

# CSE446: Decision Trees

## Spring 2017

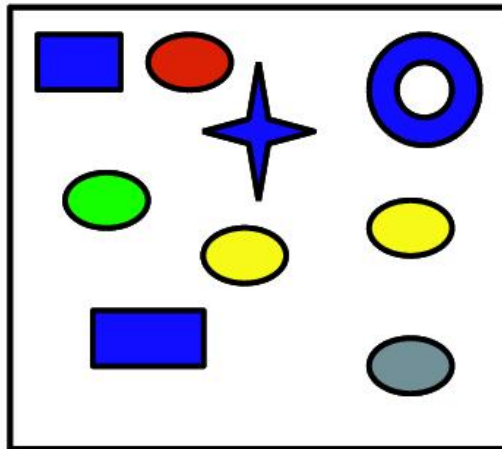
Ali Farhadi

Slides adapted from Carlos Guestrin, Andrew Moore, and Luke Zettlemoyer

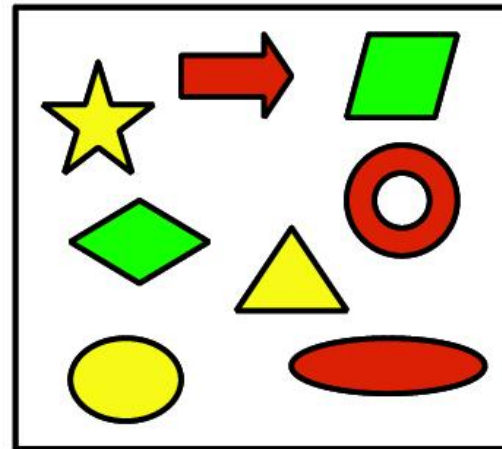
# Administrative stuff

- Office hours
- Discussion board
- Anonymous feedback form
- Contact: **[cse446-staff@cs.washington.edu](mailto:cse446-staff@cs.washington.edu)**
- No Quiz sections
- Check the webpage regularly

yes




no




# A learning problem: predict fuel efficiency

- 40 Records
- Discrete data (for now)
- Predict MPG
- Need to find:  
 $f : X \rightarrow Y$

mpg	cylinders	displacement	horsepower	weight	acceleration	modelyear	maker
good	4	low	low	low	high	75to78	asia
bad	6	medium	medium	medium	medium	70to74	america
bad	4	medium	medium	medium	low	75to78	europa
bad	8	high	high	high	low	70to74	america
bad	6	medium	medium	medium	medium	70to74	america
bad	4	low	medium	low	medium	70to74	asia
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good	4	low	medium	low	medium	75to78	europa
bad	5	medium	medium	medium	medium	75to78	europa



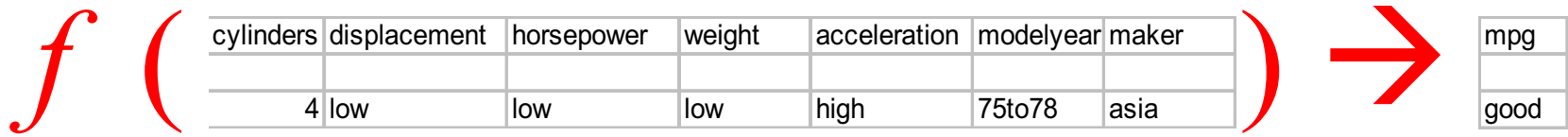
$Y$



$X$

From the UCI repository (thanks to Ross Quinlan)

# How to Represent our Function?



## Conjunctions in Propositional Logic?

maker=asia  $\wedge$  weight=low

Need to find “Hypothesis”:  $f : X \rightarrow Y$

# Restricted Hypothesis Space

- Many possible representations
- Natural choice: **conjunction** of attribute constraints
- For each attribute:
  - Constrain to a specific value: eg **maker=asia**
  - Don't care: ?
- For example

*maker cyl displace weight accel ...*  
*asia ? ? low ?*

Represents **maker=asia**  $\wedge$  **weight=low**

# Consistency

- Say an “example is consistent with a hypothesis” when the example *logically satisfies* the hypothesis

- Hypothesis: **maker=asia**  $\wedge$  **weight=low**

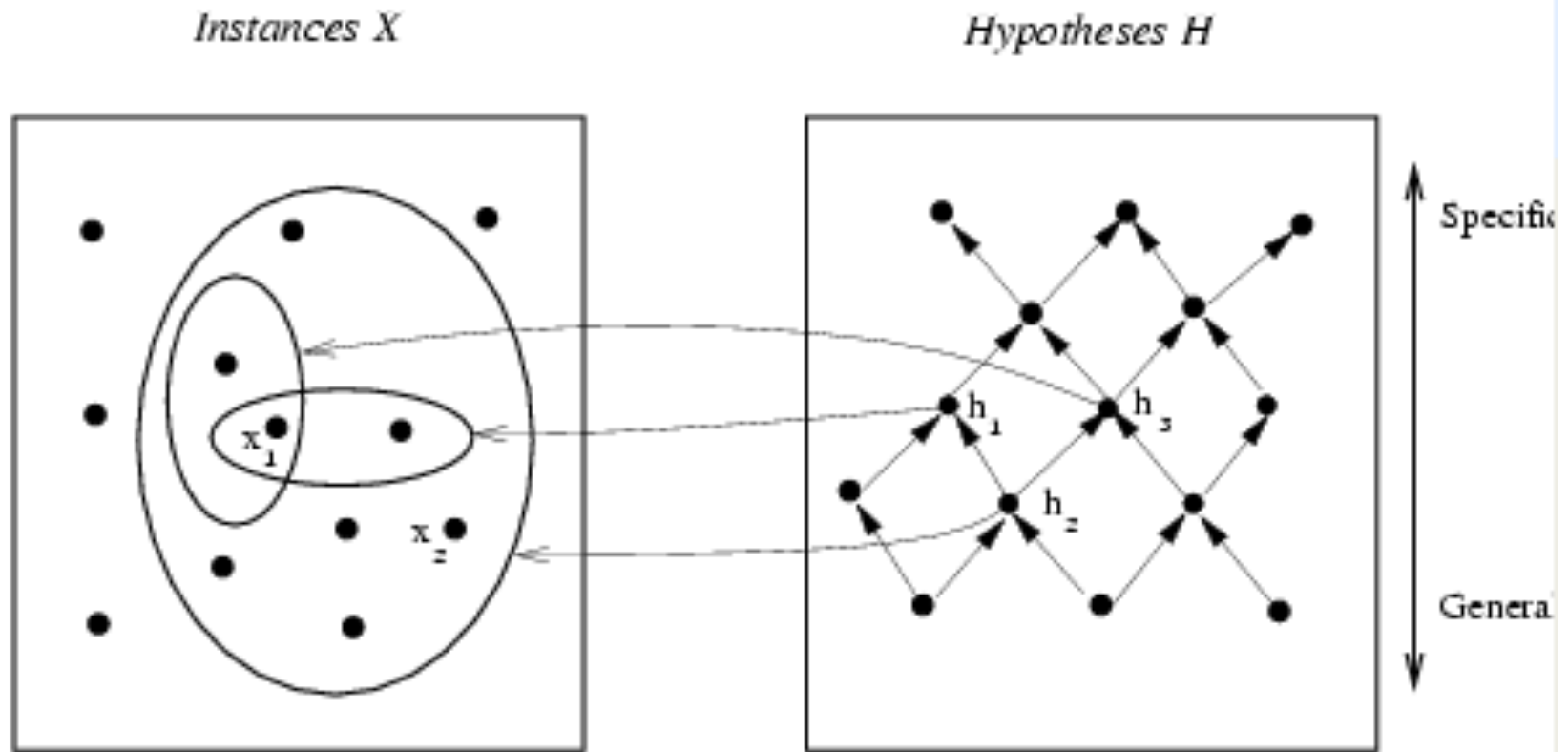
maker cyl displace weight accel ...

asia ? ? low ?

- Examples:

asia	5	low	low	low	...
usa	4	low	low	low	...

# Ordering on Hypothesis Space



$x_1$	asia	5	low	low	low
$x_2$	usa	4	med	med	med

**$h_1$ : maker=asia  $\wedge$  accel=low**

**$h_2$ : maker=asia**

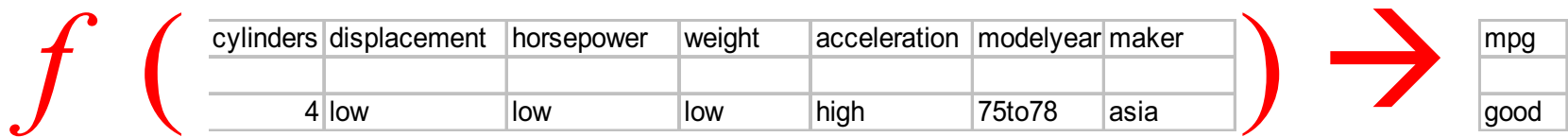
**$h_3$ : maker=asia  $\wedge$  weight=low**



# Version Space Algorithm

Ok, so how does it perform?

# How to Represent our Function?



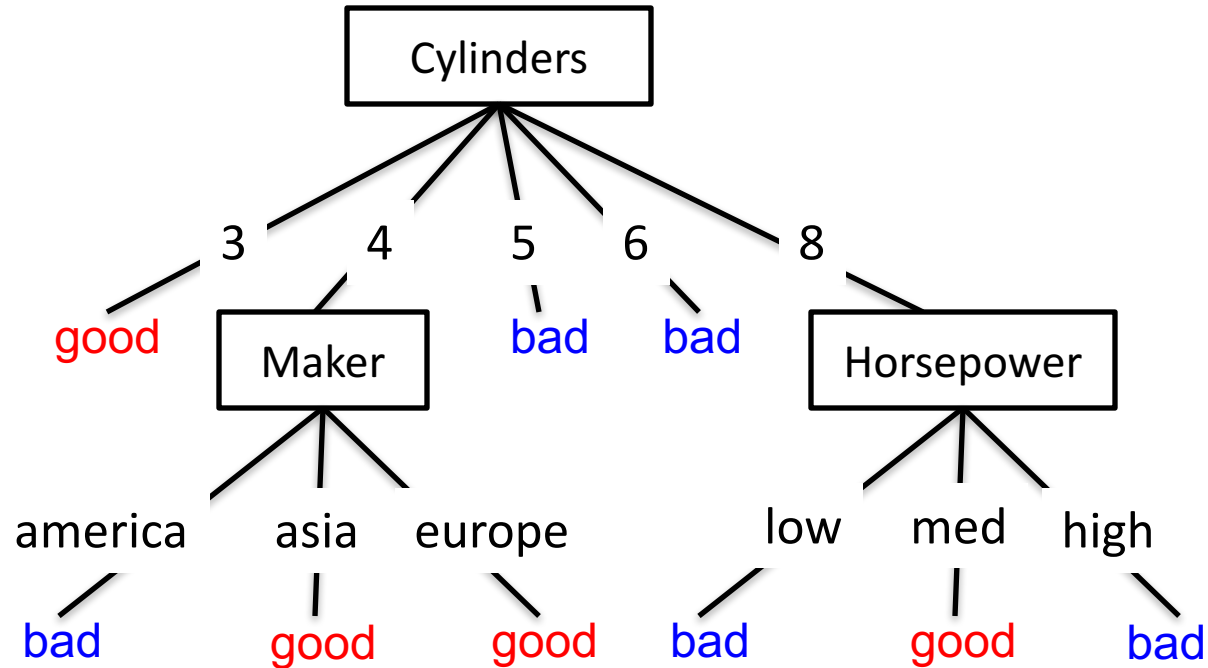
## General Propositional Logic?

