Machine Learning (CSE 446): Ensembles (continued)

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Data: $D = \langle (x_n, y_n) \rangle_{n=1}^{N}$, number of epochs E, weighted learner $\mathcal{W}$
Result: classifier
$\boldsymbol{\beta}^{(0)} = \langle \frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N} \rangle$; # initialize example weights
for $e \in \{1, \ldots, E\}$ **do**
    $f^{(e)} \leftarrow \mathcal{W}(D, \boldsymbol{\beta}^{(e-1)})$; # train the classifier on the weighted data
    $\hat{\epsilon}^{(e)} \leftarrow \sum_{n=1}^{N} \beta_n^{(e-1)} \cdot [\![f^{(e)}(x_n) \neq y_n]\!]$; # weighted error rate
    $\alpha^{(e)} \leftarrow \frac{1}{2} \log \left(\frac{1-\hat{\epsilon}^{(e)}}{\hat{\epsilon}^{(e)}} \right)$; # "adaptive" weight for $f^{(e)}$
    for $n \in \{1, \ldots, N\}$ **do**
        $\beta_n^{(e)} \leftarrow \frac{1}{Z^{(e)}} \cdot \beta_n^{(e-1)} \cdot \exp\left(-\alpha^{(e)} \cdot y_n \cdot f^{(e)}(x_n)\right)$; # update example weights
        ($Z^{(e)}$ is a normalization constant)
    end
end
return $f_{\text{boost}}(\cdot) = \text{sign}\left(\sum_{e=1}^{E} \alpha^{(e)} \cdot f^{(e)}(\cdot) \right)$;

Algorithm 1: AdaBoost

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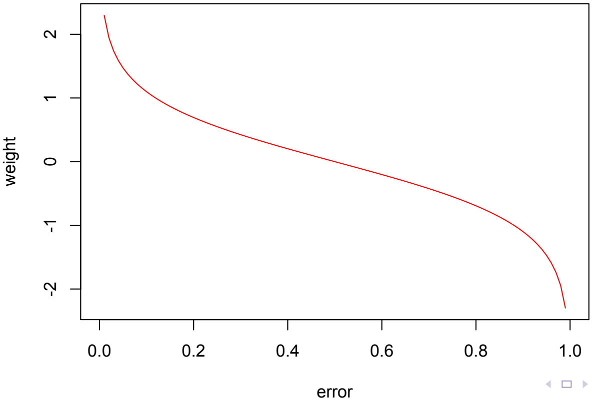

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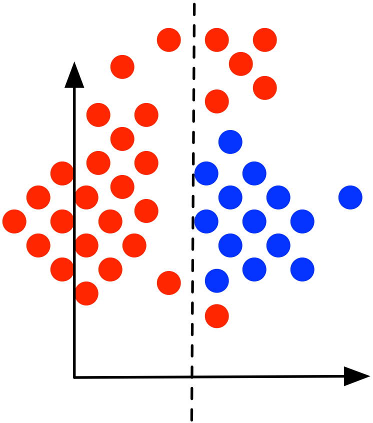
- For examples we get right ($f^{(e)}(x_n) = y_n$), the weight β_n will *decrease*; we increase the weights of examples we get wrong.
  - See the book for more insight on what happens on the first epoch with a *very* simple $\mathcal{W}$. Each successive $f^{(e)}$ is intended to work harder wherever previous classifiers have been failing (hence, "adaptive").

Boosting Example



(This is a contrived example; it may take many more iterations to achieve $\hat{\epsilon} = 0$.)

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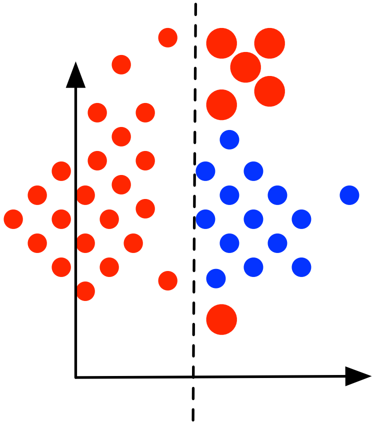

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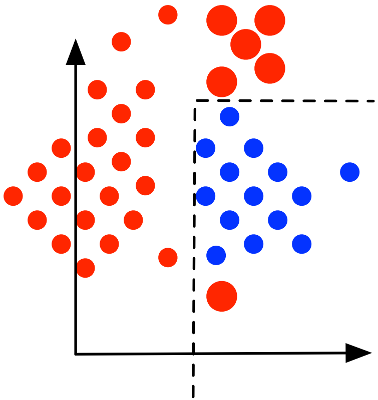

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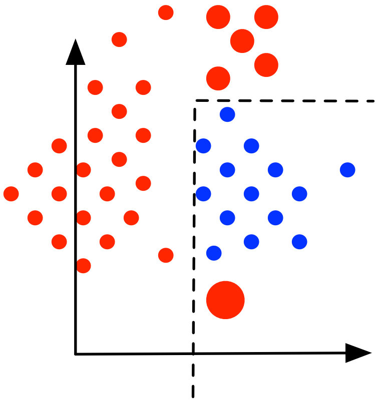

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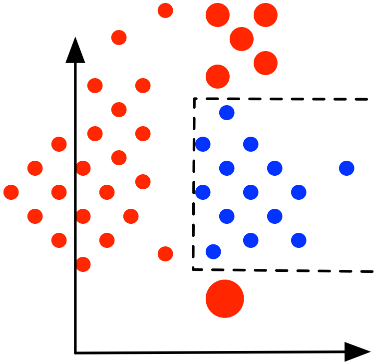

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Weak Learners

Formally, a **weak** learner is one with $\epsilon < \frac{1}{2}$.
(These tend to be high-bias, low-variance classifiers.)

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$$\overbrace{\frac{1}{N}\sum_{n=1}^{N}[\![f_{\text{boost}}(x_n) \neq y_n]\!]}^{\text{training error}} \leq \frac{1}{N}\sum_{n=1}^{N}\underbrace{\exp\left(-y_n \cdot \sum_{e=1}^{E} \alpha^{(e)} \cdot f^{(e)}(x_n)\right)}_{\text{“loss”}}$$

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- Our update of $\alpha^{(e)}$ on each round is provably the choice that minimizes this loss.
- Assuming each $\epsilon^{(e)} < \frac{1}{2}$, it's possible to prove:

$$\ldots \leq \exp -2\sum_{e=1}^{E}\left(\frac{1}{2} - \hat{\epsilon}^{(e)}\right)^2$$

(i.e., as E goes up, training error decreases *exponentially*!)

Theory and Practice

Boosting tends to be very robust to overfitting, with out-of-sample error continuing to decrease even when training error stabilizes.

Eventually, it will overfit.

Theory gives some insight about this; PAC-style generalization bound is:

$$\epsilon \leq \hat{\epsilon} + \tilde{O}\left(\sqrt{\frac{E \cdot d}{N}}\right)$$

where d measures the size of the hypothesis class.

Boosting as Loss Minimization (Exponential Loss)

The above analysis leads to another insight: boosting is minimizing yet another loss function!

Let $a(x)$ denote a score (or activation function) for input x—the value whose sign we take for binary classification.

$$\exp\left(-y \cdot a(x)\right)$$

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If a were (sub)differentiable with respect to continuous parameters, you could directly minimize exponential loss using SGD.
That's not the case if $\mathcal{W}$ is, say, a decision tree learner.

Palate Cleanser: Random Forests

Fix tree structure; randomly fill in features.

Do this E times; let them vote.

With large enough E, useless trees will cancel each other out.