Support Vector Machines & Kernels

Doing *really* well with linear decision surfaces

These slides were assembled by Byron Boots, with only minor modifications from Eric Eaton's slides and grateful acknowledgement to the many others who made their course materials freely available online. Feel free to reuse or adapt these slides for your own academic purposes, provided that you include proper attribution.

Adapted from slides by Tim Oates

Strengths of SVMs

- Good generalization
  - in theory
  - in practice
- Works well with few training instances
- Find globally best model
- Efficient algorithms
- Amenable to the kernel trick

Minor Notation Change

To better match notation used in SVMs

...and to make matrix formulas simpler

We will drop using superscripts for the i th instance

i th instance	$\boldsymbol{x}^{(i)}$	→	$\mathbf{x}_i$
i th instance label	$y^{(i)}$	→	y_i
j th feature of i th instance	$x_j^{(i)}$	→	x_{ij}

Bold denotes vector

Non-bold denotes scalar

Linear Separators

- Training instances

$$\mathbf{x} \in \mathbb{R}^{d+1}, x_0 = 1$$

$$y \in \{-1, 1\}$$

- Model parameters

$$\boldsymbol{\theta} \in \mathbb{R}^{d+1}$$

- Hyperplane

$$\boldsymbol{\theta}^\intercal \mathbf{x} = \langle \boldsymbol{\theta}, \mathbf{x} \rangle = 0$$

- Decision function

$$h(\mathbf{x}) = \text{sign}(\boldsymbol{\theta}^\intercal \mathbf{x}) = \text{sign}(\langle \boldsymbol{\theta}, \mathbf{x} \rangle)$$

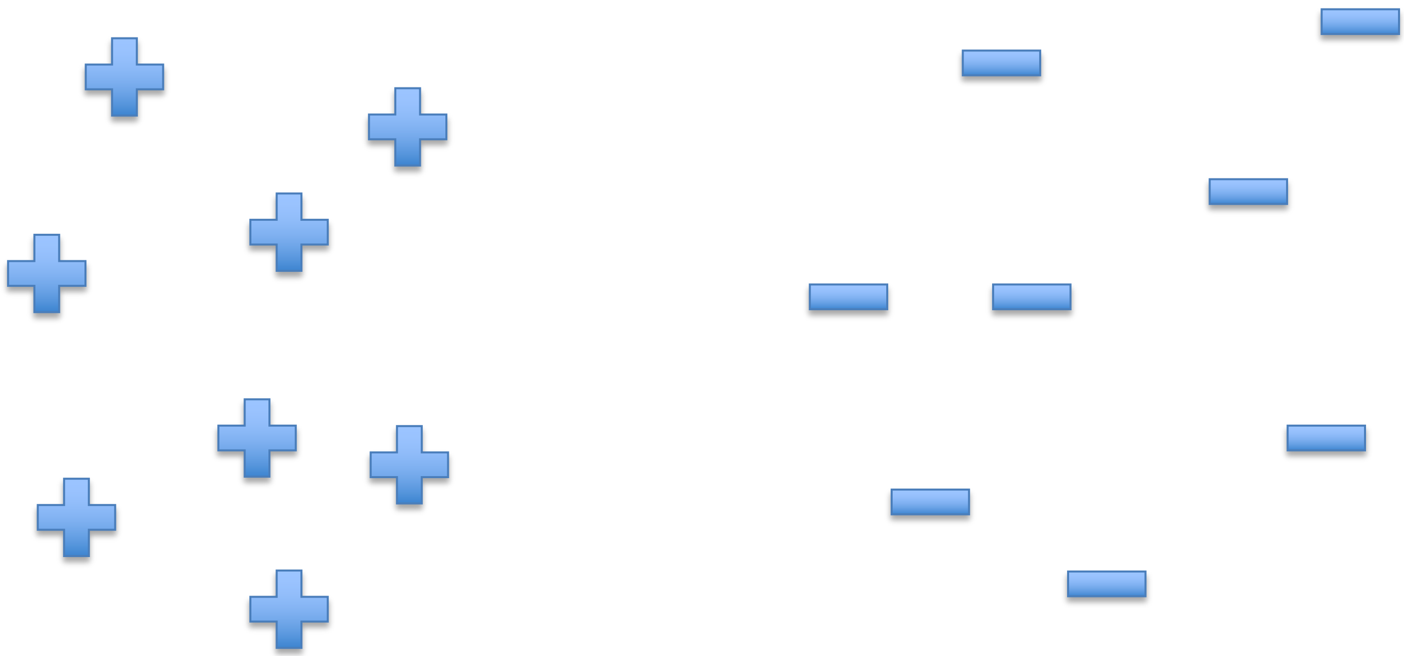
<u>Recall:</u>

Inner (dot) product:

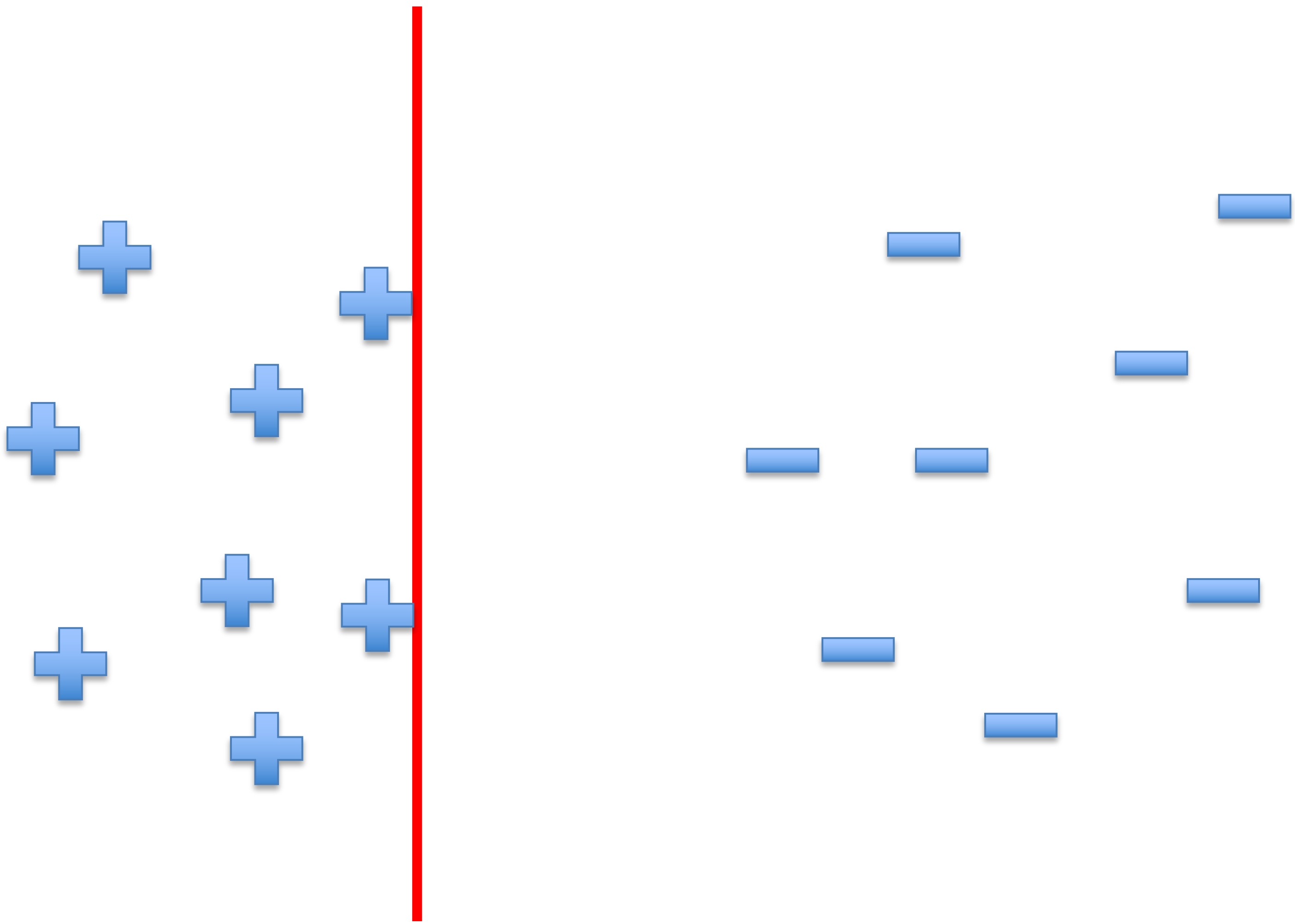
$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^\intercal \mathbf{v}$$