maker=asia  $\vee$  weight=low

Need to find “Hypothesis”:  $f : X \rightarrow Y$

# Hypotheses: decision trees $f : X \rightarrow Y$

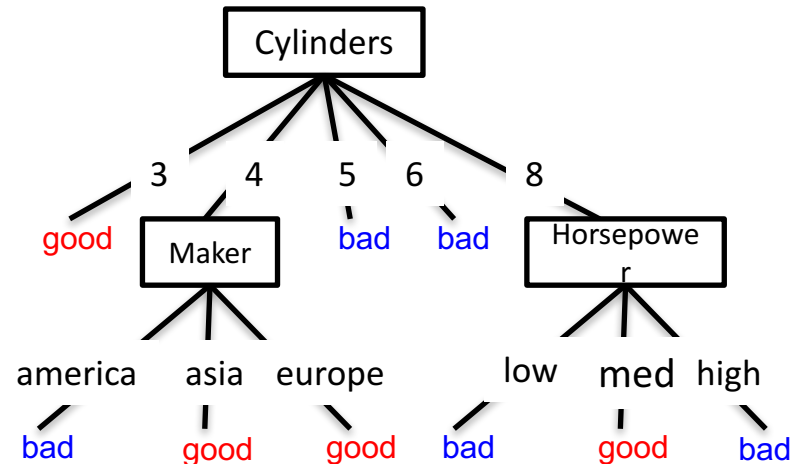
- Each internal node tests an attribute  $x_i$
- Each branch assigns an attribute value  $x_i=v$
- Each leaf assigns a class  $y$
- To classify input  $x$ : traverse the tree from root to leaf, output the labeled  $y$



# Hypothesis space

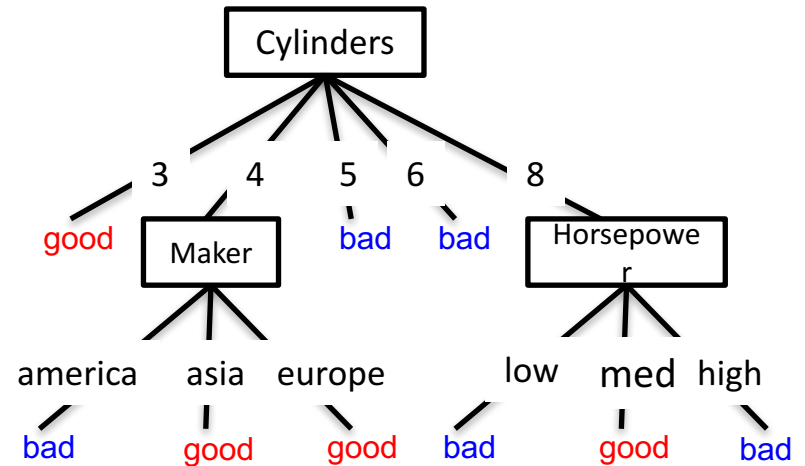
- How many possible hypotheses?
- What functions can be represented?

mpg	cylinders	displacement	horsepower	weight	acceleration	modelyear	maker
good	4	low	low	low	high	75to78	asia
bad	6	medium	medium	medium	medium	70to74	america
bad	4	medium	medium	medium	low	75to78	europa
bad	8	high	high	high	low	70to74	america
bad	6	medium	medium	medium	medium	70to74	america
bad	4	low	medium	low	medium	70to74	asia
bad	4	low	medium	low	low	70to74	asia
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bad	6	medium	medium	medium	high	75to78	america
good	4	medium	low	low	low	79to83	america
good	4	low	low	medium	high	79to83	america
bad	8	high	high	high	low	70to74	america
good	4	low	medium	low	medium	75to78	europa
bad	5	medium	medium	medium	medium	75to78	europa



# What functions can be represented?

- Decision trees can represent any boolean function!
- But, could require exponentially many nodes...

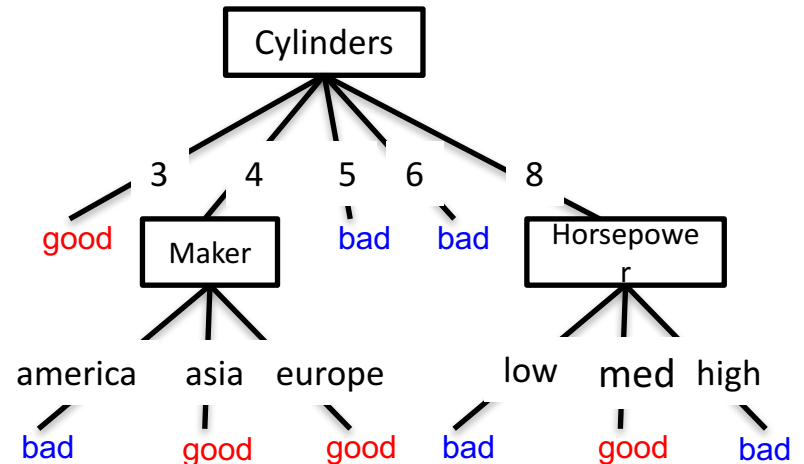


$\text{cyl}=3 \vee (\text{cyl}=4 \wedge (\text{maker}=\text{asia} \vee \text{maker}=\text{europe})) \vee \dots$

# Hypothesis space

- How many possible hypotheses?
- What functions can be represented?
- How many will be consistent with a given dataset?
- How will we choose the best one?
  - Lets first look at how to split nodes, then consider how to find the best tree

mpg	cylinders	displacement	horsepower	weight	acceleration	modelyear	maker
good	4	low	low	low	high	75to78	asia
bad	6	medium	medium	medium	medium	70to74	america
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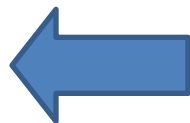
# What is the Simplest Tree?

predict  
mpg=bad

mpg	cylinders	displacement	horsepower	weight	acceleration	modelyear	maker
good	4	low	low	low	high	75to78	asia
bad	6	medium	medium	medium	medium	70to74	america
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bad	8	high	high	high	low	70to74	america
bad	6	medium	medium	medium	medium	70to74	america
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bad	5	medium	medium	medium	medium	75to78	europa

## Is this a good tree?

[22+, 18-]

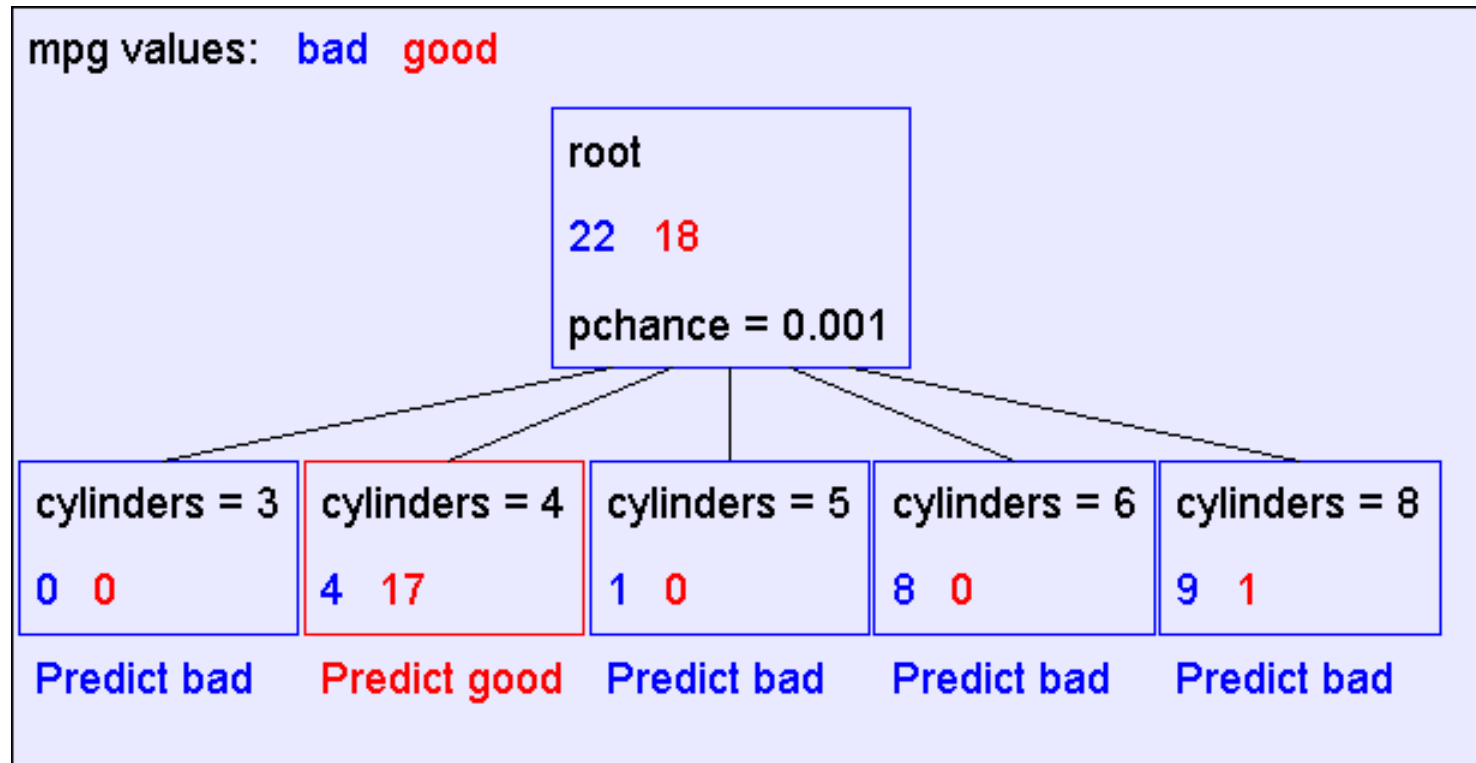


Means:

correct on 22 examples

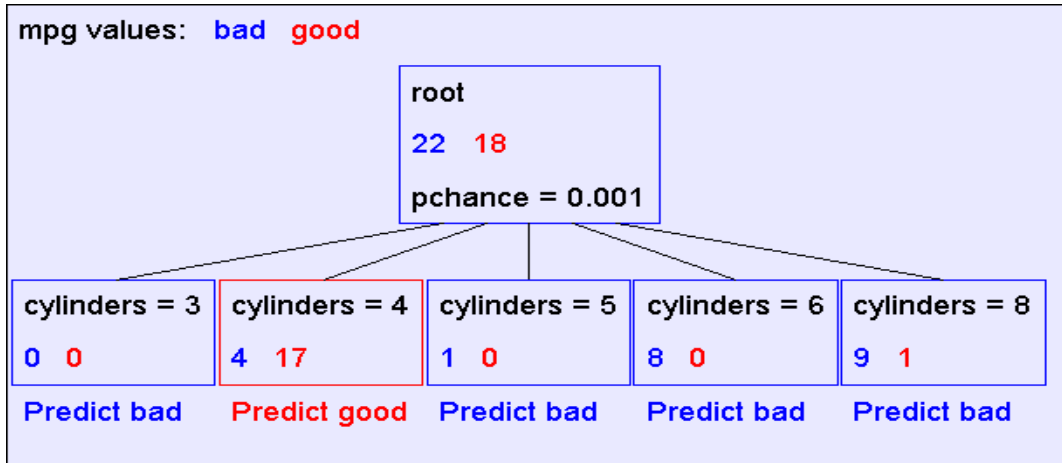
incorrect on 18 examples

# A Decision Stump





# Recursive Step



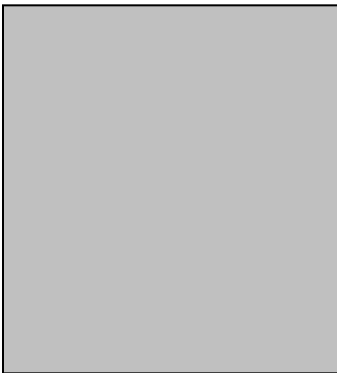
Records in which cylinders = 4

Records in which cylinders = 5

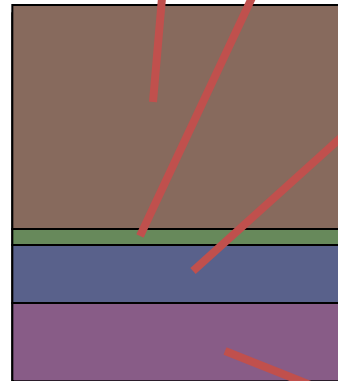
Records in which cylinders = 6

Records in which cylinders = 8

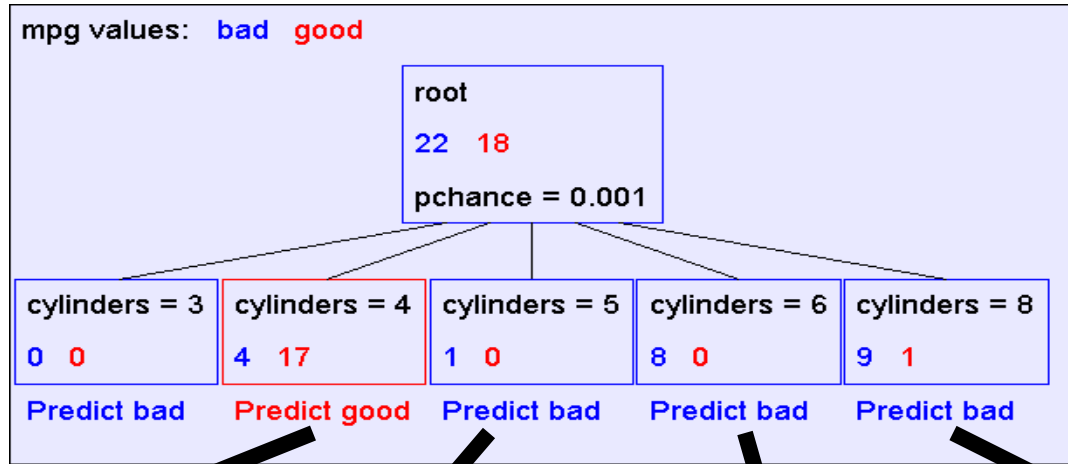
Take the Original Dataset..



