Every 3-coloring corresponds to a truth assignment for \( \phi \):

1. If \( f(\phi) \) is 3-colorable, one outer vertex per clause must be an \( F \) so corresponding literal must be a \( T \).
2. This has one \( T \) literal per clause.

If \( \phi \) is \( SAT \), color literal under via the truth value. At least one outer node in each clause is opposite of \( T \) literal node color and each outer node with \( F \) role with \( O \) and then \( T \). The inner triangle with \( O \) opposite the \( F \) and \( T \) \( F \) opposite the other two.
HAMPATH = \{ \langle G \rangle : \text{directed graph } G \text{ has a path touching each node exactly once} \}

HAMCYCLE = \{ \langle G \rangle : \text{G has a cycle} \}

Therefore, HAMPATH, HAMCYCLE are NP-complete.

Let $\text{ENP}$ be $\text{coNP} = \text{pa} \text{RPC}$. Check $\text{SAT} \leq^p \text{HAM PATH (CYCLE)}$

Proof idea:

For any path/cycle, the diamond is either traversed

- from L to R ($x_i = \text{true}$)
- from R to L ($x_i = \text{false}$)

Replace by

- $x_i$ can visit node $C_1$
- $x_{i+1}$ can visit node for $R$ to $L$

Each of node $C_i$ node

- $x_i$ can return from
- $x_{i+1}$ can come from

Each of node $C_i$ node

- $x_i$ can return from
- $x_{i+1}$ can come from
**UNAHMUCYCLE**

- **Def**: \( \text{coNP} = \Sigma^P_3 \) is the analog of \( \text{coNP} \) with \( \text{NP} \).
- **coNP problems**: \( \text{UNSAT} = \{ \varphi \mid \varphi \text{ is an unsatisfiable Boolean formula} \} \)
- \( \text{TAUT} = \{ \varphi \mid \varphi \text{ is a propositional logic tautology} \} \)

**Note**: \( \varphi \) is a tautology \( \implies \neg \varphi \) is unsatisfiable.

- \( \Sigma \) complete \( \implies \forall A \in \text{NP}, A \leq^p_\text{coNP} \overline{B} \implies B \in \text{coNP} \)
- \( \overline{\text{TAUT}}, \text{UNSAT} \) an \( \text{coNP} \) complete

- \( \Delta \) complete

- \( \text{NP} \neq \text{coNP} \) open

- \( \text{NP} = \text{coNP} \) if every tautology \( \varphi \) has a proof of polytime in some proof system.

- \( \text{NP} \cap \text{coNP} \neq \emptyset \) open.
So far we know:

\[ p \subseteq \text{NP} \subseteq \text{EXP} = \bigcup_{k} \text{TIME}(2^{O(n^k)}) \]

We find an important and natural class of problems in between here.

**Space Complexity**

We define this using 2-tape NTMs where the input is in read-only memory.

![Diagram of a 2-tape NFTM]

Defn. The space used by a 2-tape NTM \( M \)

\[ S(M) = \max \{ \# \text{ of work tape cells that } M \text{ uses on any input } w \in \Sigma^* \text{ and any computation path} \} \]

Defn. \( \text{SPACE}(S(n)) = \{ A : A \text{ is decided by a TM with read-only input with space used } O(S(n)) \} \)

\[ \text{NSPACE}(S(n)) = \{ A : A \text{ is decided by an NTM with space used } O(S(n)) \} \]
Note: A is regular $\iff A \in \text{SPACE}(1)$

Run: SAT $\in \text{SPACE}(n)$

Proof: On input a formula $\varphi$:

Run brute force algorithm that tries all possible truth assignments $Y$ and evaluates $\varphi$ on each one.

$|Y| \leq |\varphi| < 1$ (copy from step 2)

and reuse space for each assignment, total space only a constant factor more than $|\varphi|$.

$\text{PSPACE} = \bigcup_n \text{SPACE}(n^k)$

$\text{NPSPACE} = \bigcup_n \text{SPACE}(n^2)$

Thus, $\text{EQ_NFA} \in \text{NPSPACE}$

Proof: On input $\langle N_1, N_2 \rangle$ where

$N_1, N_2$ are NFA's with state set $Q_1, Q_2$ respectively.
$L(N_1) + L(N_2) \iff \exists$ string $y$ s.t. set of states reachable in $N_1$ on input $y$ contains a final state of $N_1$, but set of states reachable in $N_2$ on input $y$ does not (or vice versa)

**Claim** If such a $y$ exists, then one of length

$$\leq 2\log_2 1 + 1\log_2 1 \exists y$$

**Proof of Claim:** If $y$ is longer than one of the sets of states reachable in the two machines, repeat.

Idea: Use nondeterminism to guess $y$.

But: $y$ is too long to write down in only null symbols.

Idea: Unlike time-bounded NTMs, can't convert space-bounded NTMs to guess first form symbol-by-symbol. Instead guess $y$ and write down the whole thing.
Algorithm

On input \( \langle N_1, N_2 \rangle \)
start at \( q_0, q_1 \) states of \( N_1, N_2 \)
For \( 2^{10^3} \) steps
Given next symbol \( y \)
keep track of current set of states reached so far on \( y \) in both \( N_1, N_2 \)
if one of these sets but not the other contains an accepting state then accept

Storage
- \( \{ Q, 1, 1 \} \) bits for sets of states reached
- \( \{ Q, 1 \} + 10^3 \) bits for a timer

Time (a) \( \text{TIME}(T(n)) \leq \text{SPACE}(T(n)) \)
\( \text{NTIME}(T(n)) \leq \text{NSPACE}(T(n)) \)

(b) For \( s(n) \gg \log_2 n \),
\( \text{SPACE}(s(n)) \leq \text{TIME}(2^{O(s(n))}) \)

(c) For \( s(n) \gg \log_2 n \),
\( \text{NSPACE}(s(n)) \leq \text{TIME}(2^{O(s(n))}) \)

Proof (a) If \( M \) runs for \( T(n) \) steps it can only use \( T(n) \) memory cell

(1) just like solution for ALBA:
If \( M \) has space \( s(n) \) then for some \( d \)
if has only \( n \cdot 2^{d s(n)} \) configurations
Why \( n \cdot 2^d \cdot S(n) \) ?

Possible

read-only input head

states, work tape contents

and work type head.

Now for \( S(n) \geq \log_2 n \)

\[ n \leq 2^{S(n)} \]

so total is \( 2^{d \cdot S(n)} \)

Now just simulate the \( S(n) \) space-bounded machine for \( 2^{d \cdot S(n)} \) steps if accepted run accept.

If not accepted yet then reject.

Actually simulate also still takes space \( O(S(n)) \)

- original space + counter.

(c) Issues with doing this for NTMs:

- each path has length \( 2^{d \cdot S(n)} \)

- exponential many paths to try.

next time.