Solutions to some sample 431 final exam questions

1. Consider the following list of properties that might apply to the stated language.

T-rec: The language is Turing-recognizable.
Dec: The language is decidable.
NP: The language is in NP.
NP-c: The language is NP-complete.
$\mathcal{P}$: The language is in $\mathcal{P}$.

Circle all the properties that you are certain are true.
$\times$ out all the properties that you are certain are false.
NOTE: You may not be able to do either for some properties.

(a)	$\{\langle M, w\rangle \mid$ Turing machine M accepts $w\}$	T-rec	~~Dec~~	~~NP-c~~	~~NP~~	~~$\mathcal{P}$~~
(b)	$\{\langle M, w\rangle \mid$ Turing machine M accepts w in at most $\lvert w\rvert$ steps$\}$	T-rec	Dec	NP-c	NP	$\mathcal{P}$
(c)	$\{\langle M, w\rangle \mid$ Turing machine M accepts w in at most $2^{\lvert w\rvert}$ steps$\}$	T-rec	Dec	NP-c	NP	~~$\mathcal{P}$~~
(d)	$\{\langle M, w\rangle \mid$ Turing machine M does not accept $w\}$	~~T-rec~~	~~Dec~~	~~NP-c~~	~~NP~~	~~$\mathcal{P}$~~
(e)	$L(\alpha)$ for some regular expression α.	T-rec	Dec	NP-c	NP	$\mathcal{P}$
(f)	$\{\langle F\rangle \mid F$ is a 3-CNF formula which evaluates to true on some truth assignment$\}$	T-rec	Dec	NP-c	NP	$\mathcal{P}$
(g)	$\{\langle F, x\rangle \mid F$ is a 3-CNF formula which evaluates to true on truth assignment $x\}$	T-rec	Dec	NP-c	NP	$\mathcal{P}$
(h)	$\{\langle F\rangle \mid F$ is a propositional logic tautology$\}$	T-rec	Dec	NP-c	NP	$\mathcal{P}$
(i)	$\{\langle G, H\rangle \mid G$ and H are isomorphic graphs$\}$	T-rec	Dec	NP-c	NP	$\mathcal{P}$

5. We show that $SET\text{-}PARTITION$ in NP-complete.

1. $SET\text{-}PARTITION \in$ NP:
   (a) Guess a binary string of length n representing a subset $S \subseteq \{1, \ldots, n\}$.
   (b) Verify that $\sum_{i\in S} x_i = \sum_{i\notin S} x_i$.
   (c) This polynomial time to check since we can compute the two sums in polynomial times.
2. We show that $SET\text{-}PARTITION$ is NP-hard by showing that $SUBSET\text{-}SUM \leq_m^p SET\text{-}PARTITION$.
   (a) On input $\langle x_1, \ldots, x_m, t\rangle$ for $SUBSET\text{-}SUM$, define $M = \sum_{i=1}^{n} x_i$. Assume that $t \leq M$ – if not we simply map the input to $\{1, 2\}$. Otherwise, using the hint, remove t, let $n = m + 2$ and add two extra numbers $x_{n-1} = M + t$ and $x_n = 2M - t$.
   (b) The computation is clearly polynomial time since it simply requires the computation of M and the two extra numbers.
   (c) Correctness ($\Rightarrow$): Suppose that $\langle x_1, \ldots, x_m, t\rangle \in SUBSET\text{-}SUM$. Then there is a subset $S' \in \{1, \ldots, m\}$ such that $\sum_{i\in S'} x_i = t$ and $t \leq M$. Therefore the output has $n = m + 2$ values and we define $S \subseteq \{1, \ldots, n\}$ by $S = S' \cup \{n\}$. Then $\sum_{i\in S} x_i = t + 2M - t = 2M$ and $\sum_{i\notin S} x_i = M + t + \sum_{i\leq m, i\notin S'} = M + t + M - t = 2M$ and hence $\langle x_1, \ldots, x_n\rangle \in SET\text{-}PARTITION$ as required.
   (d) Correctness ($\Leftarrow$): Suppose that $\langle x_1, \ldots, x_n\rangle \in SET\text{-}PARTITION$ Then we know that we have a subset $S \in \{1, \ldots, n\}$ such that $\sum_{i\in S} x_i = \sum_{i\notin S} x_i$. By definition of the reduction we also

know that the sum of all the elements is $4M$ so the sum of each side is $2M$. Because x_{n-1} and x_n add up to $3M$, which is too large, we can't have both elements in S or both elements not in S. Therefore, one of S or $\overline{S}$ contains n but not $n-1$. Assume, without loss of generality that S does. (If not, simply complement S.) Then define $S'' = S - \{n\}$ and observe that $S'' \subseteq \{1, \ldots, m\}$. Then $2M = \sum_{i \in S} x_i = 2M - t + \sum_{i \in S''} x_i$. It follows that $\sum_{i \in S''} x_i = t$ and hence $\langle x_1, \ldots, x_m, t \rangle \in \mathit{SUBSET}\text{-}\mathit{SUM}$ as required.