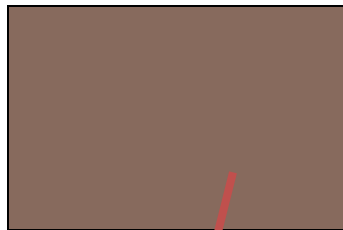
And partition it according to the value of the attribute we split on



# Recursive Step



Build tree from  
These records..



Records in  
which cylinders  
= 4

Build tree from  
These records..



Records in  
which cylinders  
= 5

Build tree from  
These records..



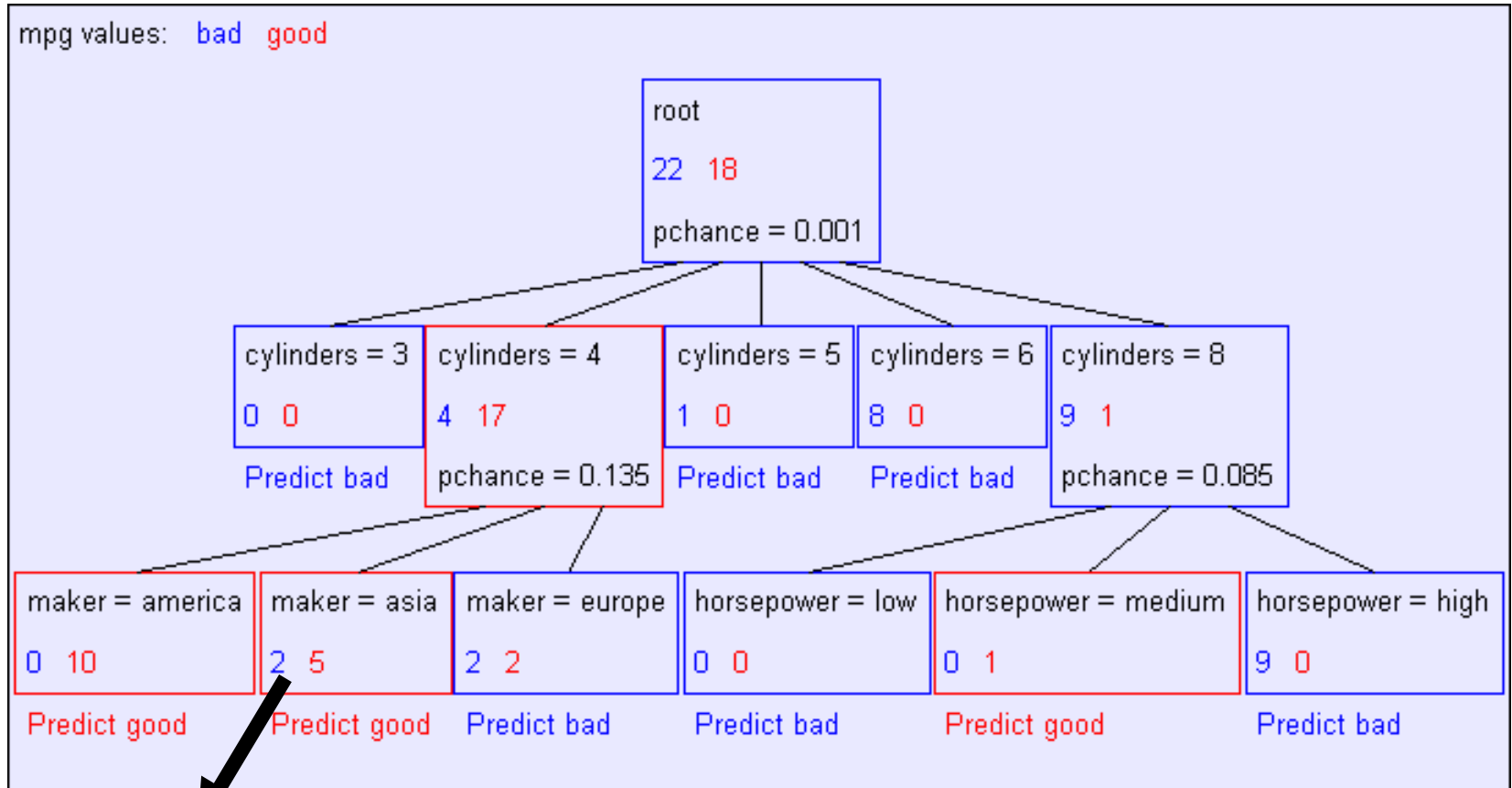
Records in  
which cylinders  
= 6

Build tree from  
These records..



Records in  
which cylinders  
= 8

# Second level of tree

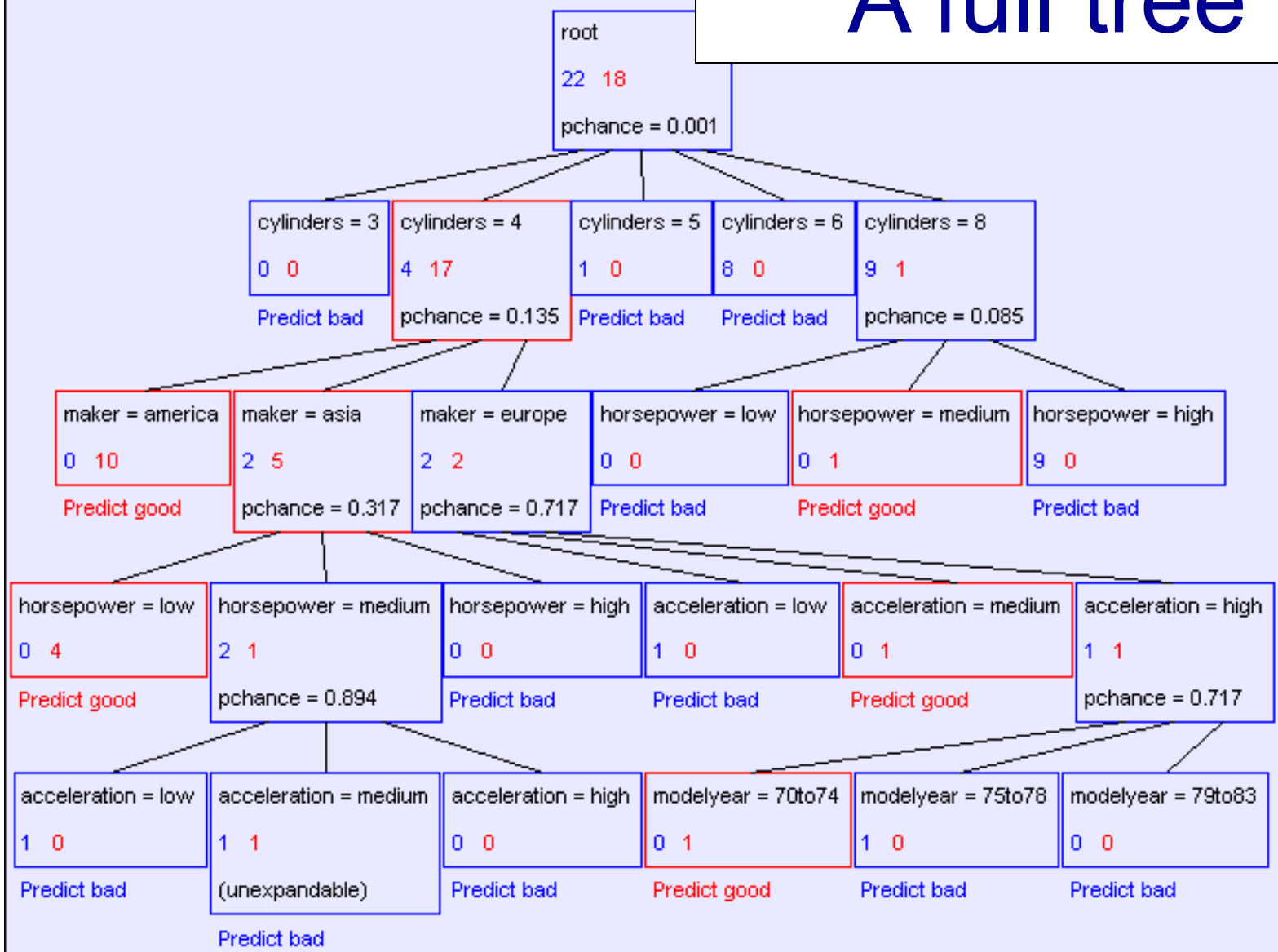


Recursively build a tree from the seven records in which there are four cylinders and the maker was based in Asia

(Similar recursion in the other cases)

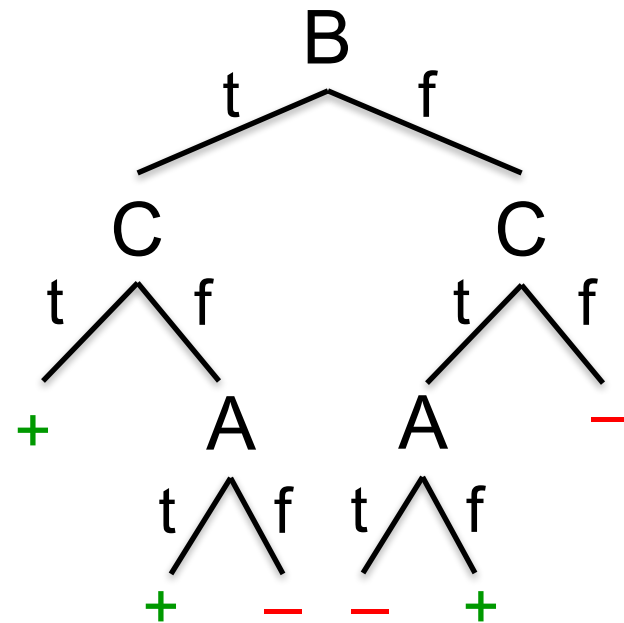
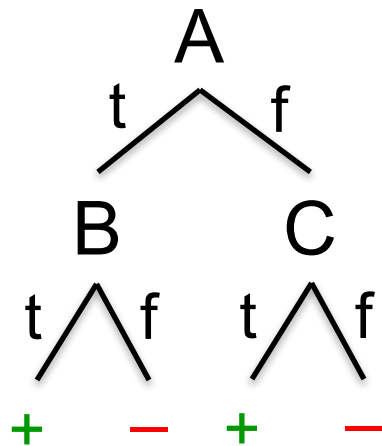
# A full tree

mpg values: bad good



# Are all decision trees equal?

- Many trees can represent the same concept
- But, not all trees will have the same size!
  - e.g.,  $\phi = (A \wedge B) \vee (\neg A \wedge C) \dashv\vdash ((A \text{ and } B) \text{ or } (\text{not } A \text{ and } C))$



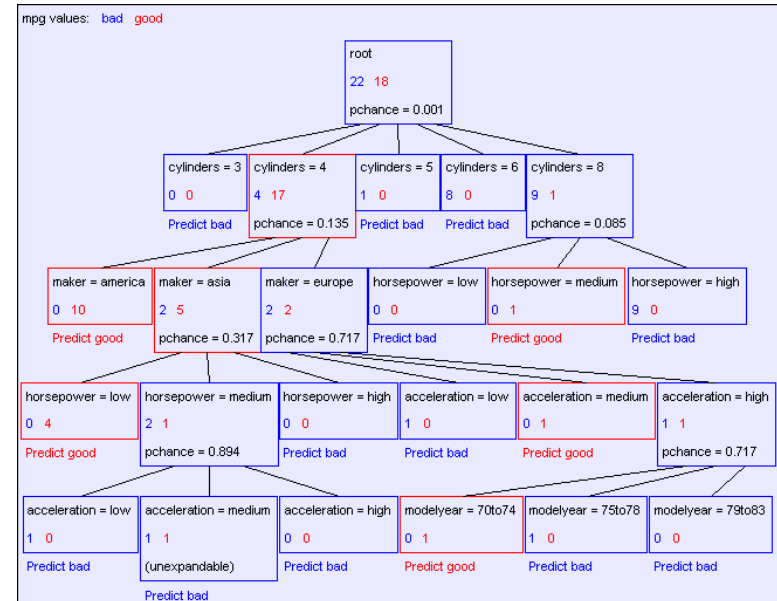
- Which tree do we prefer?
  - Smaller tree has more examples at each leaf!

