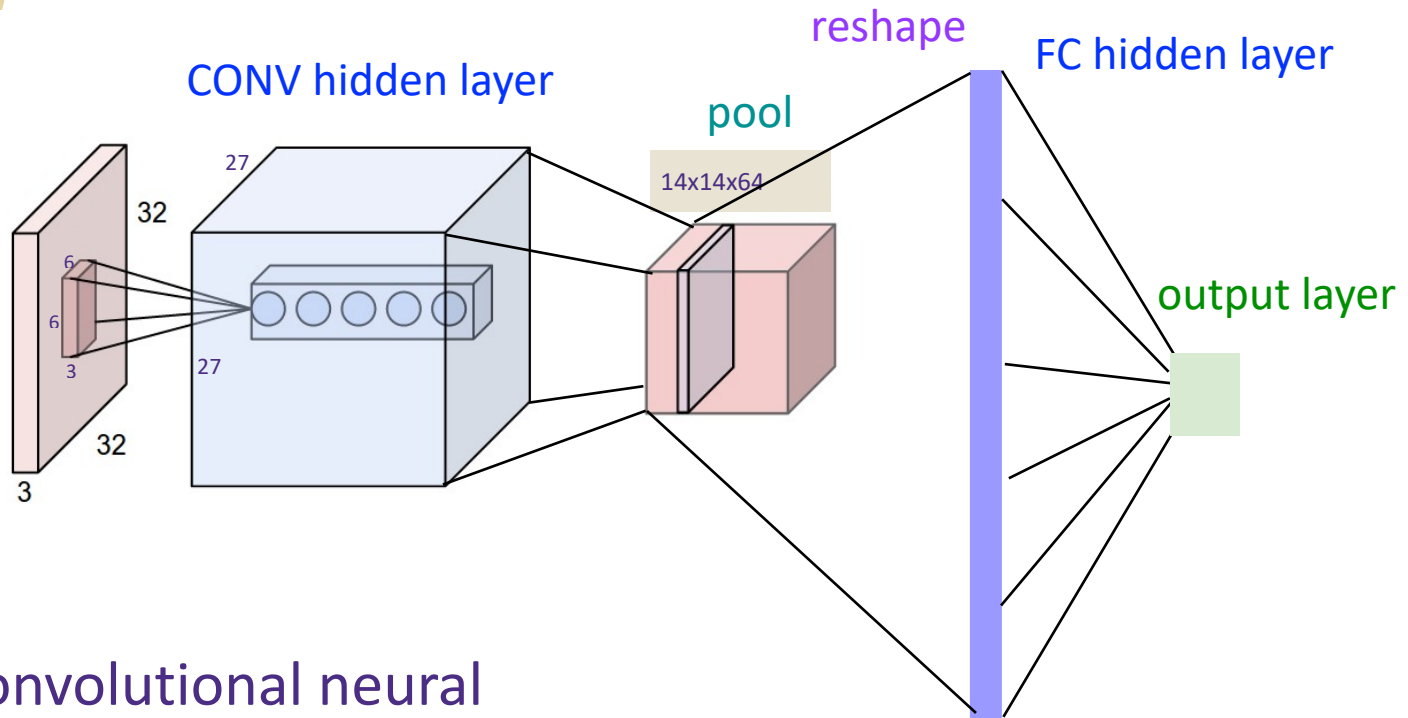
Pooling Convolution layer



Flattening

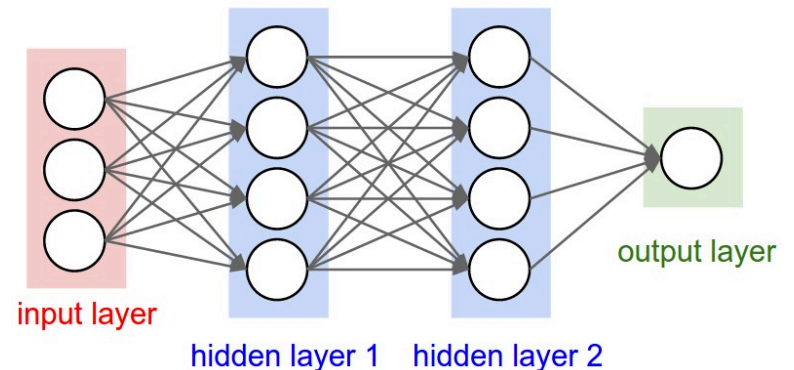


Training Convolutional Networks