$$= \sum_i u_i v_i$$

Intuitions



Intuitions



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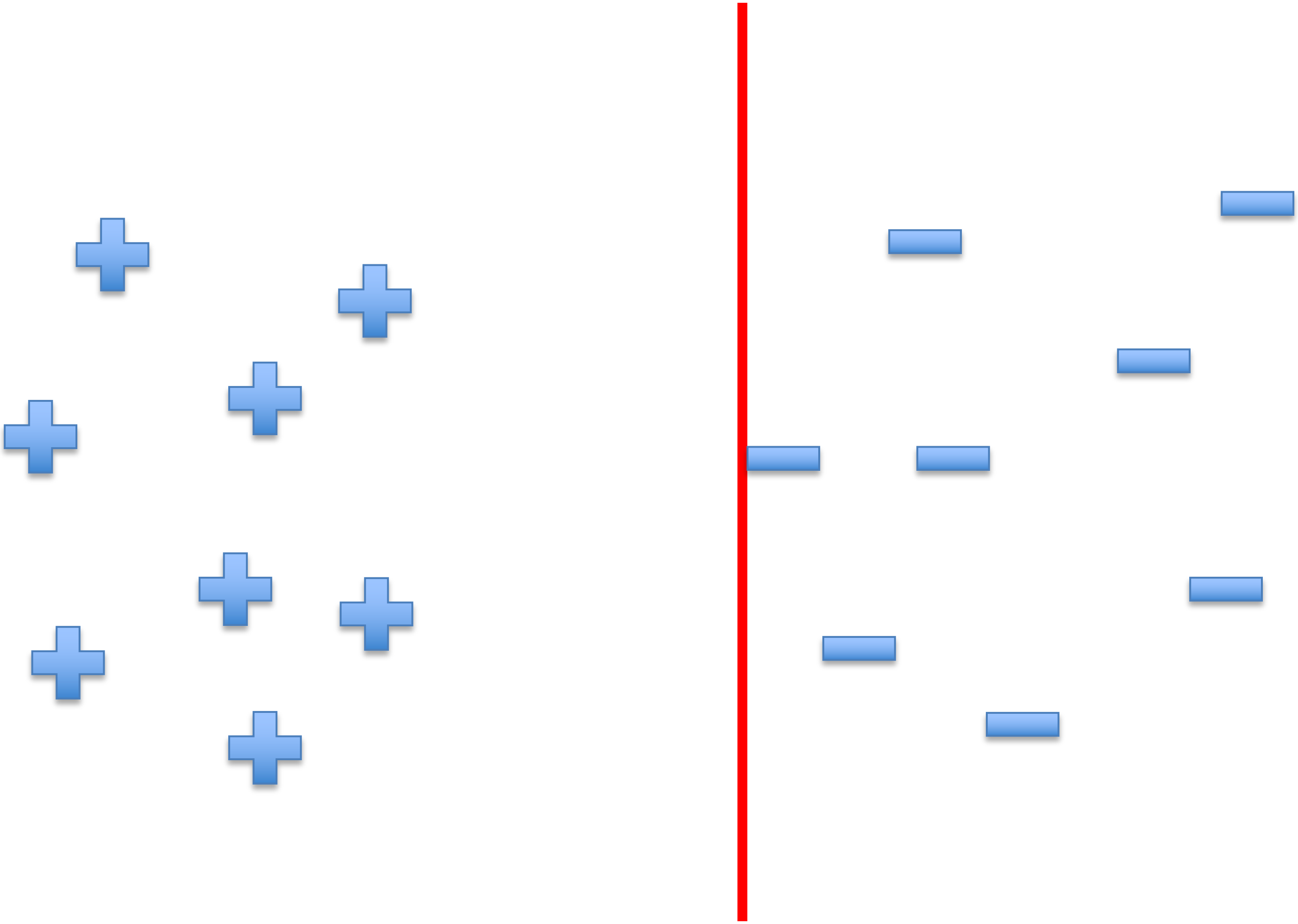
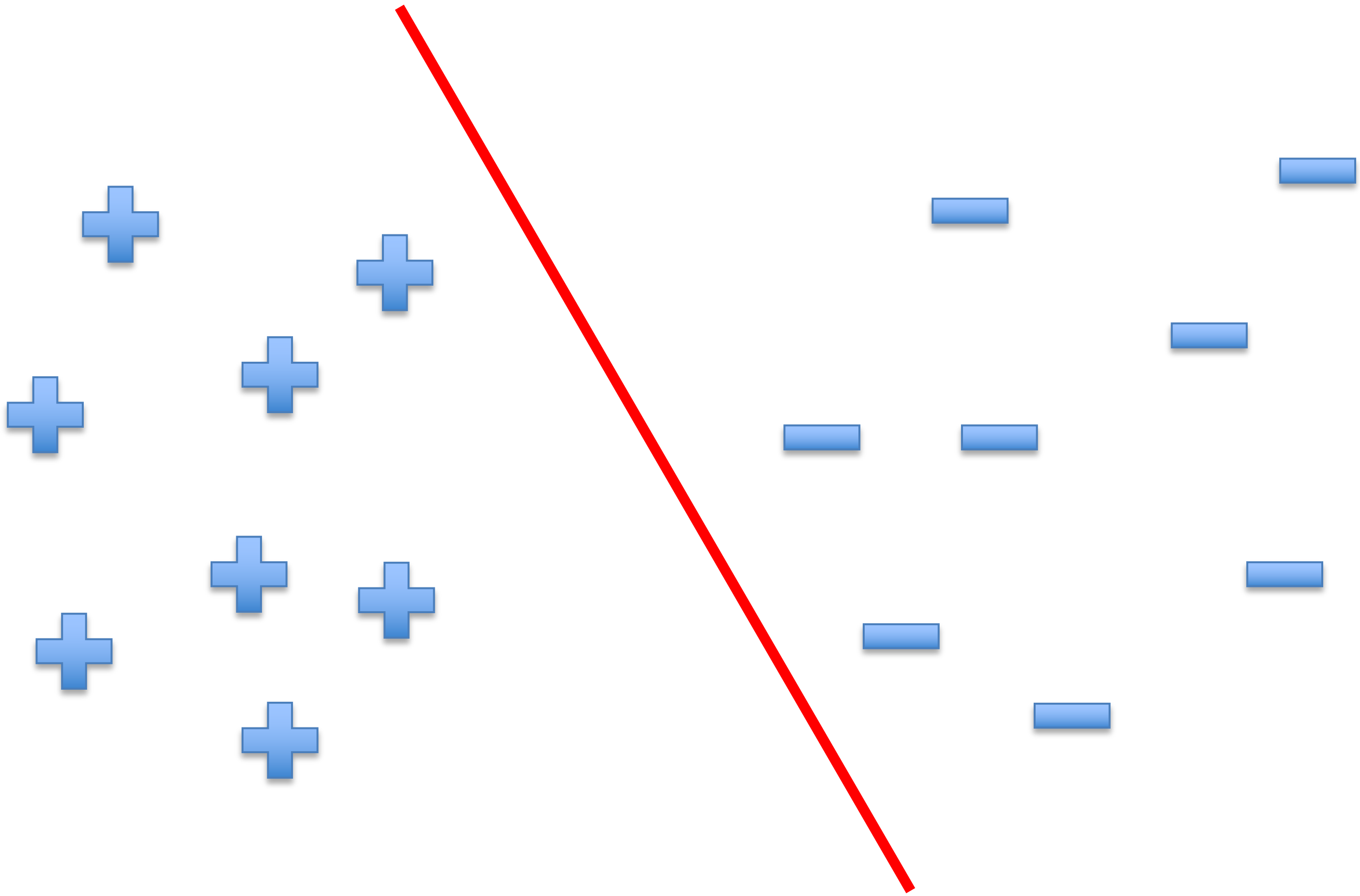
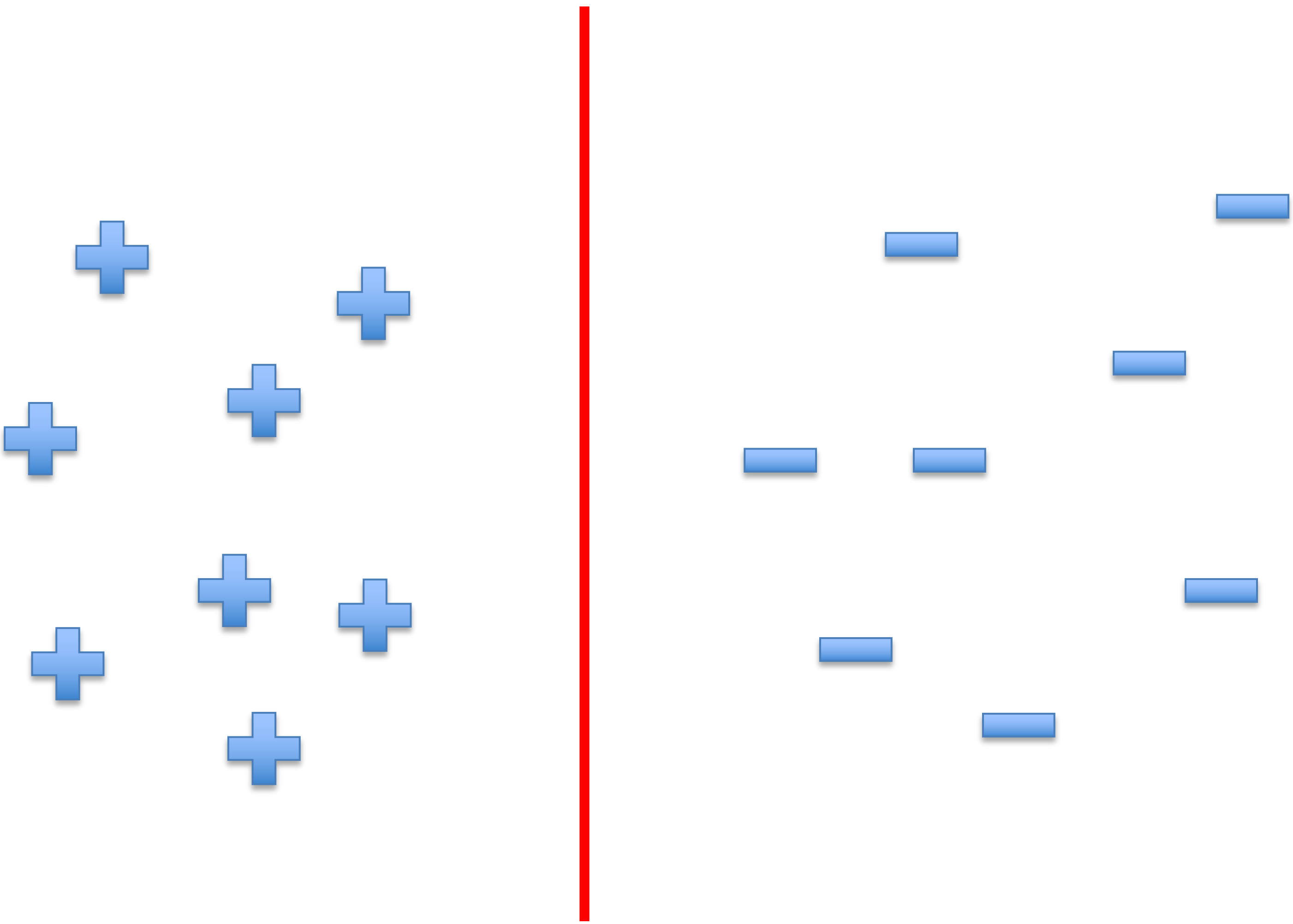