# Learning decision trees is hard!!!

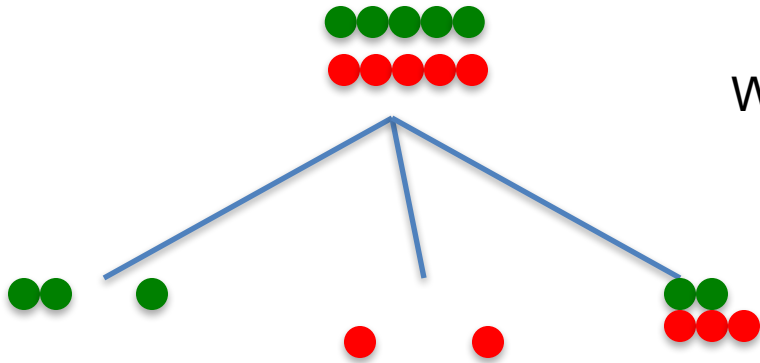
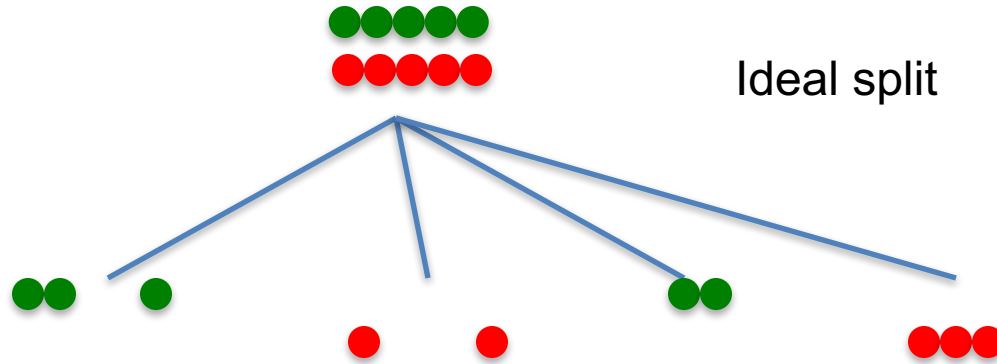
- Learning the simplest (smallest) decision tree is an NP-complete problem [Hyafil & Rivest '76]
- Resort to a greedy heuristic:
  - Start from empty decision tree
  - Split on **next best attribute (feature)**
  - Recurse

# So far ...

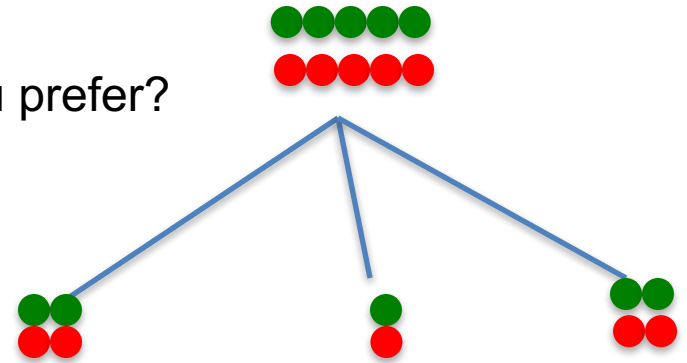
- Decision trees
- They will overfit
- How to split?
- When to stop?



# What defines a good attribute?



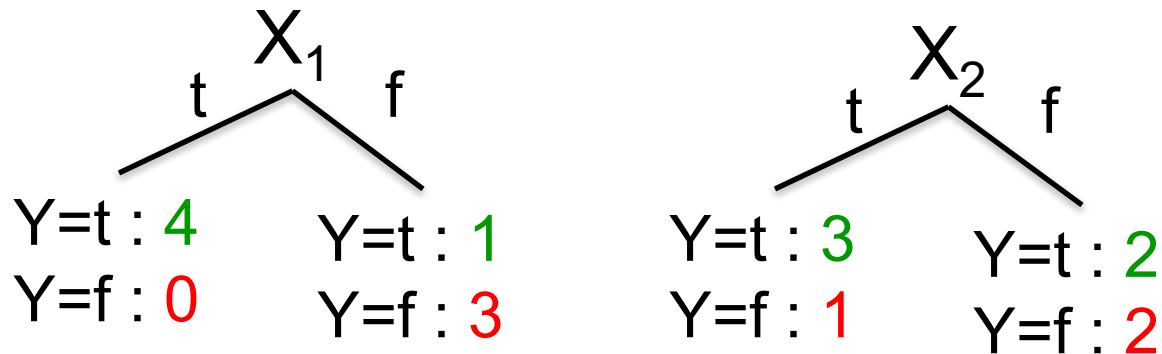
Which one do you prefer?





# Splitting: choosing a good attribute

Would we prefer to split on  $X_1$  or  $X_2$ ?



**Idea:** use counts at leaves to define probability distributions, so we can measure uncertainty!

$X_1$	$X_2$	Y
T	T	T
T	F	T
T	T	T
T	F	T
F	T	T
F	F	F
F	T	F
F	F	F

# Measuring uncertainty

- Good split if we are more certain about classification after split
  - Deterministic good (all true or all false)
  - Uniform distribution bad
  - What about distributions in between?

$P(Y=A) = 1/2$	$P(Y=B) = 1/4$	$P(Y=C) = 1/8$	$P(Y=D) = 1/8$
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$P(Y=A) = 1/4$	$P(Y=B) = 1/4$	$P(Y=C) = 1/4$	$P(Y=D) = 1/4$
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# Entropy

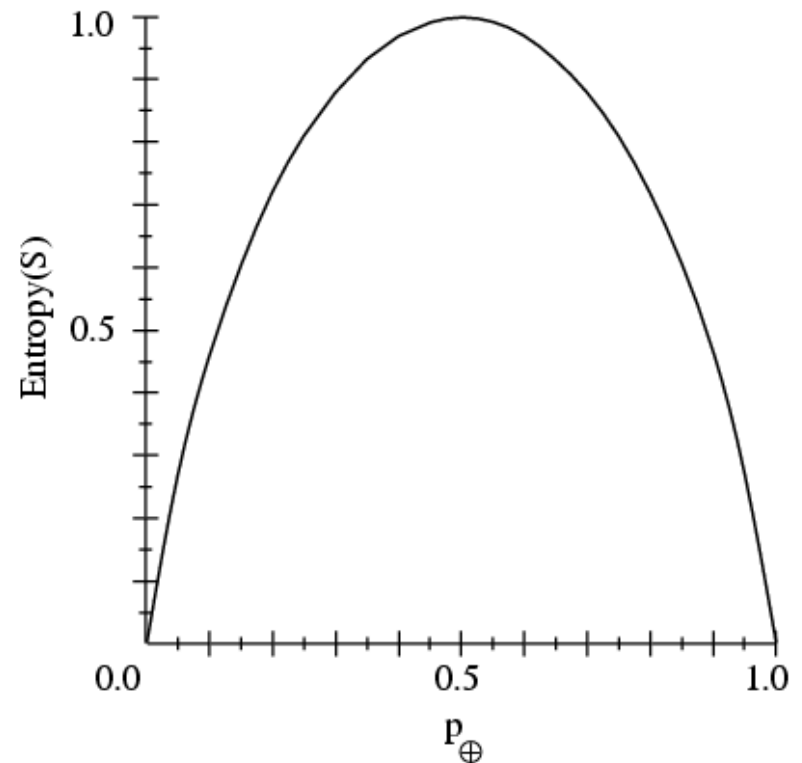
Entropy  $H(Y)$  of a random variable  $Y$

$$H(Y) = - \sum_{i=1}^k P(Y = y_i) \log_2 P(Y = y_i)$$

**More uncertainty, more entropy!**

*Information Theory interpretation:*

$H(Y)$  is the expected number of bits needed to encode a randomly drawn value of  $Y$  (under most efficient code)



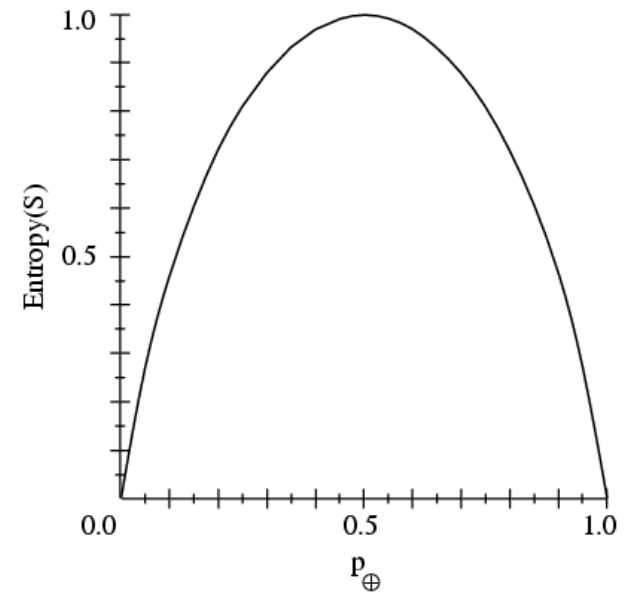
# Entropy Example

$$H(Y) = - \sum_{i=1}^k P(Y = y_i) \log_2 P(Y = y_i)$$

$$P(Y=t) = 5/6$$

$$P(Y=f) = 1/6$$

$$\begin{aligned} H(Y) &= - 5/6 \log_2 5/6 - 1/6 \log_2 1/6 \\ &= 0.65 \end{aligned}$$



$X_1$	$X_2$	Y
T	T	T
T	F	T
T	T	T
T	F	T
F	T	T
F	F	F

# Conditional Entropy

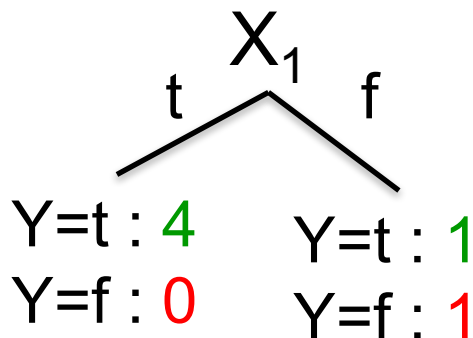
Conditional Entropy  $H(Y|X)$  of a random variable  $Y$  conditioned on a random variable  $X$

$$H(Y|X) = - \sum_{j=1}^v P(X = x_j) \sum_{i=1}^k P(Y = y_i | X = x_j) \log_2 P(Y = y_i | X = x_j)$$

Example:

$$P(X_1=t) = 4/6$$

$$P(X_1=f) = 2/6$$



$$\begin{aligned} H(Y|X_1) &= - 4/6 (1 \log_2 1 + 0 \log_2 0) \\ &\quad - 2/6 (1/2 \log_2 1/2 + 1/2 \log_2 1/2) \\ &= 2/6 \end{aligned}$$

$X_1$	$X_2$	Y
T	T	T
T	F	T
T	T	T
T	F	T
F	T	T
F	F	F

# Information gain

Decrease in entropy (uncertainty) after splitting

$$IG(X) = H(Y) - H(Y | X)$$

- $IG(X)$  is non-negative ( $\geq 0$ )
- Prove by showing  $H(Y|X) \leq H(X)$ , with Jensen's inequality

In our running example:

$$\begin{aligned} IG(X_1) &= H(Y) - H(Y|X_1) \\ &= 0.65 - 0.33 \end{aligned}$$

$IG(X_1) > 0 \rightarrow$  we prefer the split!

