1. (20 points) Prove that a language is decidable if and only if there is an enumerator that enumerates it in lexicographic order. (Hint: Handle the case where the language is finite separately from the case where it is infinite.)

2. (10 points) Use the result of question 1 to show that any infinite Turing-recognizable language contains an infinite decidable subset.

3. (30 points) Suppose that $A$ and $B$ are decidable languages. Prove that the following languages are also decidable. (The definitions here are from Chapter 1 and are included for convenience.)

   (a) $A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$.

   (b) $AB = \{ x \mid \exists y \in A \text{ and } z \in B \text{ such that } x = yz \}$.

   (c) $A^* = \{ x \mid \exists k \geq 0 \text{ and } y_1, \ldots, y_k \in A \text{ such that } x = y_1 \cdots y_k \}$.

4. (30 points) Suppose that $A$ and $B$ are Turing-recognizable languages. Prove that (a) $A \cap B$, (b) $AB$, and (c) $A^*$ are all also Turing-recognizable.

5. (Extra credit) Let $C$ be a language. Prove that $C$ is Turing-recognizable if and only if there is a decidable language $D$ such that $C = \{ x \mid \exists y \text{ such that } \langle x, y \rangle \in D \}$. 