Classification
Logistic Regression

Thus far, regression:

predict a continuous value given some inputs

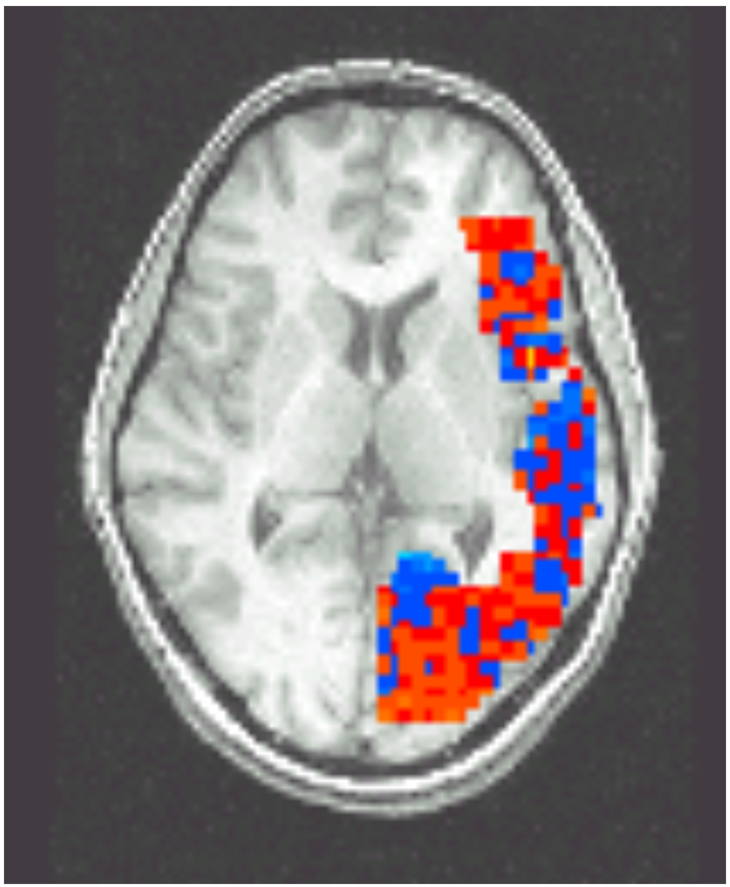
Reading Your Brain, Simple Example

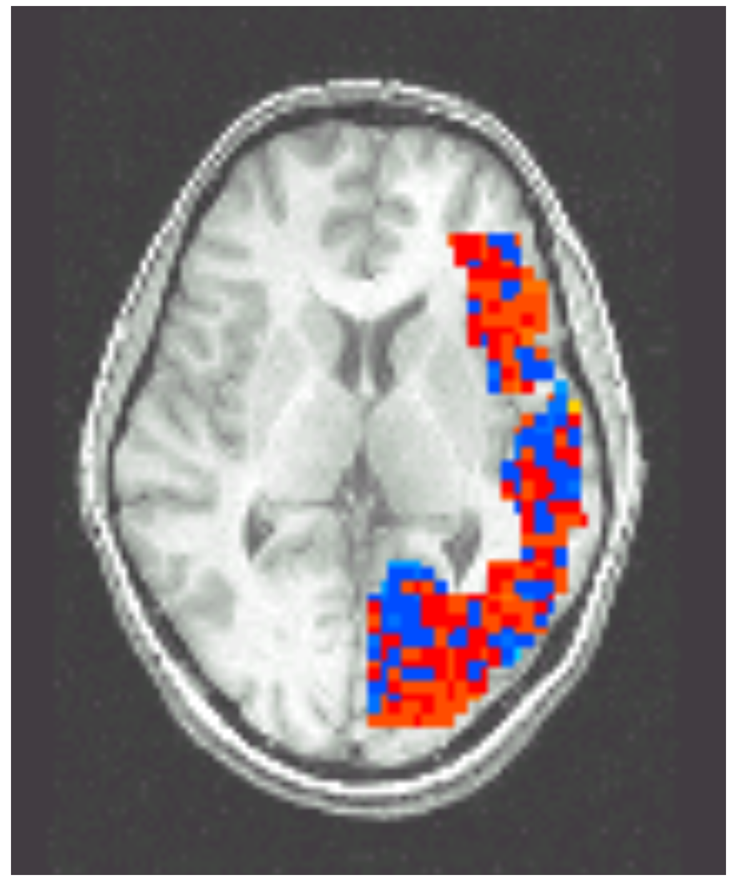
[Mitchell et al.]

Pairwise classification accuracy: 85%

Person

Animal





Classification

- Learn f: X -> Y
  - X - features
  - Y - target classes

- Loss Function
- Expected loss of f:

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- Expected loss of f:

$$\mathbb{E}_{XY}[\mathbf{1}\{f(X) \neq Y\}] = \mathbb{E}_X[\mathbb{E}_{Y|X}[\mathbf{1}\{f(x) \neq Y\}|X = x]]$$

$$\mathbb{E}_{Y|X}[\mathbf{1}\{f(x) \neq Y\}|X = x] = \sum_i P(Y = i|X = x)\mathbf{1}\{f(x) \neq i\} = \sum_{i \neq f(x)} P(Y = i|X = x)$$

$$= 1 - P(Y = f(x)|X = x)$$

- Suppose you knew P(Y|X) exactly, how should you classify?

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  - Bayes-Optimal classifier:

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Bayes Optimal Binary Classifier

$$Y \in \{0, 1\}$$

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$$\{(x_i, y_i)\}_{i=1}^n$$

- What is a natural estimator for P(Y | X)?