$X_1$	$X_2$	Y
T	T	T
T	F	T
T	T	T
T	F	T
F	T	T
F	F	F

# Learning decision trees

- Start from empty decision tree
- Split on **next best attribute (feature)**
  - Use, for example, information gain to select attribute:

$$\arg \max_i IG(X_i) = \arg \max_i H(Y) - H(Y | X_i)$$

- Recurse

Suppose we want to predict MPG

Look at all the information gains...

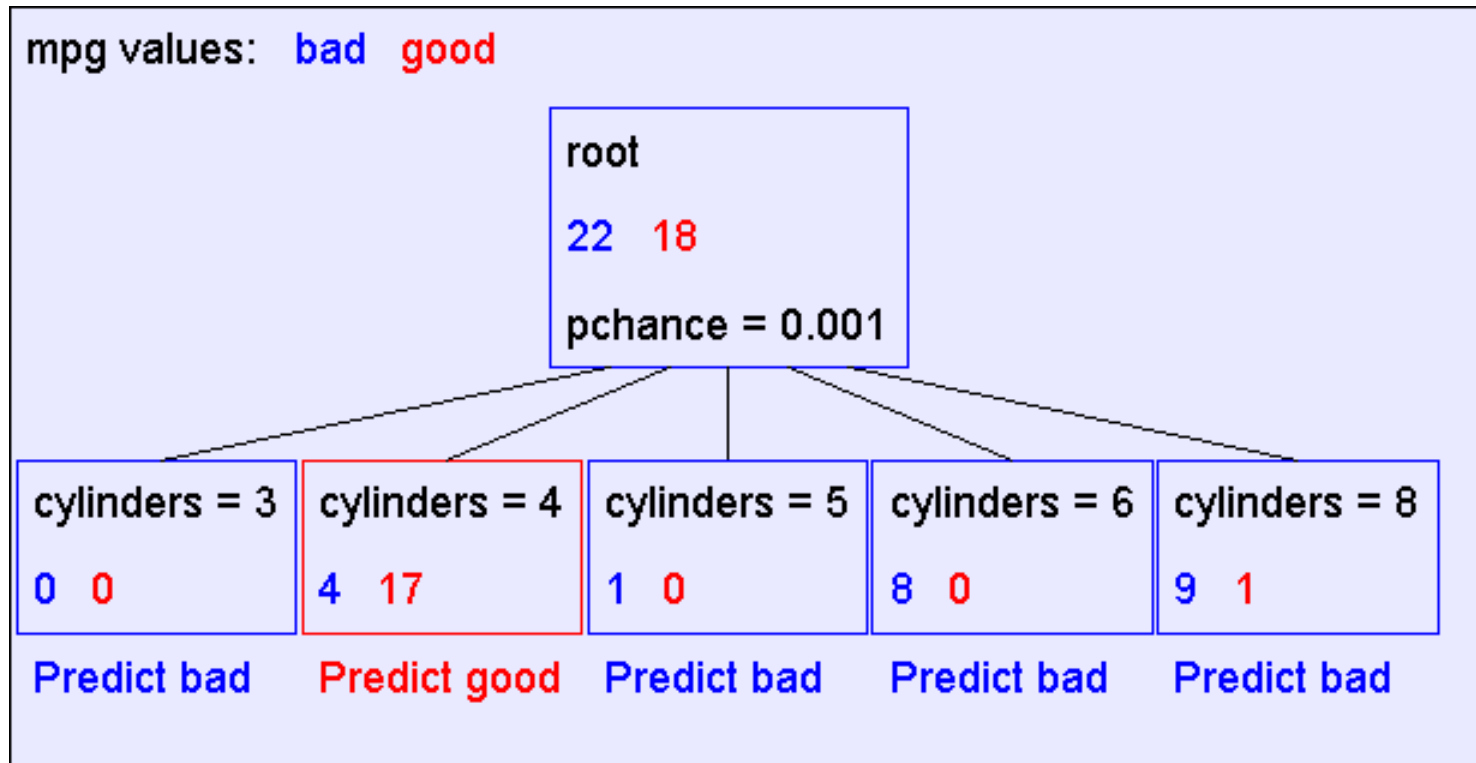
Information gains using the training set (40 records)

mpg values: bad good

Input	Value	Distribution	Info Gain
cylinders	3		0.506731
	4		
	5		
	6		
	8		
displacement	low		0.223144
	medium		
	high		
horsepower	low		0.387605
	medium		
	high		
weight	low		0.304018
	medium		
	high		
acceleration	low		0.0642088
	medium		
	high		
modelyear	70to74		0.267964



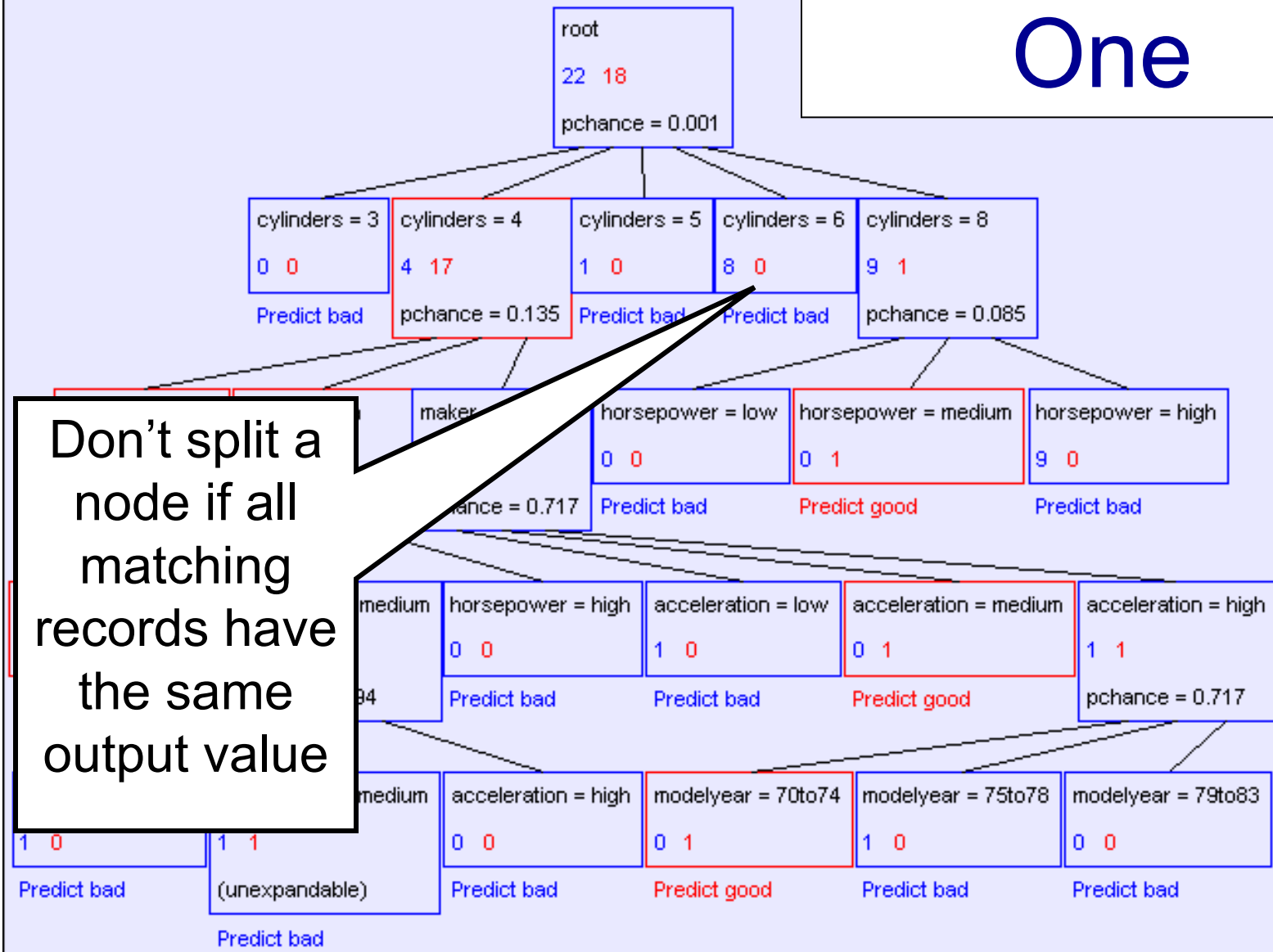
# A Decision Stump



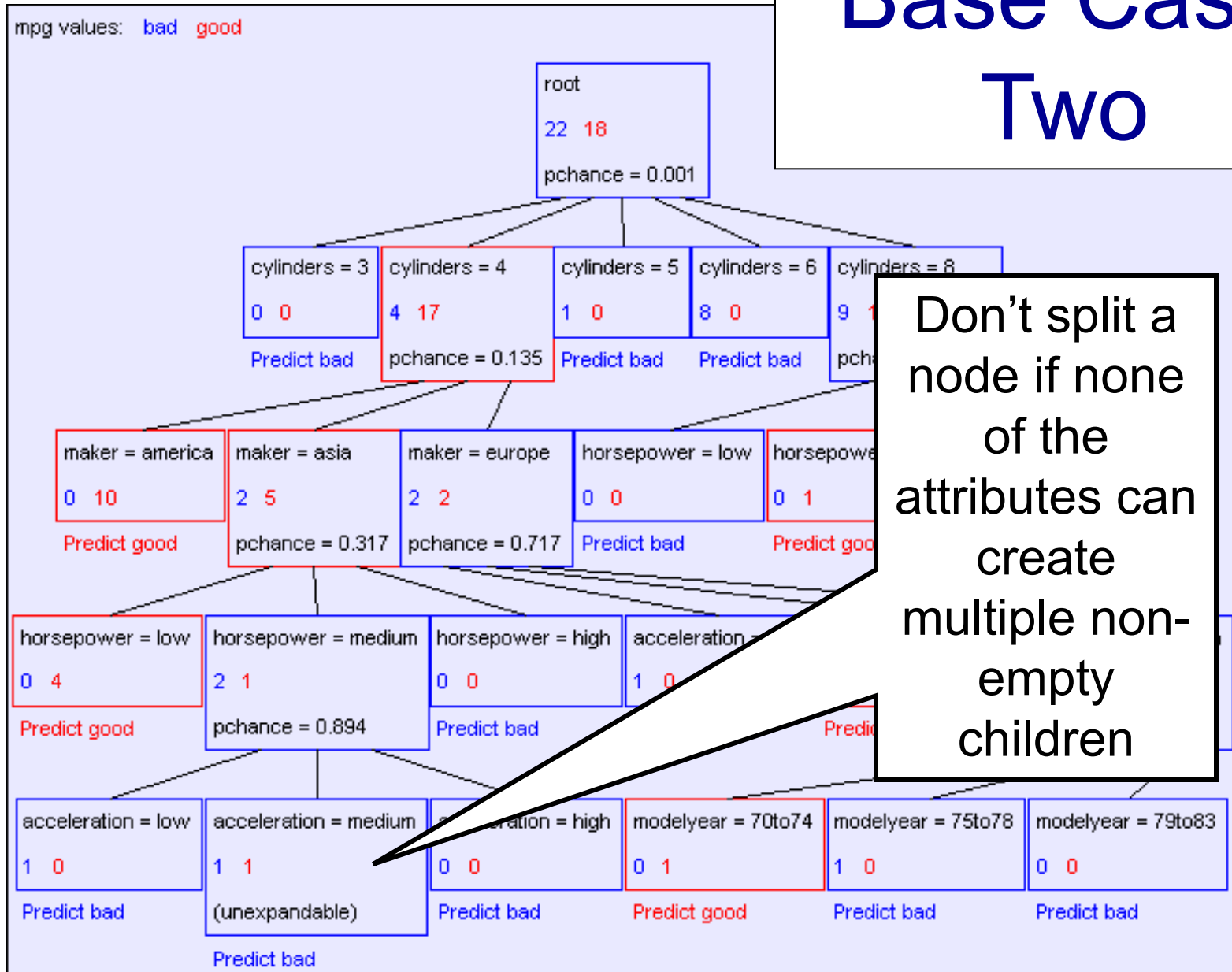
First split looks good! But, when do we stop?

