Machine Learning (CSE 446): The Perceptron Algorithm

Sham M Kakade

University of Washington
cse446-staff@cs.washington.edu

Announcements

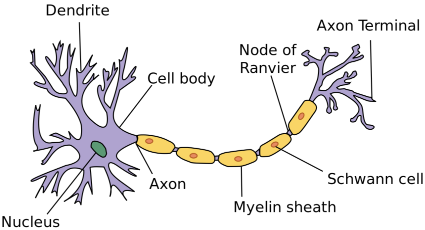
- HW1 posted.
- Today: the perceptron algo

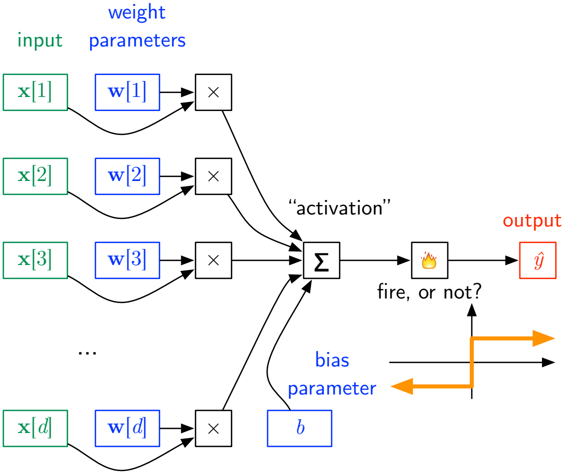
Is there a happy medium?

- Decision trees (that aren't too deep): use relatively few features to classify.
- If we want to use 'all' the features, then we need a deep (high complexity) decision tree (so overfitting will be more of a concern).
- Is there a 'low complexity' method (something which is better at keeping the generalization error small) which is able to utilize all the features for prediction?
- One idea:
  A 'linear classifier': use all features, but weight them.

Inspiration from Neurons

Image from Wikimedia Commons.





Input signals come in through dendrites, output signal passes out through the axon.

A "parametric" Hypothesis

- Consider using a "linear classifier" :

$$f(\mathbf{x}) = \text{sign}\left(\mathbf{w} \cdot \mathbf{x} + b\right)$$

where $w \in \mathbb{R}^d$ and b is a scalar.

- Here sign(z) is the function which is 1 if $z \geq 0$ and -1 otherwise. (Let us say that $\mathcal{Y} = \{-1, 1\}$.)
- Notation: $x \in \mathbb{R}^d$; $x[i]$ denotes $i - th$ coordinate; remember that:

$$w \cdot x = \sum_{j=1}^{d} w[j] \cdot x[j]$$

- Learning requires us to set the weights $\mathbf{w}$ and the bias b.
- (convenience) we can always append 1 to the vector x through the concatenation

$$x \leftarrow (x, 1)\,.$$

This transformation allows us to absorb the bias b into the last component of the weight vector.

Geometrically...

- What does the decision boundary look like?

What would we like to do?

$$\widehat{\epsilon}(w) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\{y_i \neq \mathsf{sign}(w \cdot x_i)\}$$

- **Optimization problem:** find a classifier which minimizes the classification loss on the training dataset D:
$$\min_w \widehat{\epsilon}(w)$$
- Problem: (in general) this is an NP-Hard problem.
- Let's ignore this, and think of an algorithm.

This is the general approach of loss function minimization: find parameters which make our training error "small" (and which also generalizes)

The Perceptron Leaning Algorithm (Rosenblatt, 1958)

- Let's think of an **online** algorithm, where we try to update w as we examine each training point (x_i, y_i), one at a time.
- Consider the 'epoch based' algorithm:
  1. For all (x, y) in our training set:
  2. Choose a point (x, y) without replacement from D:
     Let $\widehat{y} = \text{sign}(w \cdot x)$
     If $\widehat{y} = y$, then do not update:
     $$w_{t+1} = w_t$$
     If $\widehat{y} \neq y$,
     $$w_{t+1} = w_t + yx$$