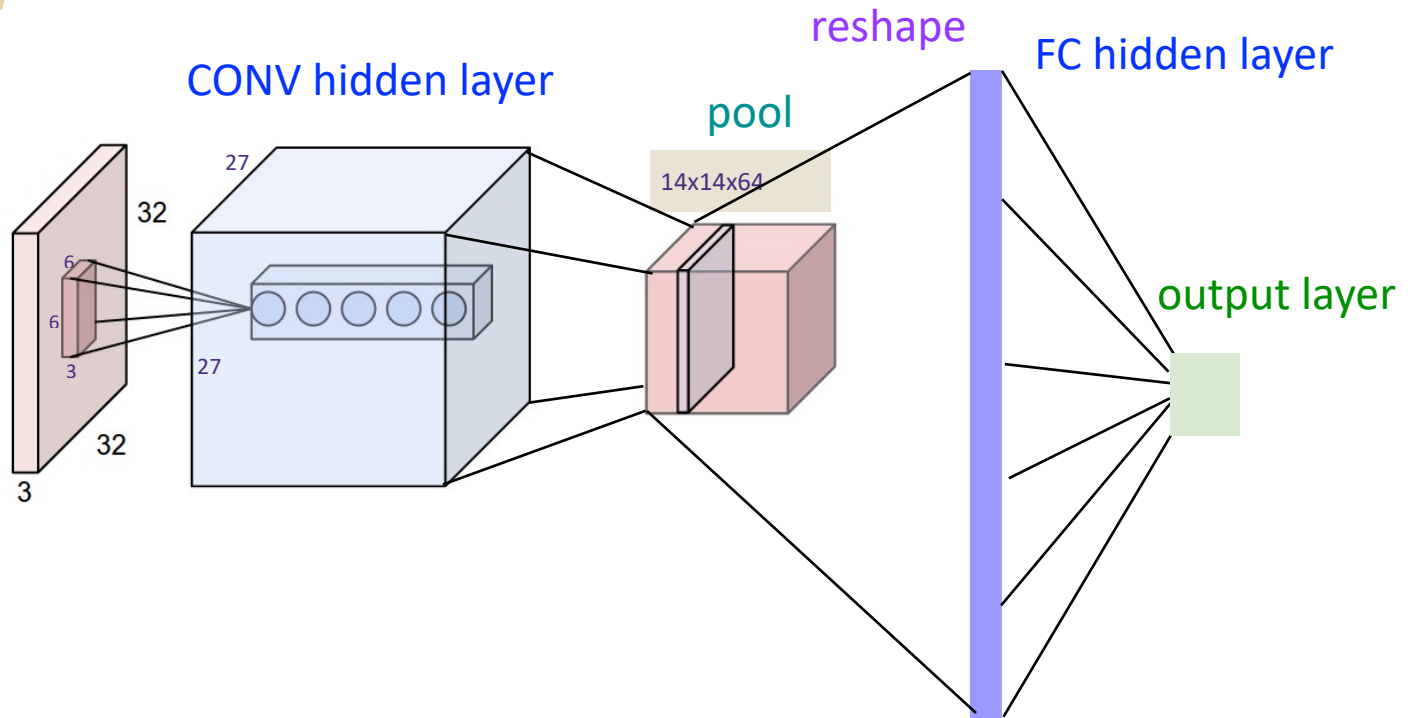


Recall: Convolutional neural networks (CNN) are just regular fully connected (FC) neural networks with some connections removed.