# Base Case One

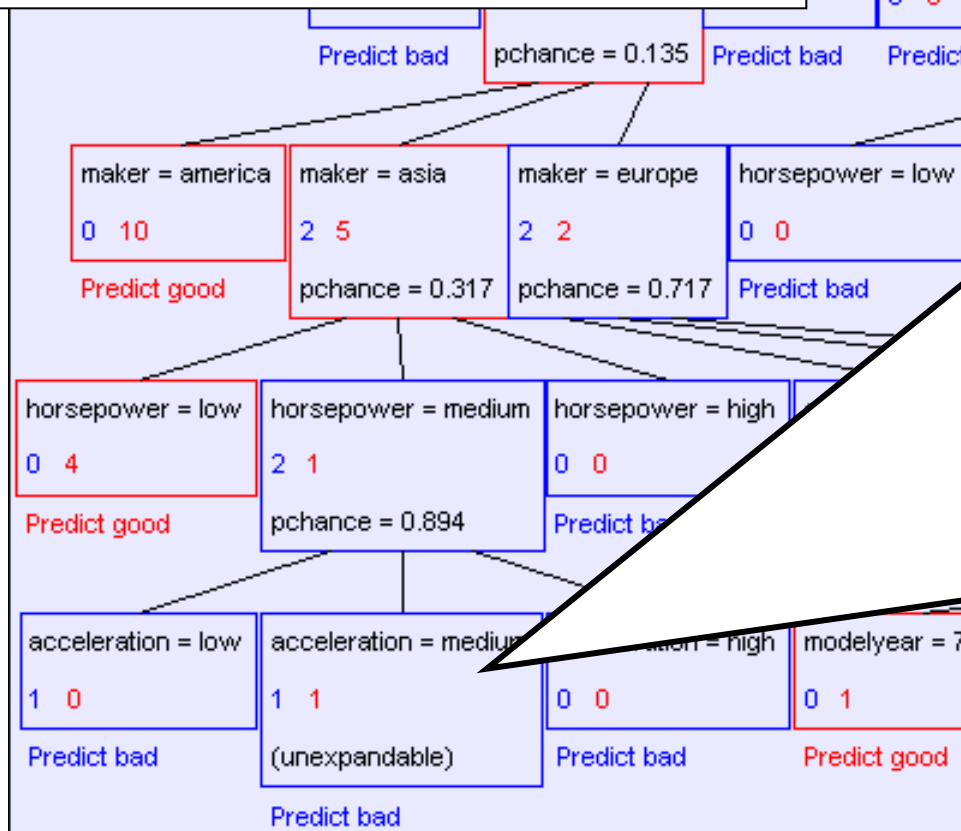
mpg values: bad good



# Base Case Two



# Base Case Two: No attributes can distinguish



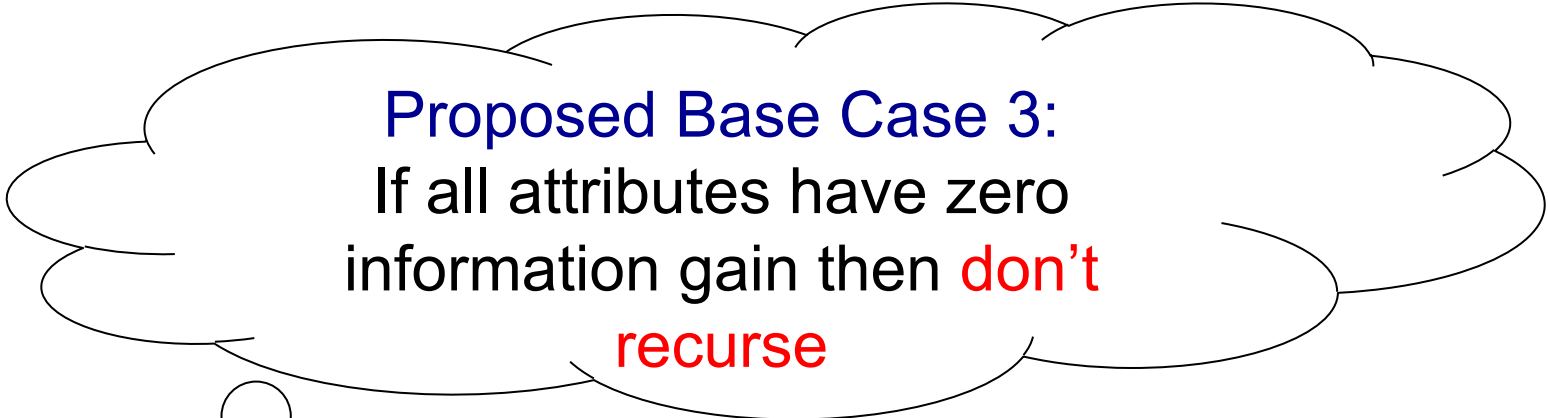
Information gains using the training set (2 records)

mpg values: bad good

Input	Value	Distribution	Info Gain
cylinders	3		0
	4		
	5		
	6		
displacement	low		0
	medium		
	high		
horsepower	low		0
	medium		
	high		
weight	low		0
	medium		
	high		
acceleration	low		0
	medium		
	high		
modelyear	70to74		0
	75to78		
	79to83		
maker	america		0
	asia		
	europe		

# Base Cases: An idea

- **Base Case One:** If all records in current data subset have the same output then **don't recurse**
- **Base Case Two:** If all records have exactly the same set of input attributes then **don't recurse**



**Proposed Base Case 3:**  
If all attributes have zero  
information gain then **don't  
recurse**

• *Is this a good idea?*





# The problem with Base Case 3

$$y = a \text{ XOR } b$$

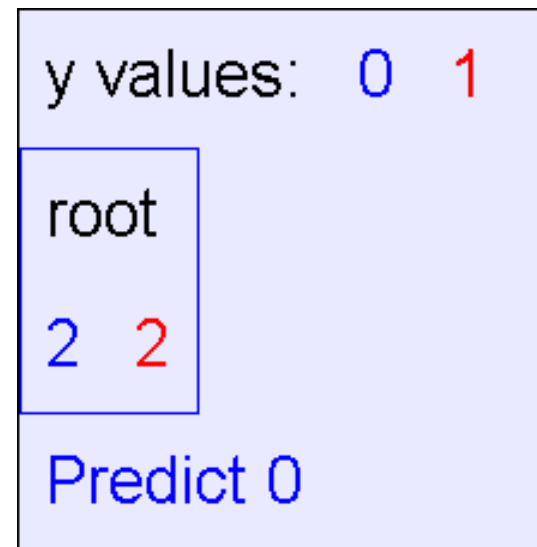
a	b	y
0	0	0
0	1	1
1	0	1
1	1	0

The information gains:

Information gains using the training set (4 records)  
y values: 0 1

Input	Value	Distribution	Info Gain
a	0		0
	1		0
b	0		0
	1		0

The resulting decision tree:



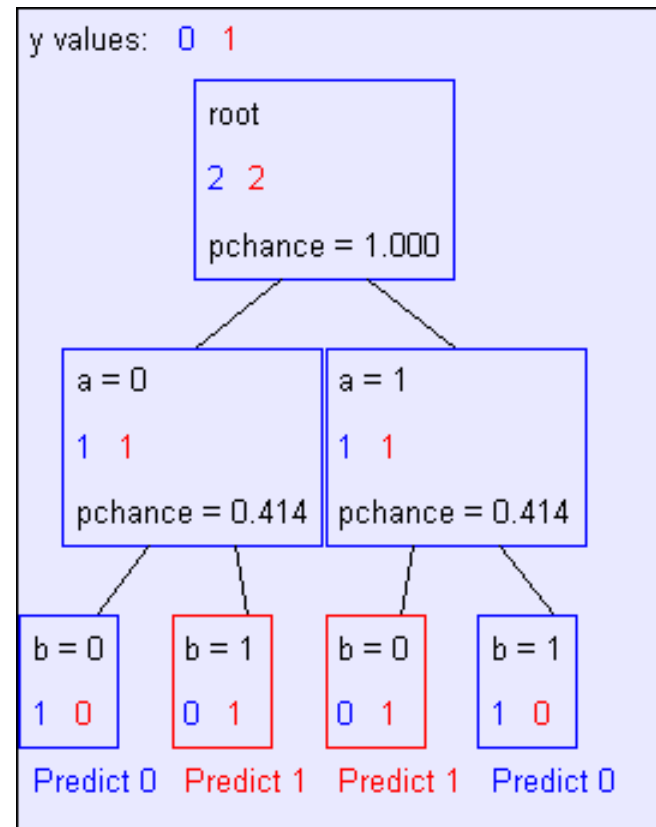
# If we omit Base Case 3:

$$y = a \text{ XOR } b$$

a	b	y
0	0	0
0	1	1
1	0	1
1	1	0

Is it OK to omit Base Case 3?

The resulting decision tree:



# Summary: Building Decision Trees

$\text{BuildTree}(\text{DataSet}, \text{Output})$

- If all output values are the same in  $\text{DataSet}$ , return a leaf node that says “predict this unique output”
- If all input values are the same, return a leaf node that says “predict the majority output”
- Else find attribute  $X$  with highest Info Gain
- Suppose  $X$  has  $n_X$  distinct values (i.e.  $X$  has arity  $n_X$ ).
  - Create a non-leaf node with  $n_X$  children.
  - The  $i$ 'th child should be built by calling

$\text{BuildTree}(\text{DS}_i, \text{Output})$

Where  $\text{DS}_i$  contains the records in  $\text{DataSet}$  where  $X = i$ th value of  $X$ .



# MPG Test set error

mpg values: bad good

root  
22 18  
pchance = 0.001

	Num Errors	Set Size	Percent Wrong
Training Set	1	40	2.50
Test Set	74	352	21.02

horsepower = high

Predict bad

horsepower = low   horsepower = medium   horsepower = high   acceleration = low   acceleration = medium   acceleration = high

0 4

0 4

0 0

4 0

0 4

4 4

= 0.717

= 79to83

The test set error is much worse than the training set error...

...why?