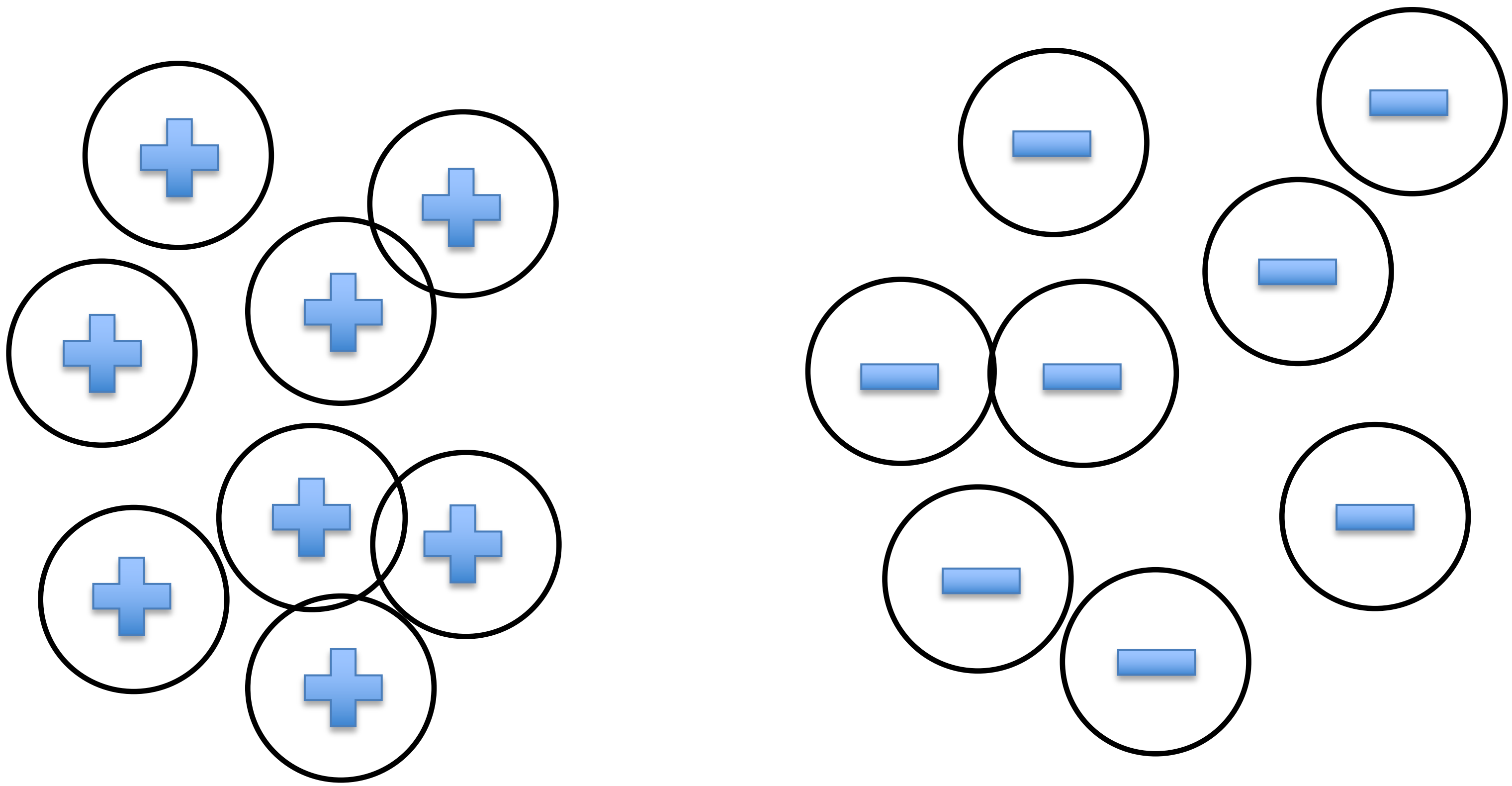
Intuitions



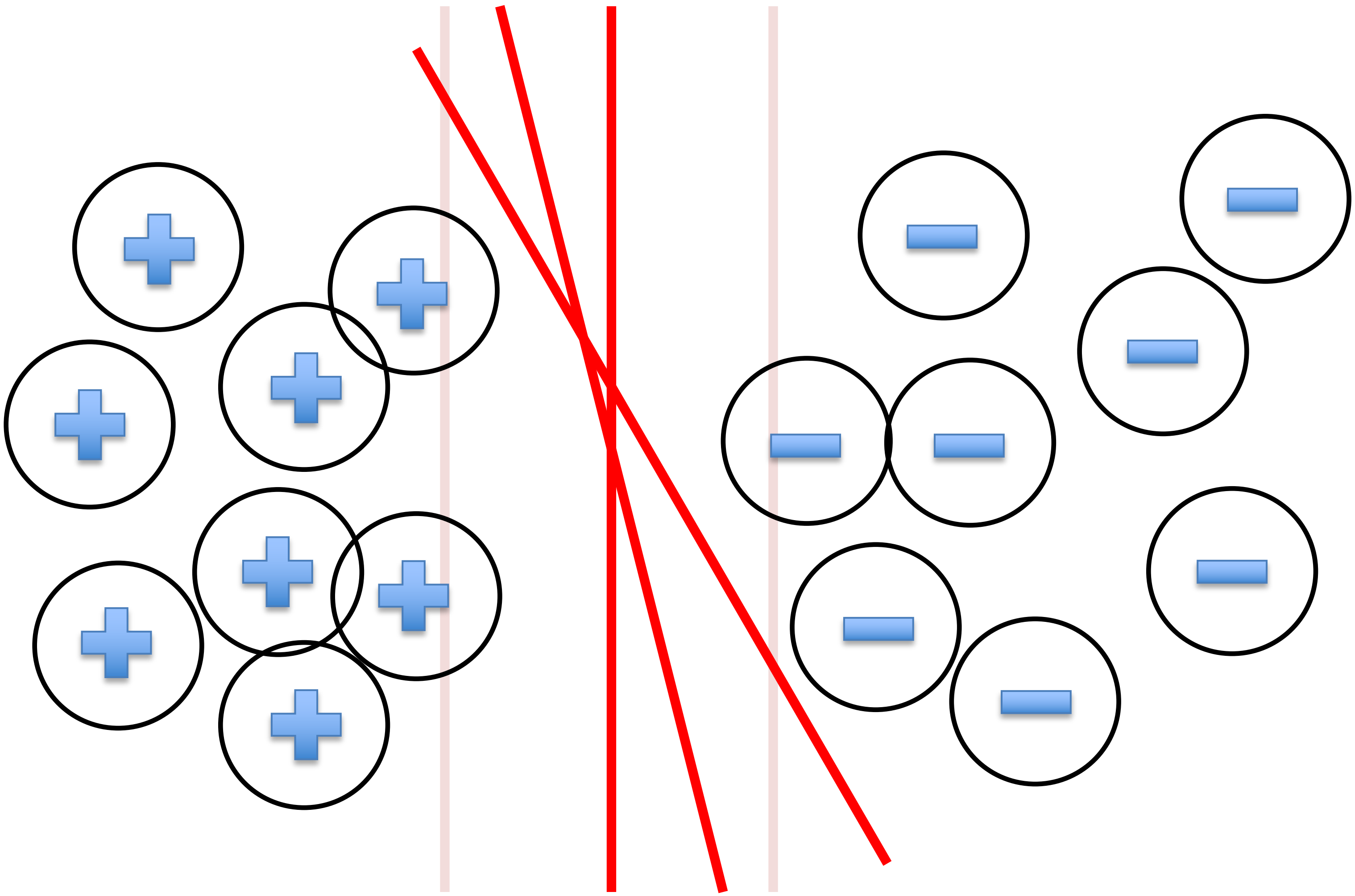
A “Good” Separator



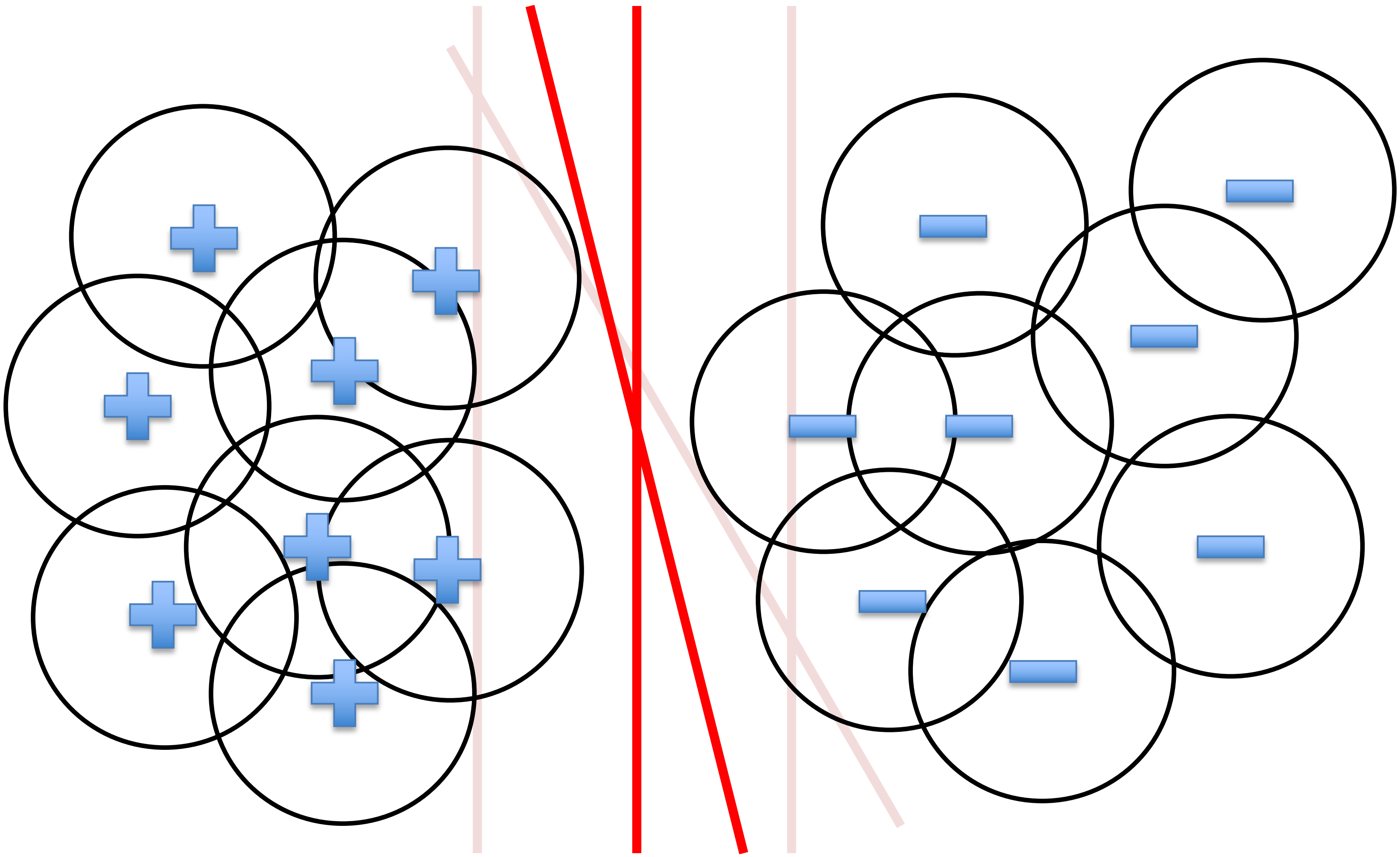
Noise in the Observations



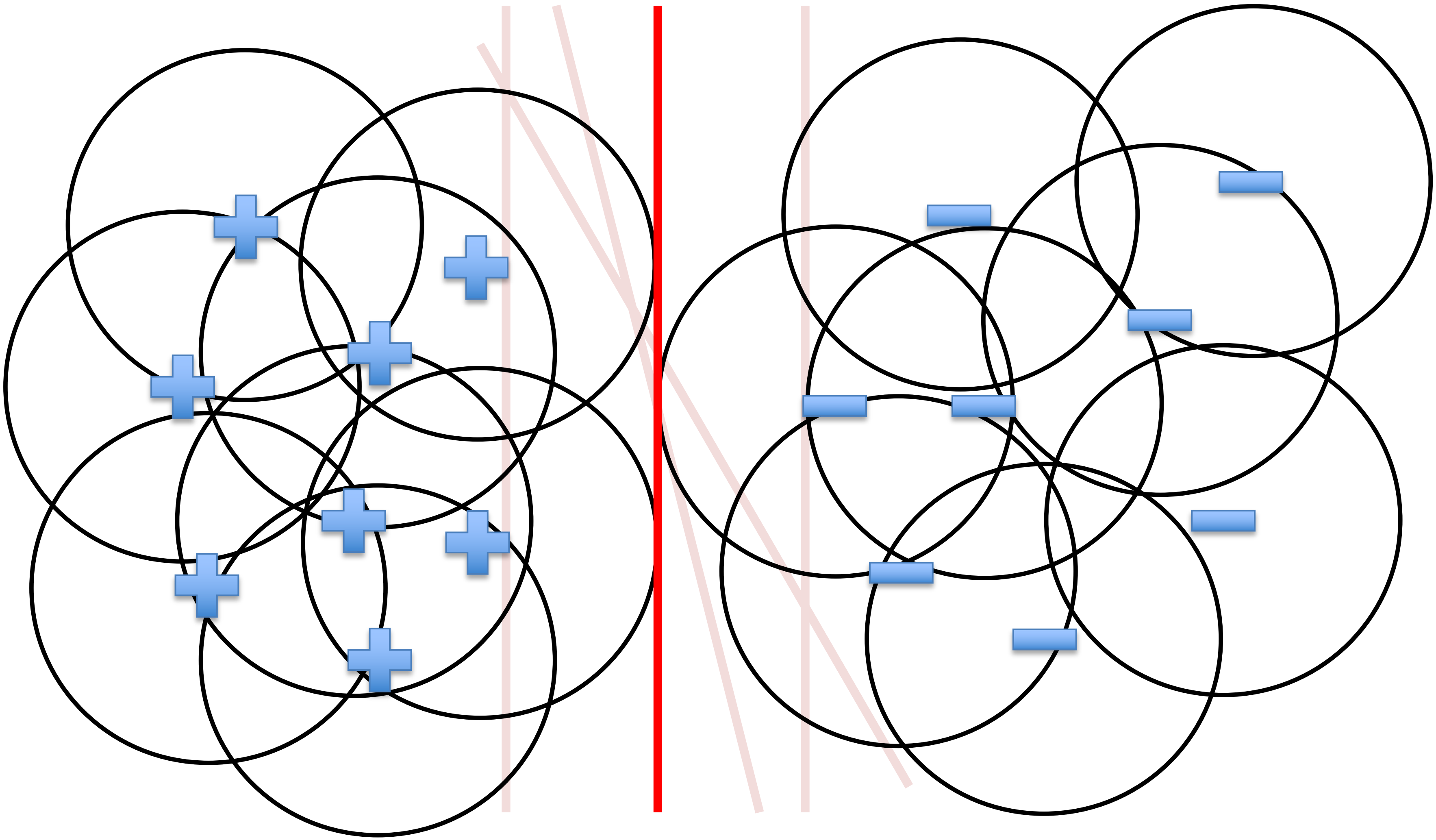
Ruling Out Some Separators



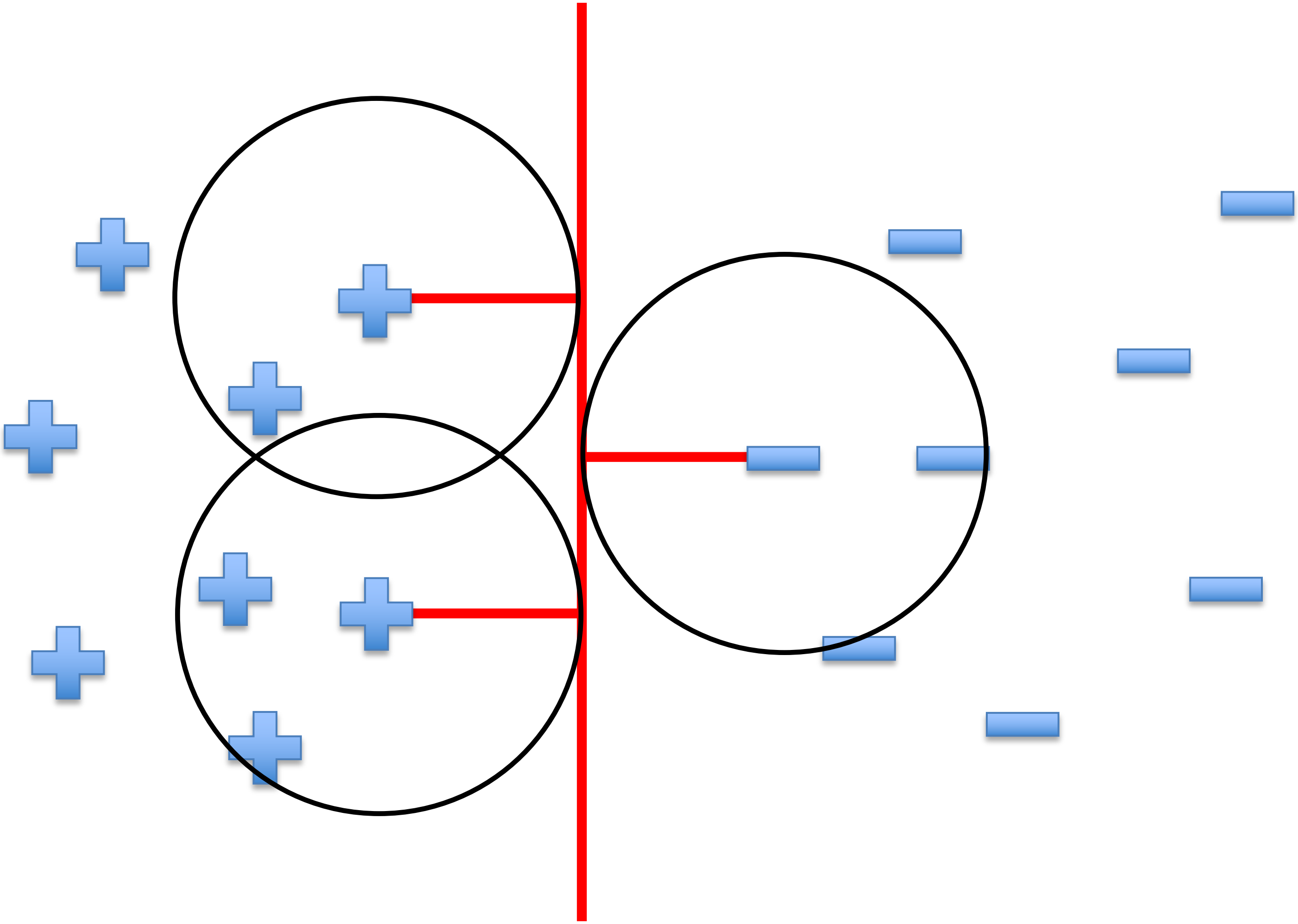
Lots of Noise



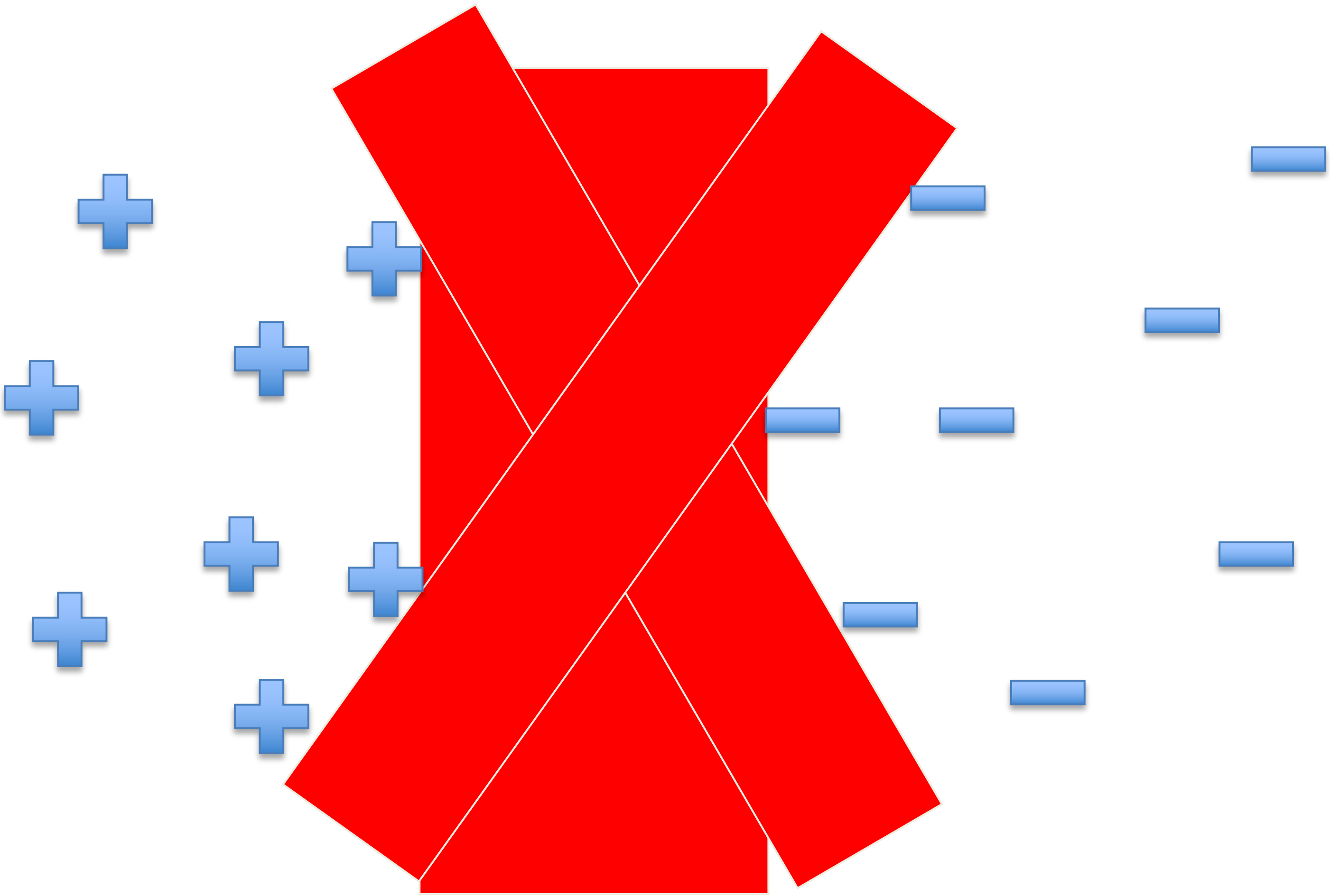
Only One Separator Remains