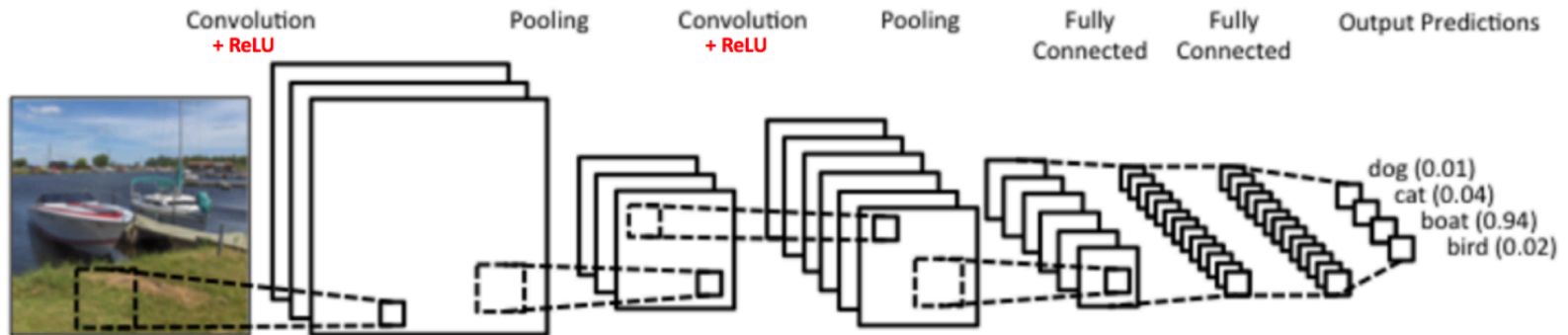
Train with SGD!

