Logistic Regression

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Classification Based on Probability

• Instead of just predicting the class, give the probability of the instance being that class
  – i.e., learn $p(y \mid x)$

• Comparison to perceptron:
  – Perceptron doesn’t produce probability estimate
Logistic Regression

• Takes a probabilistic approach to learning discriminative functions (i.e., a classifier)

• $h_\theta(x)$ should give $p(y = 1 \mid x; \theta)$
  
  – Want $0 \leq h_\theta(x) \leq 1$

• Logistic regression model:

  
  $h_\theta(x) = g(\theta^T x)$

  
  $g(z) = \frac{1}{1 + e^{-z}}$

  
  $h_\theta(x) = \frac{1}{1 + e^{-\theta^T x}}$
Interpretation of Hypothesis Output

\[ h_\theta(x) = \text{estimated } p(y = 1 \mid x; \theta) \]

Example: Cancer diagnosis from tumor size

\[ x = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \text{tumorSize} \end{bmatrix} \]

\[ h_\theta(x) = 0.7 \]

→ Tell patient that 70% chance of tumor being malignant

Note that: \[ p(y = 0 \mid x; \theta) + p(y = 1 \mid x; \theta) = 1 \]

Therefore, \[ p(y = 0 \mid x; \theta) = 1 - p(y = 1 \mid x; \theta) \]
Another Interpretation

• Equivalently, logistic regression assumes that

\[
\log \frac{p(y = 1 \mid x; \theta)}{p(y = 0 \mid x; \theta)} = \theta_0 + \theta_1 x_1 + \ldots + \theta_d x_d
\]

odds of \( y = 1 \)

**Side Note**: the odds in favor of an event is the quantity \( p / (1 - p) \), where \( p \) is the probability of the event

E.g., If I toss a fair dice, what are the odds that I will have a 6?

• In other words, logistic regression assumes that the log odds is a linear function of \( x \)
Logistic Regression

\[ h_\theta(x) = g(\theta^T x) \]

\[ g(z) = \frac{1}{1 + e^{-z}} \]

- Assume a threshold and...
  - Predict \( y = 1 \) if \( h_\theta(x) \geq 0.5 \)
  - Predict \( y = 0 \) if \( h_\theta(x) < 0.5 \)

Based on slide by Andrew Ng
Non-Linear Decision Boundary

- Can apply basis function expansion to features, same as with linear regression

\[ \mathbf{x} = \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1 x_2 \\ x_1^2 \\ x_2^2 \\ x_1 x_2^2 \\ x_1^2 x_2 \\ \vdots \end{bmatrix} \]
Logistic Regression (continued)

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Last Time: Logistic Regression

• Given \( \left\{ (x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \ldots, (x^{(n)}, y^{(n)}) \right\} \)

where \( x^{(i)} \in \mathbb{R}^d \), \( y^{(i)} \in \{0, 1\} \)

• Model: \( h_\theta(x) = g(\theta^T x) \)

\[
g(z) = \frac{1}{1 + e^{-z}}
\]
Logistic Regression Objective Function

• Shouldn't use squared loss as in linear regression:

$$J(\theta) = \frac{1}{2n} \sum_{i=1}^{n} \left( h_\theta \left( x^{(i)} \right) - y^{(i)} \right)^2$$

– Using the logistic regression model

$$h_\theta(x) = \frac{1}{1 + e^{-\theta^\top x}}$$

results in a non-convex optimization
Deriving the Cost Function via MLE

- Likelihood of data is given by: 
  \[ l(\theta) = \prod_{i=1}^{n} p(y^{(i)} | x^{(i)}; \theta) \]

- So, looking for the \( \theta \) that maximizes the likelihood
  \[ \theta_{\text{MLE}} = \arg \max_{\theta} l(\theta) = \arg \max_{\theta} \prod_{i=1}^{n} p(y^{(i)} | x^{(i)}; \theta) \]

- Can take the log without changing the solution:
  \[ \theta_{\text{MLE}} = \arg \max_{\theta} \log \prod_{i=1}^{n} p(y^{(i)} | x^{(i)}; \theta) \]
  \[ = \arg \max_{\theta} \sum_{i=1}^{n} \log p(y^{(i)} | x^{(i)}; \theta) \]
Deriving the Cost Function via MLE

• Expand as follows:

$\theta_{\text{MLE}} = \arg \max_\theta \sum_{i=1}^n \log p(y^{(i)} \mid x^{(i)}; \theta)$

$$= \arg \max_\theta \sum_{i=1}^n \left[ y^{(i)} \log p(y^{(i)} = 1 \mid x^{(i)}; \theta) + (1 - y^{(i)}) \log (1 - p(y^{(i)} = 1 \mid x^{(i)}; \theta)) \right]$$

• Substitute in model, and take negative to yield

Logistic regression objective:

$$\min_\theta J(\theta)$$

$$J(\theta) = - \sum_{i=1}^n \left[ y^{(i)} \log h_\theta(x^{(i)}) + (1 - y^{(i)}) \log (1 - h_\theta(x^{(i)})) \right]$$
Intuition Behind the Objective

\[
J(\theta) = -\sum_{i=1}^{n} \left[ y^{(i)} \log h_{\theta}(x^{(i)}) + (1 - y^{(i)}) \log (1 - h_{\theta}(x^{(i)})) \right]
\]

