So far

- \( \text{CONTIME}(f(n)) \subseteq \text{SPACE}(f(n)) \subseteq \text{NSPACE}(f(n)) \) for \( f(n) = \log^2 n \)

- \( \text{NSPACE}(f(n)) \subseteq \text{SPACE}(f^2(n)) \) for \( f(n) > \log^2 n \)

In particular,

\[
P \subseteq \text{NP} \subseteq \text{PSPACE} \subseteq \text{EXP} \subseteq \text{NSPACE}
\]

**Def**\( ^n \) B is **PSPACE-hard** iff \( \forall A \in \text{PSPACE}, A \leq^P_B \)

**Def**\( ^n \) B is **PSPACE-complete** iff

- \( B \in \text{PSPACE} \)
- \( B \) is **PSPACE-hard**

**Def**\( ^n \) \( \text{TQBF} = \exists \phi : \phi \) is a fully quantified Boolean formula that evaluates to true

- \( \exists x_1, \forall x_2, \exists x_3 ((x_1 \rightarrow x_2) \land (x_2 \rightarrow x_3) \land (x_3 \rightarrow x_2)) \)
- true, \( x_1 = 0, x_3 = x_2 \)

**Thm** \( \text{TQBF} \) is **PSPACE-complete**
Proof 1: \textbf{Claim: TAQBE \textit{PSPACE}}

Write \( \phi = Q_1 x_1 \cdots Q_n x_n \psi(x_1 \cdots x_n) \)

Imagine a full binary tree on the \( x_1, \ldots, x_n \) variables.

Consider an alg that does a DFS on this tree evaluating the formula:

The value of the leaf is easy polynomial to compute.

We can evaluate each node as we backtrack from the DFS.

If \( x_i \) is labelled by \( \exists \):
- evaluate left child
- if left child’s value is \( 1 \) return \( 1 \)
- else evaluate right child and return its value
If $x_i$ is labeled by $A$:
- evaluate left-child
  - if left-child's value is 0 return 0
  - else evaluate right-child and return $1 + \text{value}$

What storage is required:
- DFS stack: height $n$
- Enough to evaluate $G$ at a leaf
- Total $m + \langle G \rangle \leq \text{linear space}$

2) $\text{TQBF is PSPACE-hard}$:

Let $A \in \text{PSPACE}$

- $A$ is decided by some TM $M$ such
  - space $S \leq cn^k$ for some constant $c, k$

Recall: $x \in A \Leftrightarrow \exists \text{ path from } C_0 \text{ to } \text{accept in } G_{M,x}$

- Configurational Graph of $M$ on input $x$
- $G_{M,x}$ has at most $T = 2S$ nodes
- each node of $G_{M,x}$ is a configuration of $M$ on input $x$ and can be described by $O(S)$ bits.

$O(c n^k)$.
Recall \( \text{CANYIELD}_t(C,D) \)

\[ \text{then is a path from } C \text{ to } D \]

in \( G_{mix} \) of length \( \leq t \).

\( \text{CANYIELD}_0(C,D) \equiv \text{"C=D"} \)

\( \text{CANYIELD}_1(C,D) \equiv \text{"C=D"} \lor \text{"C\rightarrow D"} \)

\( \text{"yield in one step"} \)

\( \text{CANYIELD}_t(C,D) \equiv \exists \text{mid} . (\text{CANYIELD}^t_{t+1?}(C,\text{Even}) \land \text{CANYIELD}^t_{t+1?}(\text{mid},D)) \)

We prove \( \alpha \preceq^\alpha \top \alpha \beta \)

**Goal:** \( x \in A \rightarrow \langle \Phi_{mix} \rangle \)

where \( \Phi_{mix} \equiv 1 \) iff \( \text{CANYIELD}_+(C_0, C_{\text{accept}}) \)

We will define formula \( \Phi_t(\overline{C}, \overline{D}) \) s.t.

\( \Phi_t(\overline{C}, \overline{D}) \) iff \( \text{CANYIELD}_+(C_1, D) \)

where \( \overline{C}, \overline{D} \) are binary vectors of variable correspondingly to entry \( C, D \)

since space \( s \leq S \), \( \overline{C}, \overline{D} \) take \( O(S) = O(\log \alpha) \) bits.

We will set \( \Phi_{mix} = \Phi_t(\overline{C_0}, \overline{C_{\text{accept}}}) \)

(\text{constant bit-vector representation of all configurations})
\( \Phi_0 (\mathcal{C}, \bar{d}) \) is an \( \mathcal{L} \) of \( \mathcal{C}(S) \) consisting of the form \( (\mathcal{C}_i)^{\bar{d}} \).

\( \Phi_1 (\mathcal{C}, \bar{d}) = \Phi_0 (\mathcal{C}, \bar{d}) \cup \text{" } C \text{"} \)

\( \Phi_t (\mathcal{C}, \bar{d}) \) can express in logic only \( \Phi_t \) functions (just like adjacent rows or in Cook-Levin tableau).

Assume we say that we only define \( \Phi_t \) when \( t \) is a power of 2.

\textbf{Obvious: attempt based on CANFIELD}_t \( (\mathcal{C}, \bar{d}) \)

\( \Psi_t (\mathcal{C}, \bar{d}) : \exists \mathcal{C}_{\text{mid}} (\Phi_{t_1} (\mathcal{C}, \mathcal{C}_{\text{mid}}) \wedge \Phi_{t_2} (\mathcal{C}, \mathcal{C}_{\text{mid}})) \)

When we unravel this remark we realize that \( \Phi_t \)

\[ \text{size} (\Phi_t) > 2 \times \text{size} (\Phi_{t_2}) \]

So \( \text{size} (\Phi_t) > t \), which will be bad for \( \Phi_t \) since \( T \) is exponential and we need to compute in polynomial.

But we haven't used any \( A \) in this !

Our new idea will be to write \( \Phi_{t_2} \) just once and use the \( A \) quantifier to cover the two cases.
Define \( \Phi_t(\mathcal{C}, \mathcal{D}) = \bigoplus_{\mathcal{C}_{\text{mid}}} \bigvee_{\mathcal{E}, \mathcal{F}} \left( (\mathcal{C} = \mathcal{C}_{\text{mid}}) \land (\mathcal{E} = \mathcal{C}_{\text{mid}}) \right) \lor (\mathcal{E} = \mathcal{C}_{\text{mid}}) \land (\mathcal{F} = \mathcal{D}) \) 

\( \to \Phi_{t/2}(\mathcal{E}, \mathcal{F}) \)

Now \( \text{size}(\Phi_t) = c n^k + \text{size}(\Phi_{t/2}) \)

\[ \text{size}(\Phi_t) = (c n^k) \log T + c n^k \]

\[ \overset{O(n^k)}{\text{O}(n^k)} \]

\[ \text{size}(\Phi_t) = \text{O}(n^{2k}) \text{ which is polynomial} \]

\( \Phi_t \) is very easy to write down - everything but \( \Phi_t \) doesn't even depend on the details of \( M \)

\( \therefore \text{it is polynomial} \)

By continuum it satisfies correctness \( \Box \)

Next time: complexity curves inside \( P \). Is every problem in \( P \) solvable in small space?