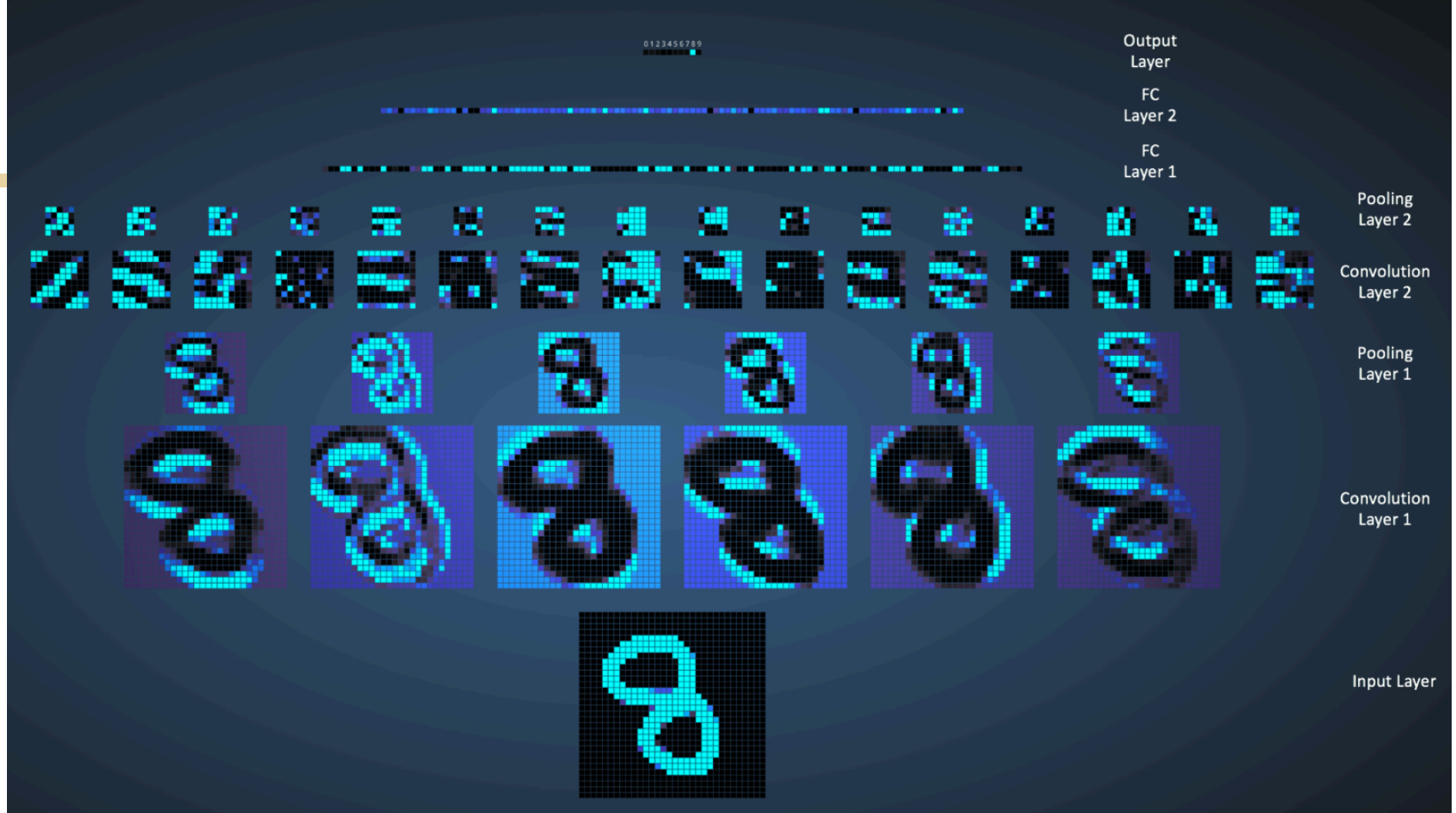


Training Convolutional Networks

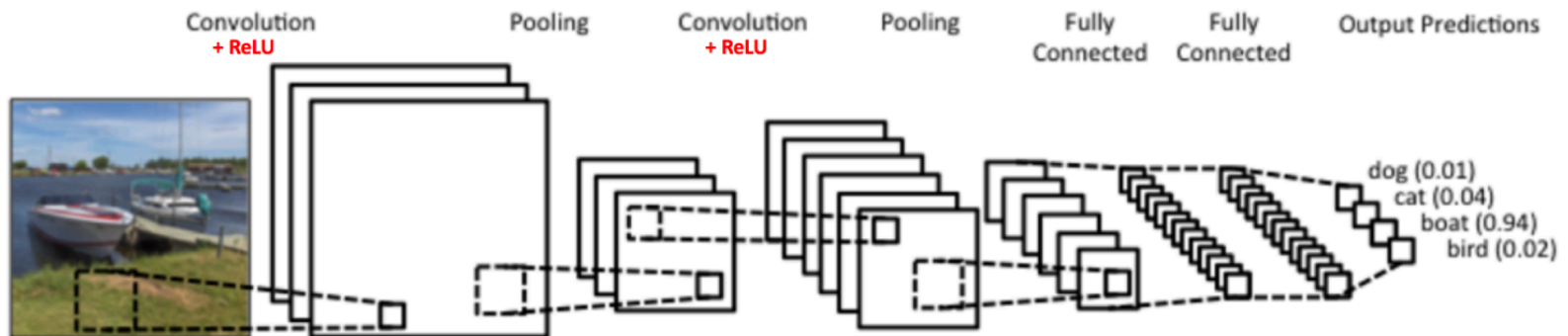


Real example network: LeNet





Real example network: LeNet



Remarks

- Convolution is a fundamental operation in signal processing. Instead of hand-engineering the filters (e.g., Fourier, Wavelets, etc.) **Deep Learning *learns* the filters and CONV layers with back-propagation**, replacing fully connected (FC) layers with convolutional (CONV) layers
- **Pooling** is a dimensionality reduction operation that summarizes the output of convolving the input with a filter
- Typically the last few layers are **Fully Connected (FC)**, with the interpretation that the CONV layers are feature extractors, preparing input for the final FC layers. Can replace last layers and retrain on different dataset+task.
- Just as hard to train as regular neural networks.
- More exotic network architectures for specific tasks

Real networks

Modern networks have
dozens of parameters to tune.

Data augmentation?
Batch norm?

ReLU leakiness
slope

Learning rate schedule

