CSE 431 Winter 2025
Assignment #3

Reading assignment: Read Chapter 5, sections 5.3 and 5.1 in Sipser's text.

Problems:

1. (20 points) Prove that A is decidable if and only if $A \leq_m \{0^n \mid n \text{ is odd}\}$.

2. (20 points) Prove that

   $$DOLLAR_{TM} = \{\langle M \rangle \mid M \text{ is a TM and there is some } w \in \Sigma^* \text{ such that } M \text{ writes symbol \$ on the tape during its computation on input } w\}$$

   is undecidable.

3. (20 points) Let

   $$J = \{w \mid w = 0x \text{ for some } x \in E_{TM} \text{ or } w = 1y \text{ for some } y \in \overline{E_{TM}}\}.$$

   Prove that neither J nor $\overline{J}$ is Turing-recognizable.

4. (20 points) A language B is called *r.e.-complete* iff the following both hold:

   - B is Turing-recognizable (equivalently, recursively enumerable).
   - For all Turing-recognizable languages A, $A \leq_m B$.

   Prove that A_{TM} is r.e.-complete.

5. (20 points) Suppose that $A \subseteq \{\langle M \rangle \mid M \text{ is a decider TM}\}$ and that A is Turing-recognizable. (That is, suppose that A only contains descriptions of TMs that are deciders, but might not include some of them.)

   Prove that there is some decidable language D that is not captured by A in that $L(M) \neq D$ for every M with $\langle M \rangle \in A$.

   (Intuitively, this means that one can't come up with some restricted easy-to-recognize format for deciders that captures all decidable languages. Such a format is essentially what researchers had been searching for until Turing proved it didn't exist.)

   Hint: Note that this does not follow from Rice's Theorem since the "is a decider" property is not about $L(M)$ but rather about the functioning of M. You may find it helpful to consider an enumerator for A.

A. (Extra Credit) Let $\Gamma = \{0, 1, blank\}$ be the tape alphabet for all TMs in this problem. Define the $BB : \mathbb{N} \to \mathbb{N}$ as follows: For each value of k, consider all k-state TMs that halt when started with a blank tape. Let $BB(k)$ be the maximum number of 1s that remain on the tape among all of these machines. Show that BB is not a computable function.