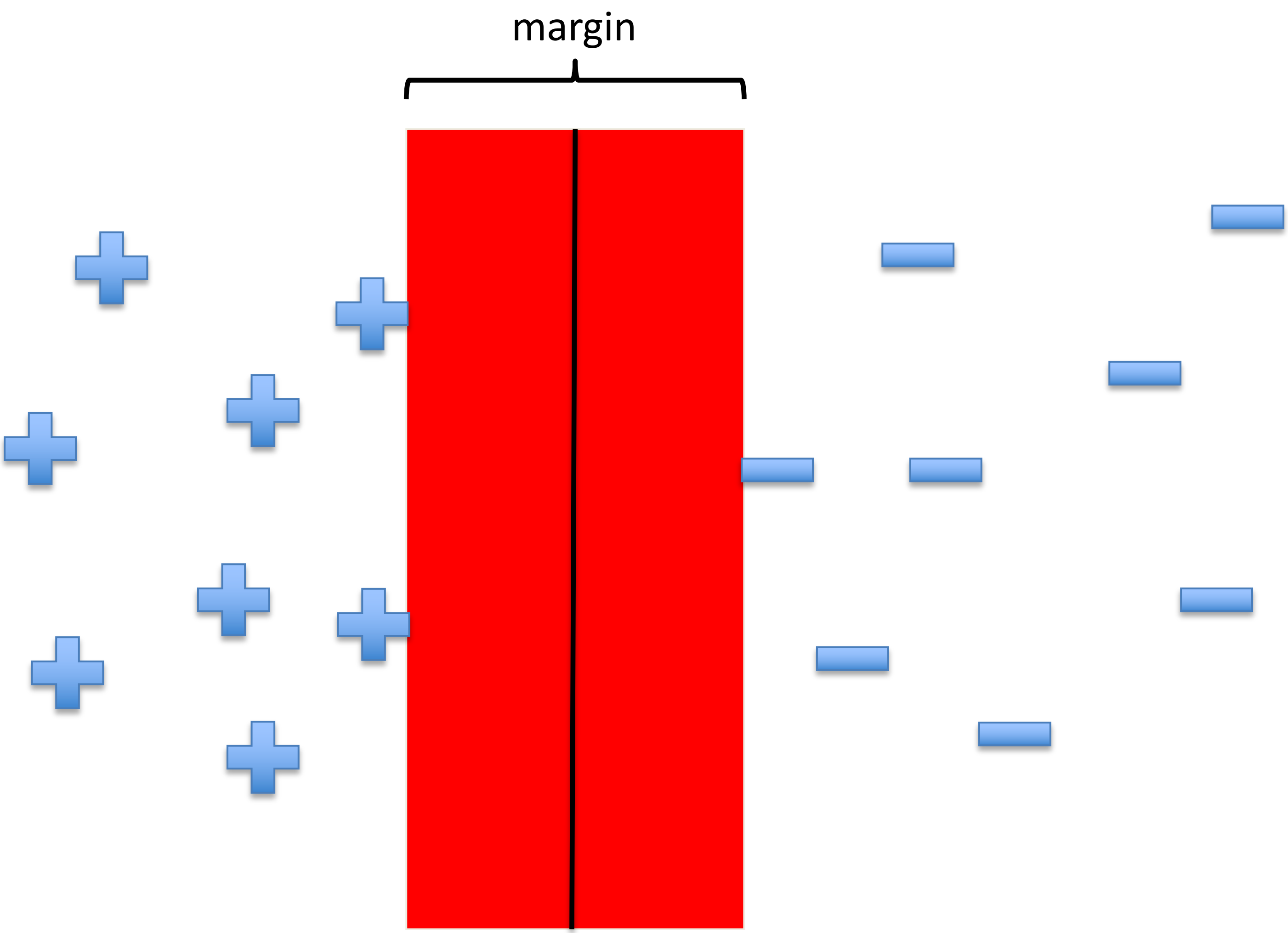
Maximizing the Margin



“Fat” Separators



“Fat” Separators



Why Maximize Margin

Increasing margin reduces *capacity*

- i.e., fewer possible models

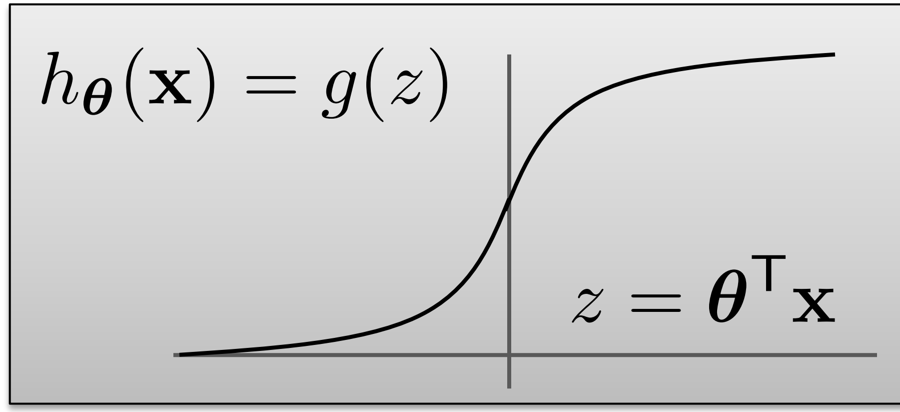
Remember Lesson from Learning Theory:

- If the following holds:
  - H is sufficiently constrained in size
  - and/or the size of the training data set n is large,

  then low training error is likely to be evidence of low generalization error

Alternative View of Logistic Regression

$$h_{\boldsymbol{\theta}}(\mathbf{x}) = \frac{1}{1 + e^{-\boldsymbol{\theta}^{\mathsf{T}}\mathbf{x}}}$$



If $y = 1$, we want $h_{\boldsymbol{\theta}}(\mathbf{x}) \approx 1, \ \boldsymbol{\theta}^{\mathsf{T}}\mathbf{x} \gg 0$

If $y = 0$, we want $h_{\boldsymbol{\theta}}(\mathbf{x}) \approx 0, \ \boldsymbol{\theta}^{\mathsf{T}}\mathbf{x} \ll 0$

$$J(\boldsymbol{\theta}) = -\sum_{i=1}^{n} [\underbrace{y_i \log h_{\boldsymbol{\theta}}(\mathbf{x}_i)}_{\text{cost}_1(\boldsymbol{\theta}^{\mathsf{T}}\mathbf{x}_i)} + \underbrace{(1 - y_i) \log (1 - h_{\boldsymbol{\theta}}(\mathbf{x}_i))}_{\text{cost}_0(\boldsymbol{\theta}^{\mathsf{T}}\mathbf{x}_i)}]$$

