Linear Regression: Basis Functions, Vectorization

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Last Time: Linear Regression

• Hypothesis:

\[ y = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_d x_d = \sum_{j=0}^{d} \theta_j x_j \]

• Fit model by minimizing sum of squared errors

![Graph showing least squares regression](image)

Figures are courtesy of Greg Shakhnarovich
Last Time: Gradient Descent

- Initialize $\theta$
- Repeat until convergence

$$\theta_j \leftarrow \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta)$$

Simultaneous update for $j = 0 \ldots d$
Regression

Given:

- Data $X = \{x^{(1)}, \ldots, x^{(n)}\}$ where $x^{(i)} \in \mathbb{R}^d$

- Corresponding labels $y = \{y^{(1)}, \ldots, y^{(n)}\}$ where $y^{(i)} \in \mathbb{R}$

Extending Linear Regression to More Complex Models

- The inputs $X$ for linear regression can be:
  - Original quantitative inputs
  - Transformation of quantitative inputs
    - e.g. log, exp, square root, square, etc.
  - Polynomial transformation
    - example: $y = \beta_0 + \beta_1 \cdot x + \beta_2 \cdot x^2 + \beta_3 \cdot x^3$
  - Basis expansions
  - Dummy coding of categorical inputs
  - Interactions between variables
    - example: $x_3 = x_1 \cdot x_2$

This allows use of linear regression techniques to fit non-linear datasets.
Linear Basis Function Models

• Generally,
  \[
  h_\theta(x) = \sum_{j=0}^{d} \theta_j \phi_j(x)
  \]

• Typically,  \( \phi_0(x) = 1 \) so that  \( \theta_0 \) acts as a bias

• In the simplest case, we use linear basis functions:
  \[
  \phi_j(x) = x_j
  \]
Linear Basis Function Models

• Polynomial basis functions:
  \[ \phi_j(x) = x^j \]
  – These are global; a small change in \( x \) affects all basis functions

• Gaussian basis functions:
  \[ \phi_j(x) = \exp \left\{ - \frac{(x - \mu_j)^2}{2s^2} \right\} \]
  – These are local; a small change in \( x \) only affect nearby basis functions. \( \mu_j \) and \( s \) control location and scale (width).

Based on slide by Christopher Bishop (PRML)
Linear Basis Function Models

- Sigmoidal basis functions:

\[ \phi_j(x) = \sigma \left( \frac{x - \mu_j}{s} \right) \]

where

\[ \sigma(a) = \frac{1}{1 + \exp(-a)} \]

- These are also local; a small change in \( x \) only affects nearby basis functions. \( \mu_j \) and \( s \) control location and scale (slope).
Example of Fitting a Polynomial Curve with a Linear Model

$y = \theta_0 + \theta_1 x + \theta_2 x^2 + \ldots + \theta_p x^p = \sum_{j=0}^{p} \theta_j x^j$
Linear Basis Function Models

- Basic linear model:
  \[ h_\theta(x) = \sum_{j=0}^{d} \theta_j x_j \]

- More general linear model:
  \[ h_\theta(x) = \sum_{j=0}^{d} \theta_j \phi_j(x) \]

- Once we have replaced the data by the outputs of the basis functions, fitting the generalized model is exactly the same problem as fitting the basic model
  - Unless we use the kernel trick – more on that when we cover support vector machines

Based on slide by Geoff Hinton