Section 07: Eigenvalues and Eigenvectors

1. Using the Eigenbasis

It's a very useful fact that for any symmetric $n \times n$ matrix A you can find a set of eiegenvectors $u_1, \ldots, u_n$ for A such that:

- $||u_i||_2 = 1$
- $u_i^T u_j = 0, \forall i \neq j$
- $u_1, \ldots, u_n$ form a basis of $\mathbb{R}^n$

One of the reasons this fact is useful is that facts about these matrices are easier to prove if you think about the vectors in terms of their "eigenbasis" components, instead of their components in the standard basis. As a trivial example, we'll show that you can calculate Ax for a vector x without having to do the matrix multiplication.

(a) Consider the matrix $A = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$. Verify that $u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $u_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ are eigenvectors and meet the definitions. Find the eigenvalues associated with u_1 and u_2

(b) Since $\{u_1, u_2\}$ are a basis, we can write any vector as a linear combination of them. Write $x = \begin{bmatrix} -1/\sqrt{2} \\ 3/\sqrt{2} \end{bmatrix}$ in this basis.

(c) Use the decomposition and the eigenvalues you calculated to calculate Ax wihtout doing matrix-vector multiplication.

This method of calculating a matrix vector product won't actually be more computationally efficient – but it's what's "really" happening when you do the multiplication, so this will be useful intuition under certain circumstances. Expressing vectors in an eigenbasis is also a useful proof technique, as we'll see in some later problems.

2. Sets of Eigenvectors

(a) Prove that if A is a symmetric matrix with n distinct eigenvalues, then its eigenvectors are orthogonal. *Hint: if u and v are eigenvectors, calculate $u^T Av$ two different ways.*

(b) Suppose that A is a symmetric matrix. Prove, without appealing to calculus, that the solution to $\arg\max_x x^T Ax$ s.t. $||x||_2 = 1$ is the eigenvector x_1 corresponding to the largest eigenvalue λ_1 of A. (Hint: the eigenvectors of a symmetric matrix can be chosen to be an orthonormal basis, i.e. unit vectors spanning all of $\mathbb{R}^n$.)

(c) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two $\mathbb{R}^{n\times n}$ symmetric matrices. Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ have the exact same set of eigenvectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n$ with the corresponding eigenvalues $\alpha_1, \alpha_2, \cdots, \alpha_n$ for $\boldsymbol{A}$, and $\beta_1, \beta_2, \cdots, \beta_n$ for $\boldsymbol{B}$. Please write down the eigenvectors and their corresponding eigenvalues for the following matrices:

(i) $\boldsymbol{D} = \boldsymbol{A} - \boldsymbol{B}$

(ii) $\boldsymbol{E} = \boldsymbol{A}\boldsymbol{B}$

(iii) $\boldsymbol{F} = \boldsymbol{A}^{-1}\boldsymbol{B}$ (assume $\boldsymbol{A}$ is invertible)

3. Positive Semi-Definite Matrices

A symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is *positive-semidefinite (PSD)* if $x^T \boldsymbol{A} x \geq 0$ for all $x \in \mathbb{R}^n$.

(a) For any $y \in \mathbb{R}^n$, show that yy^T is PSD.

(b) Let X be a random vector in $\mathbb{R}^n$ with covariance matrix $\boldsymbol{\Sigma} = \mathbb{E}[(X - \mathbb{E}[X])(X - \mathbb{E}[X])^{\mathrm{T}}]$. Show that $\boldsymbol{\Sigma}$ is PSD.

(c) Assume $\boldsymbol{A}$ is a symmetric matrix so that $\boldsymbol{A} = \boldsymbol{U}\mathrm{diag}(\alpha)\boldsymbol{U}^T$ where $\mathrm{diag}(\alpha)$ is an all zeros matrix with the entries of α on the diagonal and $\boldsymbol{U}^T\boldsymbol{U} = I$. Show that $\boldsymbol{A}$ is PSD if and only if $\min_i \alpha_i \geq 0$. (Hint: compute $x^T \boldsymbol{A} x$ and consider values of x proportional to the columns of $\boldsymbol{U}$, i.e., the orthonormal eigenvectors).

(d) Show that a real symmetrix matrix is PSD if and only if all of its eigenvalues are non-negative.