• Cost of a single instance:

\[
\text{cost} \left( h_{\theta}(x), y \right) = \begin{cases} 
- \log(h_{\theta}(x)) & \text{if } y = 1 \\
- \log(1 - h_{\theta}(x)) & \text{if } y = 0
\end{cases}
\]

• Can re-write objective function as

\[
J(\theta) = \sum_{i=1}^{n} \text{cost} \left( h_{\theta}(x^{(i)}), y^{(i)} \right)
\]
Intuition Behind the Objective

\[
\text{cost} \left( h_{\theta}(x), y \right) = \begin{cases} 
- \log(h_{\theta}(x)) & \text{if } y = 1 \\
- \log(1 - h_{\theta}(x)) & \text{if } y = 0
\end{cases}
\]

Aside: Recall the plot of \( \log(z) \)
Intuition Behind the Objective

\[
\text{cost} \left( h_\theta(x), y \right) = \begin{cases} 
- \log(h_\theta(x)) & \text{if } y = 1 \\
- \log(1 - h_\theta(x)) & \text{if } y = 0 
\end{cases}
\]

If \( y = 1 \)
- Cost = 0 if prediction is correct
- As \( h_\theta(x) \to 0 \), cost → \( \infty \)
- Captures intuition that larger mistakes should get larger penalties
  - e.g., predict \( h_\theta(x) = 0 \), but \( y = 1 \)

Based on example by Andrew Ng
**Intuition Behind the Objective**

\[
\text{cost} \left( h_\theta(\mathbf{x}), y \right) = \begin{cases} 
-\log(h_\theta(\mathbf{x})) & \text{if } y = 1 \\
-\log(1 - h_\theta(\mathbf{x})) & \text{if } y = 0 
\end{cases} 
\]

If \( y = 0 \)

- Cost = 0 if prediction is correct
- As \( (1 - h_\theta(\mathbf{x})) \to 0 \), cost \( \to \infty \)
- Captures intuition that larger mistakes should get larger penalties

Based on example by Andrew Ng
Regularized Logistic Regression

\[ J(\theta) = - \sum_{i=1}^{n} \left[ y^{(i)} \log h_{\theta}(x^{(i)}) + \left(1 - y^{(i)}\right) \log \left(1 - h_{\theta}(x^{(i)})\right) \right] \]

- We can regularize logistic regression exactly as before:

\[ J_{\text{regularized}}(\theta) = J(\theta) + \lambda \sum_{j=1}^{d} \theta_j^2 \]

\[ = J(\theta) + \lambda \| \theta_{[1:d]} \|_2^2 \]
Gradient Descent for Logistic Regression

\[ J_{\text{reg}}(\theta) = -\sum_{i=1}^{n} \left[ y^{(i)} \log h_\theta(x^{(i)}) + (1 - y^{(i)}) \log (1 - h_\theta(x^{(i)})) \right] + \lambda \left\| \theta_{[1:d]} \right\|_2^2 \]

Want \( \min_\theta J(\theta) \)

- Initialize \( \theta \)
- Repeat until convergence

\[ \theta_j \leftarrow \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta) \]

Simultaneous update for \( j = 0 \ldots d \)

Use the natural logarithm (\( \ln = \log_e \)) to cancel with the \( \exp() \) in \( h_\theta(x) \)
Gradient Descent for Logistic Regression

\[ J_{\text{reg}}(\theta) = - \sum_{i=1}^{n} \left[ y^{(i)} \log h_{\theta}(x^{(i)}) + (1 - y^{(i)}) \log (1 - h_{\theta}(x^{(i)})) \right] + \lambda \| \theta_{[1:d]} \|_2^2 \]

Want \( \min_{\theta} J(\theta) \)

- Initialize \( \theta \)
- Repeat until convergence (simultaneous update for \( j = 0 \ldots d \))

\[
\begin{align*}
\theta_0 & \leftarrow \theta_0 - \alpha \sum_{i=1}^{n} \left( h_{\theta} \left( x^{(i)} \right) - y^{(i)} \right) \\
\theta_j & \leftarrow \theta_j - \alpha \left[ \sum_{i=1}^{n} \left( h_{\theta} \left( x^{(i)} \right) - y^{(i)} \right) x_j^{(i)} - \frac{\lambda}{n} \theta_j \right]
\end{align*}
\]
Gradient Descent for Logistic Regression

• Initialize $\theta$

• Repeat until convergence
  
  \[
  \theta_0 \leftarrow \theta_0 - \alpha \sum_{i=1}^{n} \left( h_{\theta} \left( x^{(i)} \right) - y^{(i)} \right)
  \]

  \[
  \theta_j \leftarrow \theta_j - \alpha \left[ \sum_{i=1}^{n} \left( h_{\theta} \left( x^{(i)} \right) - y^{(i)} \right) x_j^{(i)} - \frac{\lambda}{n} \theta_j \right]
  \]

This looks IDENTICAL to linear regression!!!

• Ignoring the 1/n constant

• However, the form of the model is very different:

\[
 h_{\theta}(x) = \frac{1}{1 + e^{-\theta^T x}}
\]