Machine Learning (CSE 446):
(Supervised) Learning as Loss Minimization:
Linear Regression

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Announcements

- HW2 posted, milestone due this wed
- HW2 extra credit posted
- updated HW Late policy (see website)
The singular value decomposition

- Let $M$ be a **symmetric** matrix.
  SVDs also work for asymmetric matrices, with a slightly modified thm.

- SVD theorem: there exists a decomposition of the following form:

  $$M = UDU^\top$$

  where $D$ is a diagonal matrix and $U$ is an orthogonal matrix (i.e. the columns of $U$ are unit length and orthogonal to each other).

- The columns of $U$ are eigenvectors of $M$.

- For PCA, you will take $\Sigma$ to be $M$. 
Projection and Reconstruction: the one dimensional case

- Take out mean $\mu$: $x_i \leftarrow x_i - \mu$
- Find the “top” eigenvector $u_1$ of the covariance matrix, with eigenvalue $\lambda_1$
- What are your projection onto $u_1$ (i.e. writing $x_i$ in the $u_1$ basis)?
  \[(x_i \cdot u_1)\]
- What are your reconstructions, $\hat{X} = [\hat{x}_1 |\hat{x}_2| \cdots |\hat{x}_N]^\top$?
  \[
  \hat{x}_i = (x_i \cdot u_1)u_1 + \mu
  \]
- What is is your reconstruction error?
  \[
  \frac{1}{N} \sum_{i} \|x_i - \hat{x}_i\|^2 = \lambda_2 + \ldots \lambda_d
  \]
  (also, we if did nothing and projected everything to $\mu$, then:
  \[
  \frac{1}{N} \sum_{i} \|x_i - \mu\|^2 = \lambda_1 + \lambda_2 + \ldots \lambda_d
  \]
  so we ’save’ $\lambda_1$ in our error.
Today: how do we *efficiently* do supervised learning?

“Minimize training-set error rate”:

$$\min_{w, b} \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}\{y_i(w \cdot x_i + b) \leq 0\}$$

This problem is NP-hard; even for a (multiplicative) approximation. **Perceptron Algorithm**: A model and an algorithm, rolled into one.

Is there a more principled methodology to derive algorithms?
Relax!

- The mis-classification optimization problem:

\[
\min_w \frac{1}{N} \sum_{i=1}^{N} 1\{y_i (w \cdot x_i) \leq 0\}
\]

- Instead, let's try to choose a “reasonable” loss function \(\ell(y_i, w \cdot x)\) and then try to solve the relaxation:

\[
\min_w \frac{1}{N} \sum_{i=1}^{N} \ell(y_i, w \cdot x_i)
\]
What is a good “relaxation”? 

- Want that minimizing our **surrogate** loss helps with minimizing the mis-classification loss.
  - idea: try to use a (sharp) upper bound of the zero-one loss by $\ell$:
    
    $$1\{y(w \cdot x) \leq 0\} \leq \ell(y, w \cdot x)$$

- want our **relaxed** optimization problem to be easy to solve. What properties might we want for $\ell(\cdot)$?
  - differentiable? sensitive to changes in $w$?
  - **convex**?
The square loss as an upper bound

- We have:

\[ 1\{y(w \cdot x) \leq 0\} \leq (y - w \cdot x)^2 \]

- Easy to see, by plotting:
A better (convex) upper bound

- The logistic loss:
  \[ \ell_{\text{logistic}}(y, \mathbf{w} \cdot \mathbf{x}) = \log (1 + \exp(-y \mathbf{w} \cdot \mathbf{x})) \cdot \]

- We have:
  \[ 1\{y(\mathbf{w} \cdot \mathbf{x}) \leq 0\} \leq \text{constant} \cdot \ell_{\text{logistic}}(y, \mathbf{w} \cdot \mathbf{x}) \]

- Again, easy to see, by plotting:
The square loss! (and linear regression)

- The square loss: \( \ell(y, w \cdot x) = (y - w \cdot x)^2 \).
- The relaxed optimization problem:
  \[
  \min_w \frac{1}{N} \sum_{i=1}^{N} (y_i - w \cdot x_i)^2
  \]
- nice properties:
  - for binary classification, it is an upper bound on the zero-one loss.
  - It makes sense more generally, e.g. if we want to predict real valued \( y \).
  - We have a convex optimization problem.
- For classification, what is your decision rule using a \( w \)?
Remember this problem?

Data derived from https://archive.ics.uci.edu/ml/datasets/Auto+MPG

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</table>

Input: a row in this table.

Goal: predict whether mpg is < 23 ("bad" = 0) or above ("good" = 1) given the input row.

Predicting a real \( y \) (often) makes more sense.
Least squares: let’s minimize it!

- The optimization problem:

\[
\min_w \frac{1}{N} \sum_{i=1}^{N} (y_i - w \cdot x_i)^2 = \min_w \|Y - Xw\|^2
\]

where \(Y\) is an \(N\)-vector and \(X\) is our \(N \times d\) data matrix.

- How do we interpret \(Xw\)?

The solution is the least squares estimator:

\[
w^{\text{least squares}} = (X^\top X)^{-1} X^\top Y
\]
Vector calculus hints

- Suppose we have a function \( f(w) = w \cdot c = \sum_j w[j]c[j] \), where \( w \) and \( c \) is a \( d \)-dimensional vector.

- Elementary calculus tells us scalar derivatives:
  \[
  \frac{\partial f(w)}{\partial w[i]} = c[i]
  \]

- The gradient is the vector of all the partial derivatives:
  \[
  \nabla f(w) := \left( \frac{\partial f(w)}{\partial w[1]}, \frac{\partial f(w)}{\partial w[2]}, \cdots, \frac{\partial f(w)}{\partial w[d]} \right)^\top
  \]

- So we have that:
  \[
  \nabla f(w) = c
  \]
Matrix calculus hints

- Suppose we have a function

\[
  f(w) = w^\top M w = \sum_{j,k} w[j] w[k] M[j, k],
\]

where \( M \) is a symmetric \( d \times d \) matrix.

- Elementary calculus tells us scalar derivatives:

\[
  \frac{\partial f(w)}{\partial w[i]} = 2 \sum_j M[i, j] w[j]
\]

- The gradient is just the matrix of all the partial derivatives:

\[
  \nabla f(w) := \left( \frac{\partial f(w)}{\partial w[1]}, \frac{\partial f(w)}{\partial w[2]}, \ldots, \frac{\partial f(w)}{\partial w[d]} \right)
\]

- It is straightforward to see that a far more compact way to write the gradient is:

\[
  \nabla f(w) = 2Mw
\]

(just equate each coordinate with the scalar derivative).
Lots of questions:

- What could go wrong with least squares?
  - Suppose we are in “high dimensions”: more dimensions than data points.
  - Inductive bias: we need a way to control the complexity of the model.
- Optimization: how do we do this all quickly?