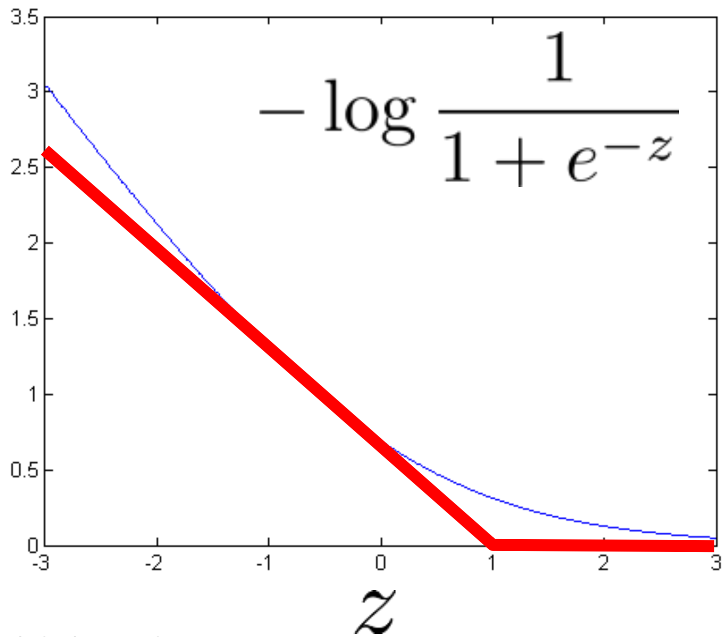
$$\min_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$$

Alternative View of Logistic Regression

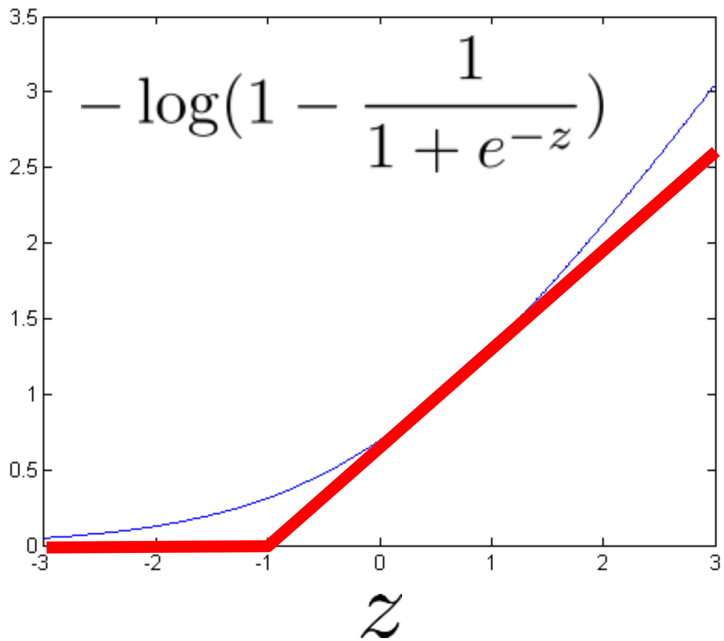
Cost of example: $-y_i \log h_{\boldsymbol{\theta}}(\mathbf{x}_i) - (1 - y_i) \log (1 - h_{\boldsymbol{\theta}}(\mathbf{x}_i))$

$$h_{\boldsymbol{\theta}}(\mathbf{x}) = \frac{1}{1 + e^{-\boldsymbol{\theta}^\mathsf{T}\mathbf{x}}} \qquad z = \boldsymbol{\theta}^\mathsf{T}\mathbf{x}$$

If $y = 1$ (want $\boldsymbol{\theta}^\mathsf{T}\mathbf{x} \gg 0$):



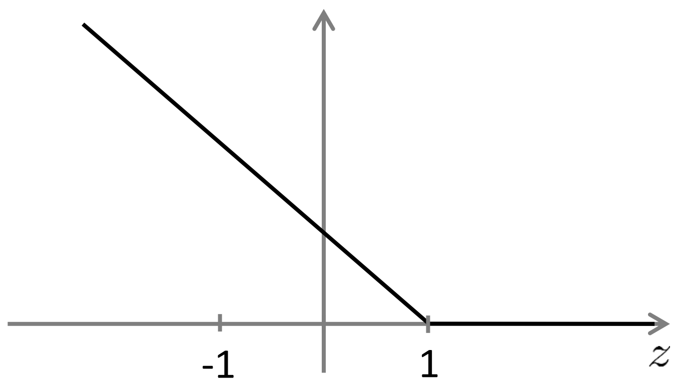
If $y = 0$ (want $\boldsymbol{\theta}^\mathsf{T}\mathbf{x} \ll 0$):



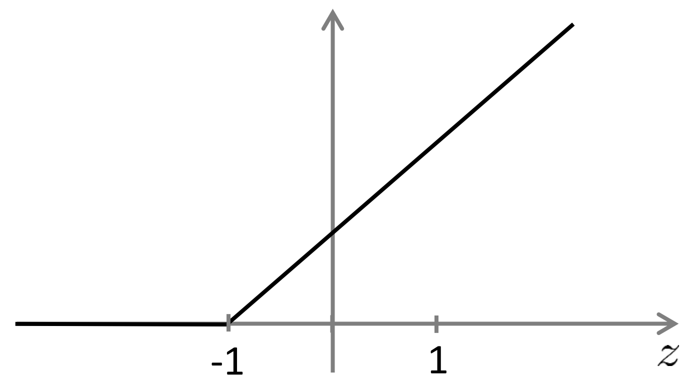
Based on slide by Andrew Ng

Support Vector Machine

If $y = 1$ (want $\boldsymbol{\theta}^{\mathsf{T}}\mathbf{x} \geq 1$):



If $y_i = -1$ (want $\boldsymbol{\theta}^{\mathsf{T}}\mathbf{x} \leq -1$):



$$\ell_{\text{hinge}}(h(\mathbf{x})) = \max(0, 1 - y \cdot h(\mathbf{x}))$$

Support Vector Machine

$$\min_{\boldsymbol{\theta}} \; C \sum_{i=1}^{n} [y_i \text{cost}_1(\boldsymbol{\theta}^\intercal \mathbf{x}_i) + (1 - y_i) \, \text{cost}_0(\boldsymbol{\theta}^\intercal \mathbf{x}_i)] + \frac{1}{2} \sum_{j=1}^{d} \theta_j^2$$

y = 1 / 0

y = +1 / -1

with C = 1

$$\min_{\boldsymbol{\theta}} \frac{1}{2} \sum_{j=1}^{d} \theta_j^2$$

$$\text{s.t.} \;\; \boldsymbol{\theta}^\intercal \mathbf{x}_i \geq 1 \quad \text{if } y_i = 1$$

$$\boldsymbol{\theta}^\intercal \mathbf{x}_i \leq -1 \quad \text{if } y_i = -1$$

$$\min_{\boldsymbol{\theta}} \frac{1}{2} \sum_{j=1}^{d} \theta_j^2$$

$$\text{s.t.} \;\; y_i(\boldsymbol{\theta}^\intercal \mathbf{x}_i) \geq 1$$