Bayes Optimal Binary Classifier

- **Suppose we don't know P(Y|X), but have n iid examples**

$$\{(x_i, y_i)\}_{i=1}^n \qquad Y \in \{0,1\}$$

- **What is a natural estimator for P(Y | X)?**

Fix some $\tilde{x} \in X$

Suppose $x_i = \tilde{x}$ for $m \leq n$ samples

What is a natural estimator for $\theta_* := \mathbb{P}(Y = 1 | X = \tilde{x})$?

If k of the m labels are equal to $Y = 1$ then

Bayes Optimal Binary Classifier

- **Suppose we don't know P(Y|X), but have n iid examples**

$$\{(x_i, y_i)\}_{i=1}^n \qquad Y \in \{0,1\}$$

- **What is a natural estimator for argmax_y P(Y = y | X)?**

If $X = \{0,1\}^d$, or is generally discrete

$$\hat{f}(x) = \arg\max_{y \in \{0,1\}} \frac{\sum_{i=1}^n \mathbf{1}[\mathbf{x_i}=\mathbf{x}, \mathbf{y_i}=\mathbf{y}]}{\sum_{i=1}^n \mathbf{1}[\mathbf{x_i}=\mathbf{x}]}$$

Issues?

Process

Collect a **dataset**

Decide on a **model**

Find the function which fits the data best

- **Choose a loss function**
- **Pick the function which minimizes loss on data**

Use function to make prediction on new examples

Decide on a model, Binary Classification

To make predictions for unseen inputs (xs),

need a **general** model for $\mathbb{P}(Y = 1|X = x)$

- What about standard linear regression model?

- Need to map real values to [0,1]
  - We call such maps "link functions"

Logistic Regression

Actually classification, not regression :)

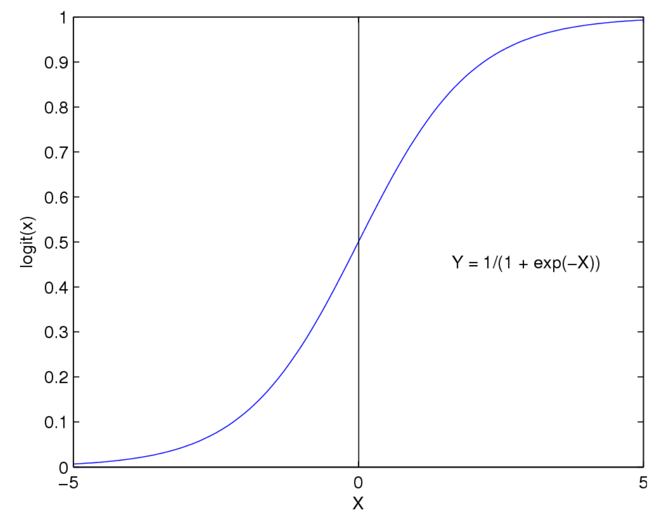
Learn $\mathbb{P}(Y = 1|X = x)$ using $\sigma(w^T x)$, for link function $\sigma =$

Logistic function(or Sigmoid): $\frac{1}{1 + exp(-z)}$

$$\mathbb{P}[Y = 1|X = x, w] = \sigma(w^T x) = \frac{1}{1 + \exp(-w^T x)}$$

$$\mathbb{P}[Y = 0|X = x, w] = 1 - \sigma(w^T x) = \frac{\exp(-w^T x)}{1 + \exp(-w^T x)}$$

$$= \frac{1}{1 + \exp(w^T x)}$$

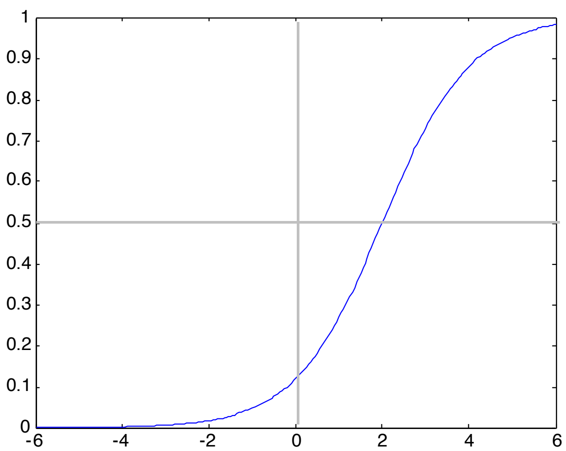


Features can be discrete or continuous!

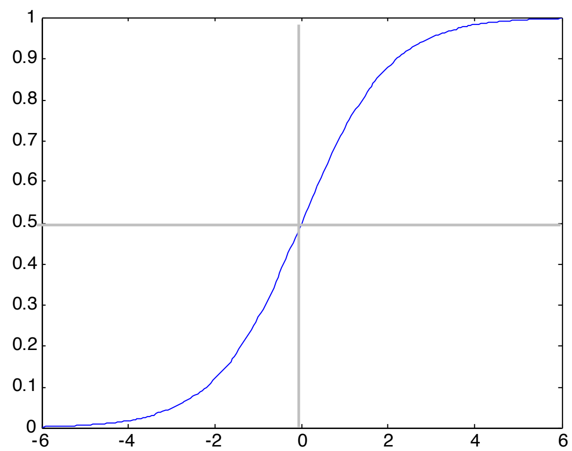
Understanding the sigmoid

$$\sigma(w_0 + \sum_k w_k x_k) = \frac{1}{1 + e^{w_0 + \sum_k w_k x_k}}$$

w_0=-2, w_1=-1



w_0=0, w_1=-1



w_0=0, w_1=-0.5