CSE 431 Winter 2025
Assignment #7

Reading assignment: Read sections 8.1-8.5, 9.3.

Problems:

1. (20 points) Show that $TQBF$ restricted to formulas where the part following the quantifiers is in conjunctive normal form is still PSPACE-complete.

2. (20 points) Define $SORTED\text{-}VERSION$ as the set $\{\langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle \mid n \in \mathbb{N}$ and $(b_1, \ldots, b_n)$ is a sorted version of $(a_1, \ldots, a_n)$ (non-decreasing)$\}$. Show that $SORTED\text{-}VERSION$ is in L.

3. (20 points) Recall that a directed graph $G = (V, E)$ is *strongly connected* iff for every pair of vertices $u, v \in V$, there exists a path in G from u to v.
   Let $STRONGLY\text{-}CONNECTED = \{\langle G \rangle \mid G$ is a strongly connected directed graph$\}$.
   In this problem you will show that $STRONGLY\text{-}CONNECTED$ is NL-complete.

   (a) Prove that $PATH \leq_m^L STRONGLY\text{-}CONNECTED$.
   Hint: for your reduction, add a number of "backward" edges.

   (b) Prove that $STRONGLY\text{-}CONNECTED \in NL$.

4. (20 points) We stated in class that $UPATH \in L$. Use this to show that $ACYCLIC \in L$ where $ACYCLIC = \{\langle G \rangle \mid$ undirected graph G does not contain a cycle$\}$

5. (20 points) The function MAJORITY:$\{0,1\}^* \rightarrow \{0,1\}$ is given by MAJORITY$(x) = 1$ iff $\geq 1/2$ the bits in x are 1. Give a circuit of size $O(n \log n)$ that computes MAJORIY$_n$. Hint: First compute the sum of the bits of x using divide and conquer.

A. (Extra Credit) In this you will show that every Boolean formula F of size s can be converted to an equivalent formula of depth $O(\log s)$.

   (a) Let T be a rooted binary tree with ℓ leaves. Argue that if you start at the root and repeatedly move to the child with the larger number of below it, you will eventually reach a node v in T such that between $\ell/3$ and $2\ell/3$ leaves of T are below v.

   (b) A Boolean formula F over $\wedge, \vee, \neg$ is a circuit that is a tree. For any gate g of F we can define the sub-formula G of F rooted at g. We can also define $F_{g=0}$ and $F_{g=1}$ to be the formulas you get from replacing the gate g by the input values 0 and 1, respectively. Describe how to write a new formula for F that adds only a constant number of gates to the three formulas: G, $F_{g=0}$, $F_{g=1}$.

   (c) Use induction and the choose gates g for part (b) corresponding to the 1/3-2/3 nodes from part (a) in order to show that any Boolean formula of size s can be replaced by one with $O(\log s)$ depth (which might be much bigger but still has size $s^{O(1)}$).

B. (Extra Credit) Creating an NC^1 circuit for MAJORITY$_n$:

(a) Given three natural numbers x, y, z in binary, show how to compute two numbers c and s such that $x + y + z = s + c$ using only constant-depth fan-in 2 circuits (think of s as the sums and c as the carries.

(b) Use part (a) recursively to compute two numbers whose total is the number of 1's in the input x and use this to produce an NC^1 circuit for MAJORITY$_n$