Predict bad

(unexpandable)

Predict bad

Predict good

Predict bad

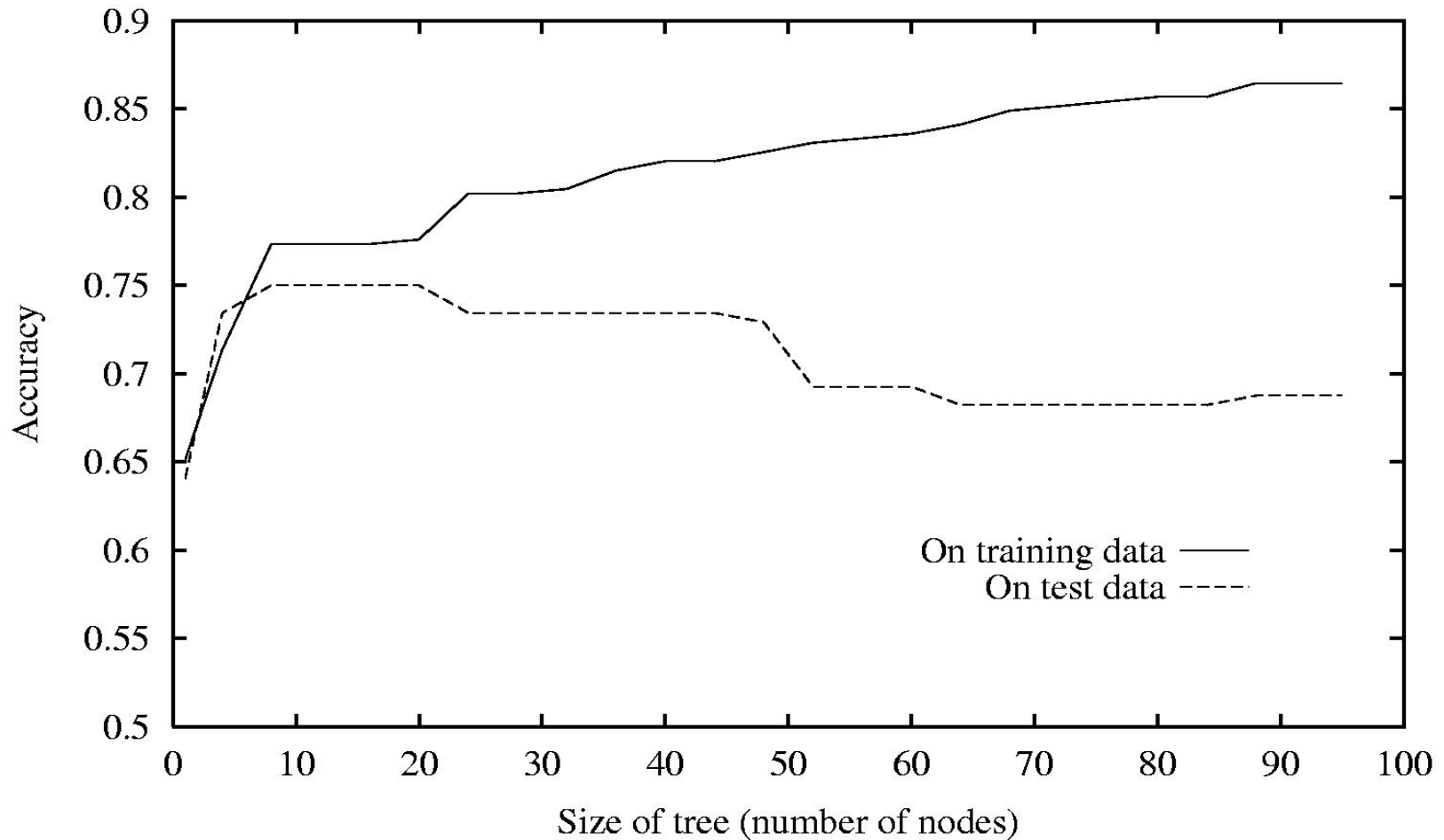
Predict bad

Predict bad

# Decision trees will overfit!!!

- Standard decision trees have no learning bias
  - Training set error is always zero!
    - (If there is no label noise)
  - Lots of variance
  - Must introduce some bias towards simpler trees
- Many strategies for picking simpler trees
  - Fixed depth
  - Fixed number of leaves
  - Or something smarter...

# Decision trees will overfit!!!



# One Definition of Overfitting

- Assume:
  - Data generated from distribution  $D(X, Y)$
  - A hypothesis space  $H$
- Define errors for hypothesis  $h \in H$ 
  - Training error:  $error_{train}(h)$
  - Data (true) error:  $error_D(h)$
- We say  $h$  **overfits** the training data if there exists an  $h' \in H$  such that:

$$error_{train}(h) < error_{train}(h')$$

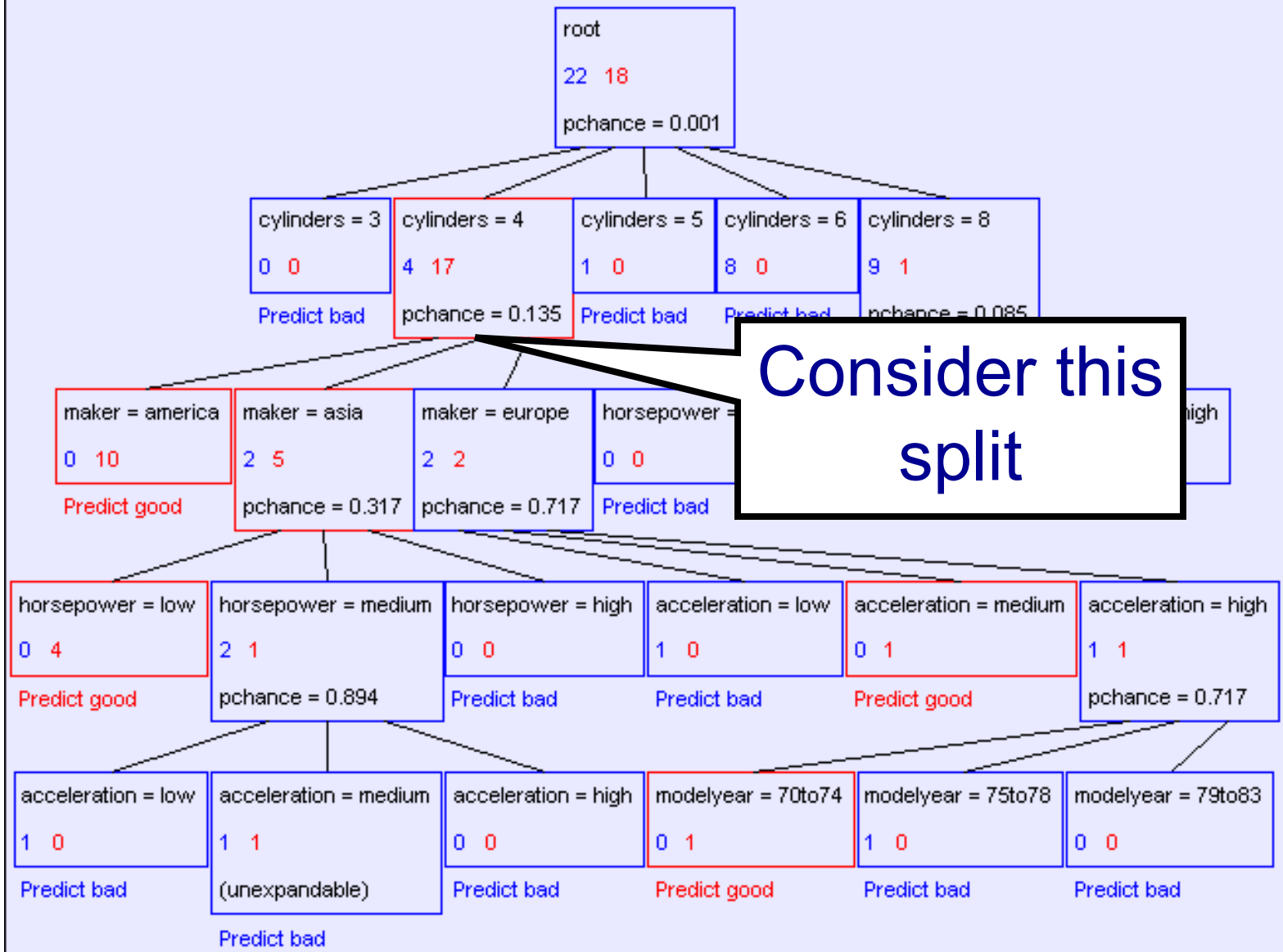
and

$$error_D(h) > error_D(h')$$

# Occam's Razor

- Why Favor Short Hypotheses?
- Arguments for:
  - Fewer short hypotheses than long ones
    - A short hyp. less likely to fit data by coincidence
    - Longer hyp. that fit data may might be coincidence
- Arguments against:
  - Argument above really uses the fact that hypothesis space is small!!!
  - What is so special about small sets based on the size of each hypothesis?

mpg values: bad good






# How to Build Small Trees

Two reasonable approaches:

- **Optimize on the held-out (development) set**
  - If growing the tree larger hurts performance, then stop growing!!!
  - Requires a larger amount of data...
- **Use statistical significance testing**
  - Test if the improvement for any split is likely due to noise
  - If so, don't do the split!

# A Chi Square Test

mpg values:		bad	good		
maker	america	0	10		$H(\text{mpg} \mid \text{maker} = \text{america}) = 0$
	asia	2	5		$H(\text{mpg} \mid \text{maker} = \text{asia}) = 0.863121$
	europa	2	2		$H(\text{mpg} \mid \text{maker} = \text{europa}) = 1$
$H(\text{mpg}) = 0.702467$		$H(\text{mpg} \mid \text{maker}) = 0.478183$			
		$IG(\text{mpg} \mid \text{maker}) = 0.224284$			

- Suppose that mpg was completely uncorrelated with maker.
- What is the chance we'd have seen data of at least this apparent level of association anyway?

By using a particular kind of chi-square test, the answer is 13.5%

We will not cover Chi Square tests in class. See page 93 of the original ID3 paper [Quinlan, 86].



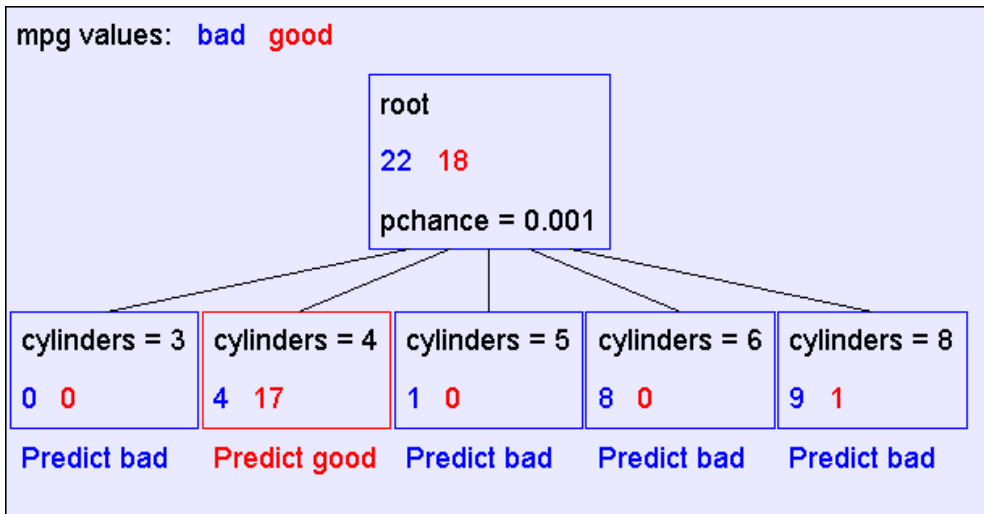
# Using Chi-squared to avoid overfitting

- Build the full decision tree as before
- But when you can grow it no more, start to prune:
  - Beginning at the bottom of the tree, delete splits in which  $p_{chance} > MaxPchance$
  - Continue working your way up until there are no more prunable nodes

*MaxPchance* is a magic parameter you must specify to the decision tree, indicating your willingness to risk fitting noise

# Pruning example

- With  $\text{MaxPchance} = 0.05$ , you will see the following MPG decision tree:



