Linear Classification: The Perceptron

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Linear Classifiers

• A hyperplane partitions $\mathbb{R}^d$ into two half-spaces
  – Defined by the normal vector $\theta \in \mathbb{R}^d$
    • $\theta$ is orthogonal to any vector lying on the hyperplane
  – Assumed to pass through the origin
    • This is because we incorporated bias term $\theta_0$ into it by
      $x_0 = 1$

• Consider classification with +1, -1 labels ...
Linear Classifiers

- **Linear classifiers**: represent decision boundary by hyperplane

\[
\theta = \begin{bmatrix}
\theta_0 \\
\theta_1 \\
\vdots \\
\theta_d \\
\end{bmatrix} \quad \mathbf{x}^T = \begin{bmatrix} 1 & x_1 & \ldots & x_d \end{bmatrix}
\]

\[h(\mathbf{x}) = \text{sign}(\theta^T \mathbf{x}) \quad \text{where} \quad \text{sign}(z) = \begin{cases} 
1 & \text{if } z \geq 0 \\
-1 & \text{if } z < 0 
\end{cases}
\]

- Note that: \(\theta^T \mathbf{x} > 0 \implies y = +1\)
  \[\theta^T \mathbf{x} < 0 \implies y = -1\]
The Perceptron

\[ h(x) = \text{sign}(\theta^T x) \]  

where

\[ \text{sign}(z) = \begin{cases} 
1 & \text{if } z \geq 0 \\
-1 & \text{if } z < 0 
\end{cases} \]

- The perceptron uses the following update rule each time it receives a new training instance \((x^{(i)}, y^{(i)})\)

\[
\theta_j \leftarrow \theta_j - \frac{\alpha}{2} \left( h_\theta (x^{(i)}) - y^{(i)} \right) x_j^{(i)}
\]

either 2 or -2

- If the prediction matches the label, make no change
- Otherwise, adjust \(\theta\)
The Perceptron

• The perceptron uses the following update rule each time it receives a new training instance \((\mathbf{x}^{(i)}, y^{(i)})\)

\[
\theta_j \leftarrow \theta_j - \frac{\alpha}{2} \left( h_\theta (\mathbf{x}^{(i)}) - y^{(i)} \right) x_j^{(i)}
\]

either 2 or -2

• Re-write as \(\theta_j \leftarrow \theta_j + \alpha y^{(i)} x_j^{(i)}\) (only upon misclassification)
  – Can eliminate \(\alpha\) in this case, since its only effect is to scale \(\theta\) by a constant, which doesn’t affect performance

Perceptron Rule: If \(\mathbf{x}^{(i)}\) is misclassified, do \(\theta \leftarrow \theta + y^{(i)} \mathbf{x}^{(i)}\)
Why the Perceptron Update Works

Based on slide by Piyush Rai
Why the Perceptron Update Works

• Consider the misclassified example \((y = +1)\)
  – Perceptron wrongly thinks that \(\theta_{\text{old}}^T x < 0\)

• Update:
  \[
  \theta_{\text{new}} = \theta_{\text{old}} + yx = \theta_{\text{old}} + x \quad \text{(since } y = +1)\]

• Note that
  \[
  \theta_{\text{new}}^T x = (\theta_{\text{old}} + x)^T x \\
  = \theta_{\text{old}}^T x + x^T x \quad \text{\|x\|}_2^2 > 0
  \]

• Therefore, \(\theta_{\text{new}}^T x\) is less negative than \(\theta_{\text{old}}^T x\)
  – So, we are making ourselves more correct on this example!

Based on slide by Piyush Rai
The Perceptron Cost Function

- The perceptron uses the following cost function

\[ J_p(\theta) = \frac{1}{n} \sum_{i=1}^{n} \max(0, -y^{(i)} x^{(i)} \theta) \]

- \( \max(0, -y^{(i)} x^{(i)} \theta) \) is 0 if the prediction is correct
- Otherwise, it is the confidence in the misprediction
Online Perceptron Algorithm

Let $\theta \leftarrow [0, 0, \ldots, 0]$
Repeat:
  Receive training example $(x^{(i)}, y^{(i)})$
  if $y^{(i)} x^{(i)} \theta \leq 0$  // prediction is incorrect
  $\theta \leftarrow \theta + y^{(i)} x^{(i)}$

**Online learning** – the learning mode where the model update is performed each time a single observation is received

**Batch learning** – the learning mode where the model update is performed after observing the entire training set
Online Perceptron Algorithm

When an error is made, moves the weight in a direction that corrects the error

Red points are labeled +

Blue points are labeled -

Decision boundary 1

Decision boundary 2

Decision boundary 3

Based on slide by Alan Fern
Batch Perceptron

Given training data \( \{(x^{(i)}, y^{(i)})\}_{i=1}^n \)

Let \( \theta \leftarrow [0, 0, \ldots, 0] \)

Repeat:

Let \( \Delta \leftarrow [0, 0, \ldots, 0] \)

for \( i = 1 \ldots n \), do

\[
\text{if } y^{(i)} x^{(i)} \theta \leq 0 \quad \text{// prediction for } i^{th} \text{ instance is incorrect}
\]

\[
\Delta \leftarrow \Delta + y^{(i)} x^{(i)}
\]

\[
\Delta \leftarrow \Delta / n \quad \text{// compute average update}
\]

\[
\theta \leftarrow \theta + \alpha \Delta
\]

Until \( \|\Delta\|_2 < \epsilon \)

- Simplest case: \( \alpha = 1 \) and don’t normalize, yields the fixed increment perceptron
- Guaranteed to find a separating hyperplane if one exists