Reading assignment: Read Chapter 4 of Sipser’s textbook.

Problems:

1. (20 points) Prove that a language is decidable if and only if there is an enumerator that enumerates it in lexicographic order.
   (Hint: Handle the case where the language is finite separately from the case where it is infinite.)

2. (10 points) Use the result of question 1 to show that any infinite Turing-recognizable language contains an infinite decidable subset.

3. (10 points) Prove that the language 
\[ \text{\textit{\text{ALL}}_{\text{DFA}} = \{\langle M \rangle \mid M \text{ is a DFA with alphabet \( \Sigma \) and } L(M) = \Sigma^*\}} \] 
is decidable.

4. (30 points) Suppose that \( A \) and \( B \) are decidable languages. Prove that the following languages are also decidable. (The definitions of the latter two are from Chapter 1 and all are included for convenience.)

   (a) \( A \cap B = \{x \mid x \in A \text{ and } x \in B\} \).
   (b) \( AB = \{x \mid \exists y \in A \text{ and } z \in B \text{ such that } x = yz\} \).
   (c) \( A^* = \{x \mid \exists k \geq 0 \text{ and } y_1, \ldots, y_k \in A \text{ such that } x = y_1 \cdots y_k\} \).

5. (30 points) Suppose that \( A \) and \( B \) are Turing-recognizable languages. Prove that the following languages are also Turing-recognizable:

   (a) \( AB \).
   (b) \( \text{Pref}(A) = \{x \mid \exists y \text{ with } xy \in A\} \), the set of all prefixes of strings in \( A \).

6. (Extra credit) Let \( C \) be a language. Prove that \( C \) is Turing-recognizable if and only if there is a decidable language \( D \) such that \( C = \{x \mid \exists y \text{ such that } \langle x, y \rangle \in D\} \).