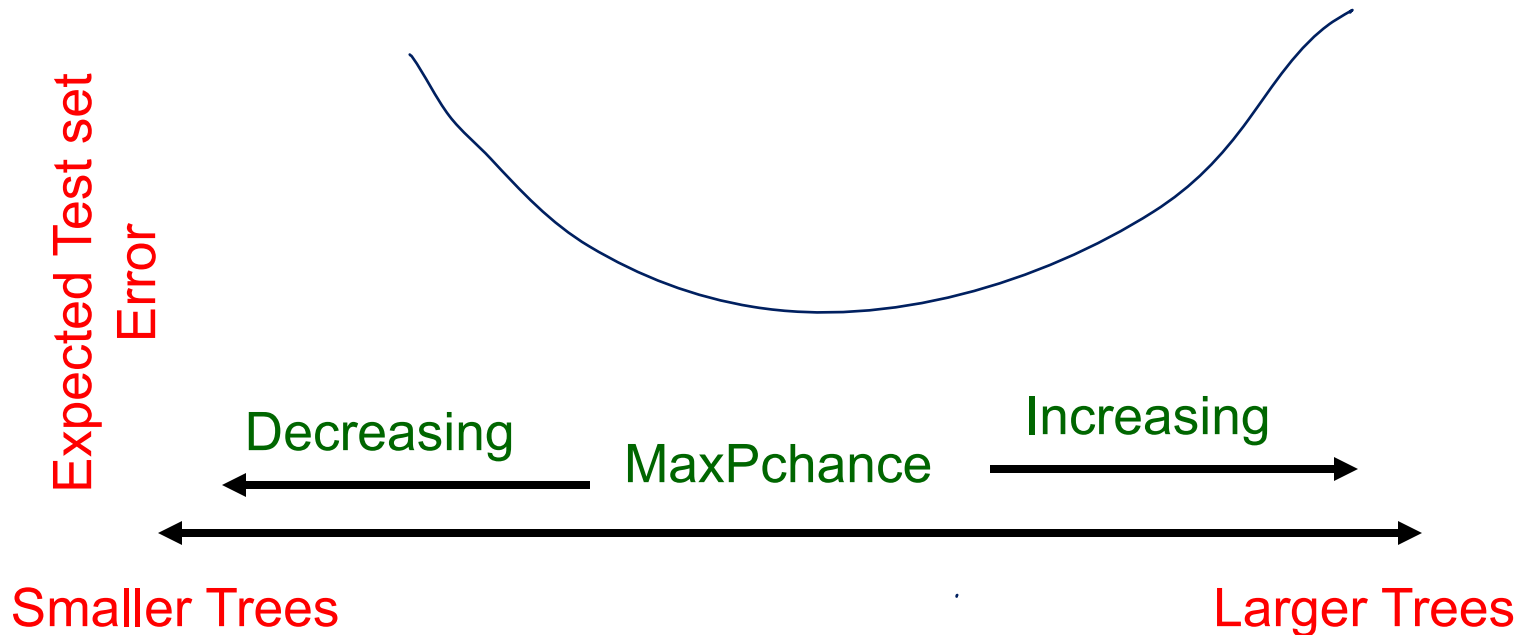
When compared to the unpruned tree

- improved test set accuracy
- worse training accuracy

	Num Errors	Set Size	Percent Wrong
Training Set	5	40	12.50
Test Set	56	352	15.91

# MaxPchance

- Technical note: MaxPchance is a regularization parameter that helps us bias towards simpler models



We'll learn to choose the value of magic parameters like this one later!

# Real-Valued inputs

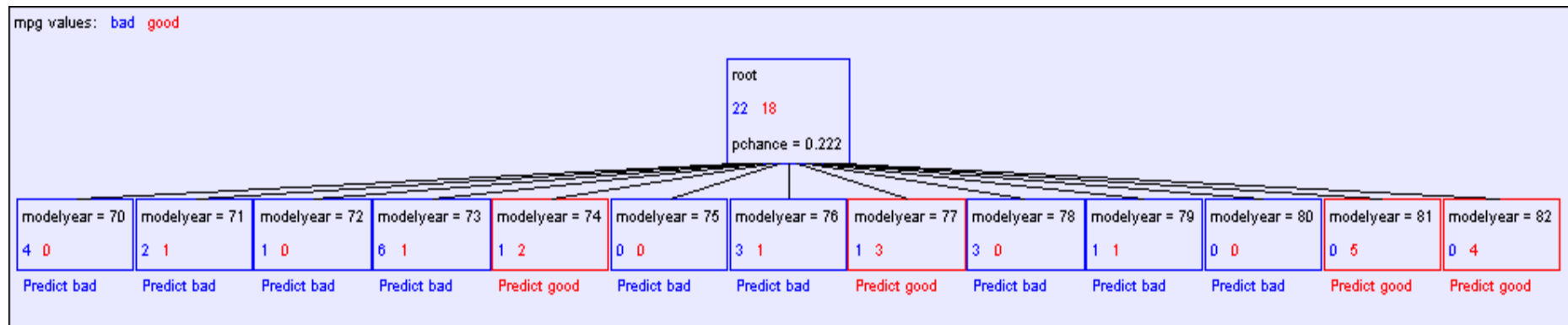
What should we do if some of the inputs are real-valued?

mpg	cylinders	displacemen	horsepower	weight	acceleration	modelyear	maker
good	4	97	75	2265	18.2	77	asia
bad	6	199	90	2648	15	70	america
bad	4	121	110	2600	12.8	77	europa
bad	8	350	175	4100	13	73	america
bad	6	198	95	3102	16.5	74	america
bad	4	108	94	2379	16.5	73	asia
bad	4	113	95	2228	14	71	asia
bad	8	302	139	3570	12.8	78	america
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
good	4	120	79	2625	18.6	82	america
bad	8	455	225	4425	10	70	america
good	4	107	86	2464	15.5	76	europa
bad	5	131	103	2830	15.9	78	europa

Infinite  
number of  
possible split  
values!!!

Finite  
dataset, only  
finite number  
of relevant  
splits!

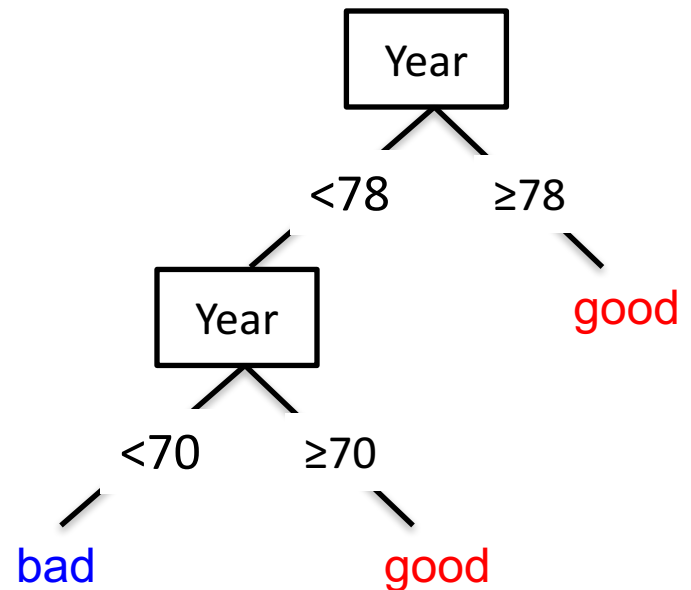
“One branch for each numeric value”  
idea:



**Hopeless:** with such high branching factor will shatter the dataset and overfit

# Threshold splits

- **Binary tree:** split on attribute  $X$  at value  $t$ 
  - One branch:  $X < t$
  - Other branch:  $X \geq t$
- **Requires small change**
  - Allow repeated splits on same variable
  - How does this compare to “branch on each value” approach?



# The set of possible thresholds

- Binary tree, split on attribute  $X$ 
  - One branch:  $X < t$
  - Other branch:  $X \geq t$
- Search through possible values of  $t$ 
  - Seems hard!!!
- But only finite number of  $t$ 's are important
  - Sort data according to  $X$  into  $\{x_1, \dots, x_m\}$
  - Consider split points of the form  $x_i + (x_{i+1} - x_i)/2$

# Picking the best threshold
















- Suppose  $X$  is real valued with threshold  $t$
- Want  $IG(Y|X:t)$ : the information gain for  $Y$  when testing if  $X$  is greater than or less than  $t$
- Define:
  - $H(Y|X:t) =$   
 $H(Y|X < t) P(X < t) + H(Y|X \geq t) P(X \geq t)$
  - $IG(Y|X:t) = H(Y) - H(Y|X:t)$
  - $IG^*(Y|X) = \max_t IG(Y|X:t)$
- Use:  $IG^*(Y|X)$  for continuous variables



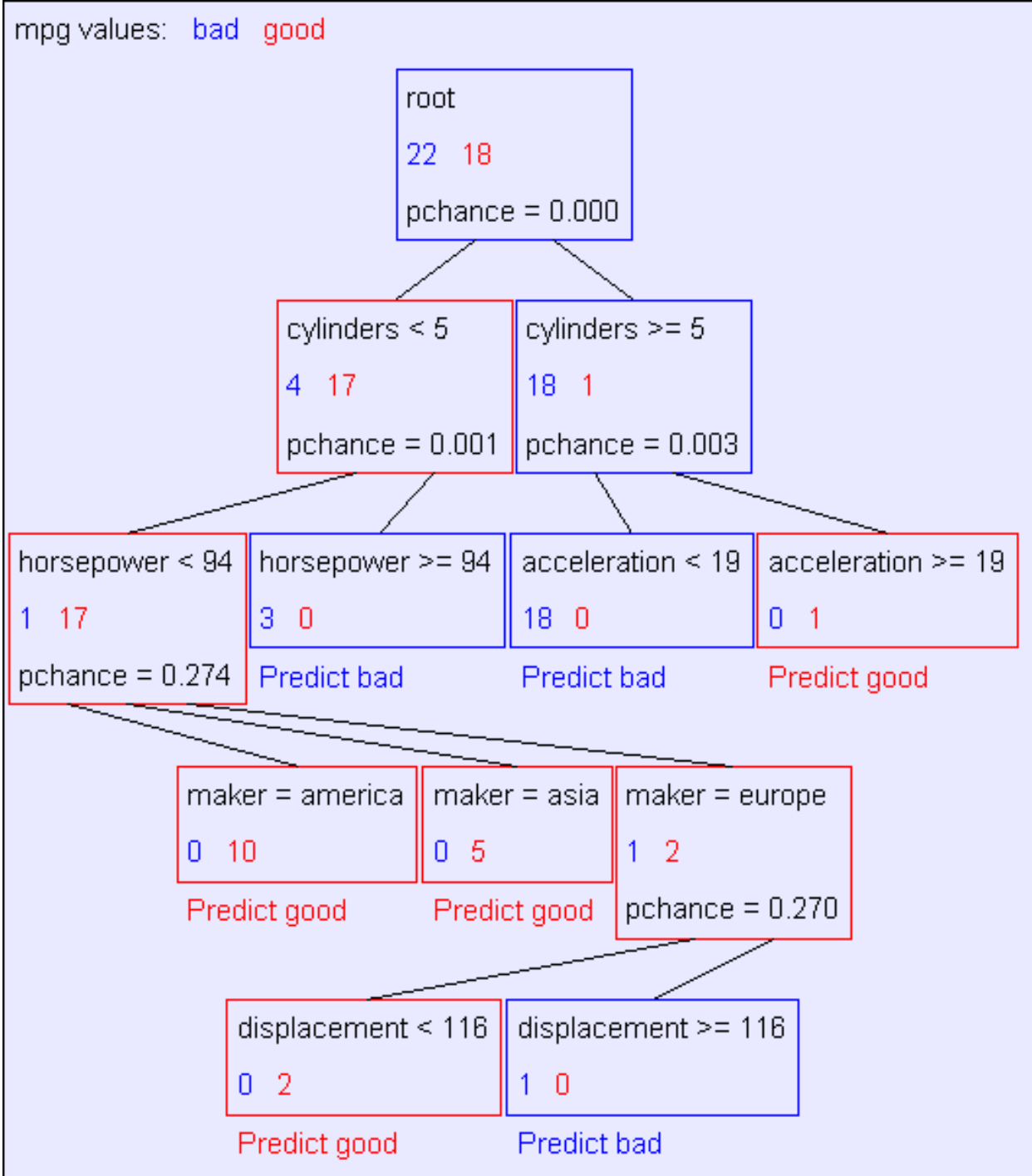
# Example with MPG

Information gains using the training set (40 records)

mpg values: bad good

Input	Value	Distribution	Info Gain
cylinders	< 5		0.48268
	>= 5		
displacement	< 198		0.428205
	>= 198		
horsepower	< 94		0.48268
	>= 94		
weight	< 2789		0.379471
	>= 2789		
acceleration	< 18.2		0.159982
	>= 18.2		
modelyear	< 81		0.319193
	>= 81		
maker	america		0.0437265
	asia		
	europe		

# Example tree for our continuous dataset



# What you need to know about decision trees

- Decision trees are one of the most popular ML tools
  - Easy to understand, implement, and use
  - Computationally cheap (to solve heuristically)
- Information gain to select attributes (ID3, C4.5,...)
- Presented for classification, can be used for regression and density estimation too
- Decision trees will overfit!!!
  - Must use tricks to find “simple trees”, e.g.,
    - Fixed depth/Early stopping
    - Pruning
    - Hypothesis testing

# Acknowledgements

- Some of the material in the decision trees presentation is courtesy of Andrew Moore, from his excellent collection of ML tutorials:
  - <http://www.cs.cmu.edu/~awm/tutorials>