Solutions to some sample 431 final exam questions

1. Consider the following list of properties that might apply to the stated language.

T-rec: The language is Turing-recognizable.
Dec: The language is decidable.
NP: The language is in NP.
NP-c: The language is NP-complete.

\[ \mathcal{P} \text{: The language is in } \mathcal{P}. \]

Circle all the properties that you are certain are true.
\[ \times \text{ out all the properties that you are certain are false.} \]

**NOTE:** You may not be able to do either for some properties.

(a) \( \{ (M, w) \mid \text{Turing machine } M \text{ accepts } w \} \)
(b) \( \{ (M, w) \mid \text{Turing machine } M \text{ accepts } w \text{ in at most } |w| \text{ steps} \} \)
(c) \( \{ (M, w) \mid \text{Turing machine } M \text{ accepts } w \text{ in at most } 2^{|w|} \text{ steps} \} \)
(d) \( \{ (M, w) \mid \text{Turing machine } M \text{ does not accept } w \} \)
(e) \( L(\alpha) \text{ for some regular expression } \alpha \)
(f) \( \{ (F) \mid F \text{ is a 3-CNF formula which evaluates to true on some truth assignment} \} \)
(g) \( \{ (F, x) \mid F \text{ is a 3-CNF formula which evaluates to true on truth assignment } x \} \)
(h) \( \{ (F) \mid F \text{ is a propositional logic tautology} \} \)
(i) \( \{ (G, H) \mid G \text{ and } H \text{ are isomorphic graphs} \} \)

5. We show that \( \text{SET-PARTITION} \) in NP-complete.

1. \( \text{SET-PARTITION} \in \text{NP}: \)

   (a) Guess a binary string of length \( n \) representing a subset \( S \subseteq \{1, \ldots, n\} \).

   (b) Verify that \( \sum_{i \in S} x_i = \sum_{i \notin S} x_i \).

   (c) This polynomial time to check since we can compute the two sums in polynomial times.

2. We show that \( \text{SET-PARTITION} \) is NP-hard by showing that \( \text{SUBSET-SUM} \leq^p \text{SET-PARTITION} \).

   (a) On input \( \langle x_1, \ldots, x_m, t \rangle \) for \( \text{SUBSET-SUM} \), define \( M = \sum_{i=1}^{n} x_i \). Assume that \( t \leq M \) – if not we simply map the input to \( \{1, 2\} \). Otherwise, using the hint, remove \( t \), let \( n = m + 2 \) and add two extra numbers \( x_{n-1} = M + t \) and \( x_n = 2M - t \).

   (b) The computation is clearly polynomial time since it simply requires the computation of \( M \) and the two extra numbers.

   (c) Correctness (\( \Rightarrow \)): Suppose that \( \langle x_1, \ldots, x_m, t \rangle \in \text{SUBSET-SUM} \). Then there is a subset \( S' \in \{1, \ldots, m\} \) such that \( \sum_{i \in S'} x_i = t \) and \( t \leq M \). Therefore the output has \( n = m + 2 \) values and we define \( S \subseteq \{1, \ldots, n\} \) by \( S = S' \cup \{n\} \). Then \( \sum_{i \in S} x_i = t + 2M - t = 2M \) and \( \sum_{i \notin S} x_i = M + t + \sum_{i \leq m, i \notin S'} = M + t + M - t = 2M \) and hence \( \langle x_1, \ldots, x_n \rangle \in \text{SET-PARTITION} \) as required.

   (d) Correctness (\( \Leftarrow \)): Suppose that \( \langle x_1, \ldots, x_n \rangle \in \text{SET-PARTITION} \) Then we know that we have a subset \( S \in \{1, \ldots, n\} \) such that \( \sum_{i \in S} x_i = \sum_{i \notin S} x_i \). By definition of the reduction we also
know that the sum of all the elements is $4M$ so the sum of each side is $2M$. Because $x_{n-1}$ and $x_n$ add up to $3M$, which is too large, we can’t have both elements in $S$ or both elements not in $S$. Therefore, one of $S$ or $\overline{S}$ contains $n$ but not $n-1$. Assume, without loss of generality that $S$ does. (If not, simply complement $S$.) Then define $S'' = S - \{n\}$ and observe that $S'' \subseteq \{1, \ldots, m\}$. Then $2M = \sum_{i \in S} x_i = 2M - t + \sum_{i \in S''} x_i$. It follows that $\sum_{i \in S''} x_i = t$ and hence $(x_1, \ldots, x_m, t) \in \text{SUBSET-SUM}$ as required.