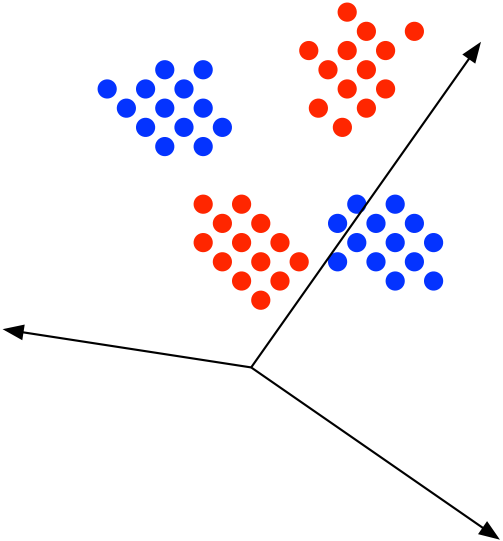
  Return to the first step.
- When to stop?

Parameters and Convergence

This is the first supervised algorithm we have seen with (no-trivial) real valued **parameters**, w.

- The perceptron learning algorithm's sole hyperparameter is E, the number of epochs (passes over the training data). How should we tune E?
  use a Dev set.
- Basic question: will the algorithm converge or not?
  - Can you say what has to occur for the algorithm to converge? (Novikoff, 1962)
  - Can we understand when it will *never* converge? (Minsky and Papert, 1969)

When does the perceptron not converge?



Linear Separability

A dataset $D = \langle (\mathbf{x}_n, y_n) \rangle_{n=1}^{N}$ is **linearly separable** if there exists some linear classifier (defined by $\mathbf{w}, b$) such that, for all n, $y_n = \text{sign}\left(\mathbf{w} \cdot \mathbf{x}_n + b\right)$.

The Perceptron Convergence

- Again taking $b = 0$ (absorbing it into w).
- Margin def: Suppose the data are linearly separable, and all data points are γ away from the separating hyperplane. Precisely, there exists a w_*, which we can assume to be of unit norm (without loss of generality), such that for all $(x, y) \in D$.

$$y\,(w_* \cdot x) \geq \gamma$$

γ is the **margin**.

Theorem: (Novikoff, 1962) Suppose the inputs bounded such that $\|x\| \leq R$. Assume our data D is linearly separable with margin γ. Then the perceptron algorithm will make at most $\frac{R^2}{\gamma^2}$ mistakes.
(This implies that at most $O(\frac{N}{\gamma^2})$ updates, after which time w_t never changes.)

Proof of the "Mistake Lemma"

- Let M_t be the number of mistakes at time t.
  If we make a mistake using w_t on (x, y), then observe that $yw_t \cdot x \leq 0$.
- Suppose we make a mistake at time t:

$$w_* \cdot w_t = w_* \cdot (w_{t-1} + yx) = w_* \cdot w_{t-1} + yw_* \cdot x \geq w_* \cdot w_{t-1} + \gamma \,.$$

  Since $w_0 = 0$ and $w_* \cdot w_t$ grows by γ every time we make a mistake, this implies that $w_* \cdot w_t \geq \gamma M_t$.
- Also, if we make a mistake at time t (using that $yw_t \cdot x \leq 0$),

$$\|w_t\|^2 = \|w_{t-1}\|^2 + 2yw_{t-1} \cdot x + ||x||^2 \leq \|w_{t-1}\|^2 + 0 + ||x||^2 \leq \|w_{t-1}\|^2 + R^2 \,.$$

  Since $\|w_t\|^2$ grows by R^2 on every mistake, this implies $\|w_t\|^2 \leq R^2 M_t$ and so $\|w_t\| \leq R\sqrt{M_t}$.
- Now we have that:

$$\gamma M_t \leq w_* \cdot w_t \leq \|w_*\| \|w_t\| \leq R\sqrt{M_t} \,.$$

  solving for M_t completes the proof.

References I

M. Minsky and S. Papert. *Perceptrons: An Introduction to Computational Geometry*. MIT Press, Cambridge, MA, 1969.

A. B. Novikoff. On convergence proofs on perceptrons. In *Proceedings of the Symposium on the Mathematical Theory of Automata*, 1962.

Frank Rosenblatt. The perceptron: A probabilistic model for information storage and organization in the brain. *Psychological Review*, 65:386–408, 1958.