CSE 431
Introduction to the Theory of Computation
Sample Final Exam

DIRECTIONS: Closed book, closed notes. Time limit 1 hour 50 minutes. Answer the problems on the exam paper.

1. Consider the following list of properties that might apply to the stated language.

   T-rec: The language is Turing-recognizable.
   Dec: The language is decidable.
   NP: The language is in NP.
   NP-c: The language is NP-complete.
   $\mathcal{P}$: The language is in $\mathcal{P}$.

   Circle all the properties that you are certain are true.
   $\times$ out all the properties that you are certain are false.
   NOTE: You may not be able to do either for some properties.

   (a) $\{\langle M, w\rangle \mid$ Turing machine M accepts $w\}$ .... T-rec Dec NP-c NP $\mathcal{P}$
   (b) $\{\langle M, w\rangle \mid$ Turing machine M accepts w in at most $|w|$ steps$\}$ .... T-rec Dec NP-c NP $\mathcal{P}$
   (c) $\{\langle M, w\rangle \mid$ Turing machine M accepts w in at most $2^{|w|}$ steps$\}$ .... T-rec Dec NP-c NP $\mathcal{P}$
   (d) $\{\langle M, w\rangle \mid$ Turing machine M does not accept $w\}$ .... T-rec Dec NP-c NP $\mathcal{P}$
   (e) $L(\alpha)$ for some regular expression α. .... T-rec Dec NP-c NP $\mathcal{P}$
   (f) $\{\langle F\rangle \mid F$ is a 3-CNF formula which evaluates to true on some truth assignment$\}$ .... T-rec Dec NP-c NP $\mathcal{P}$
   (g) $\{\langle F, x\rangle \mid F$ is a 3-CNF formula which evaluates to true on truth assignment $x\}$ .... T-rec Dec NP-c NP $\mathcal{P}$
   (h) $\{\langle F\rangle \mid F$ is a propositional logic tautology$\}$ .... T-rec Dec NP-c NP $\mathcal{P}$
   (i) $\{\langle G, H\rangle \mid G$ and H are isomorphic graphs$\}$ .... T-rec Dec NP-c NP $\mathcal{P}$

2. Use the fact that A_{TM} is undecidable to show that the following language is undecidable.

$$L_2 = \{\langle M\rangle \mid \text{Turing machine } M \text{ accepts the input string “2”}\}.$$

3. (a) Give a full formal definition of what it means for A to be polynomial-time mapping reducible to B.
   (b) Show that if $A \leq_p B$ and B is in PSPACE, (*i.e. B can be decided by a TM using only a polynomial number of tape cells*), then A is also in PSPACE.

4. (a) What is NP-completeness and why is it an interesting/useful notion?
   (b) Describe the error in the following incorrect “proof” that $P \neq NP$:

      Consider an algorithm for SAT:
      “On input $\langle F\rangle$, try all possible assignments to the variables. Accept if any satisfy F”
      This algorithm clearly requires exponential time. Thus SAT has exponential time complexity. Therefore SAT is not in P. Because SAT is in NP, it must be true that P is not equal to NP.

5. The *SET-PARTITION* problem asks, given a collection of decimal numbers $x_1, \ldots, x_n$ whether or not it is possible to partition these numbers into two groups so that the sum in each group is the same. More formally, if $\langle \ldots \rangle$ means a decimal encoding,

$$\textit{SET-PARTITION} = \{\langle x_1, \ldots, x_n \rangle \mid \text{there is a set } S \subseteq \{1, \ldots, n\} \text{ so that } \sum_{i \in S} x_i = \sum_{i \notin S} x_i\}$$

Prove that *SET-PARTITION* is NP-complete.
Hint: Use the NP-completeness of

$$\textit{SUBSET-SUM} = \{\langle x_1, \ldots, x_m, t \rangle \mid \textit{there is a set } S \subseteq \{1, \ldots, m\} \textit{ so that } \sum_{i \in S} x_i = t\}$$

Hint: Try including two large numbers whose size differs by exactly $\sum_{i=1}^{m} x_i \; - \; 2t$.

6. The Travelling Salesperson Problem, TSP, asks, given an $n \times n$ matrix C containing for each pair $i, j \in \{1, \ldots, n\}$, the integer cost c_{ij} for travelling from city i to city j, *representing, say, the cost of gasoline to drive directly from city i to city j*, as well as an integer K, *representing a total fuel budget,* whether or not there is an order (for a travelling salesperson) to visit each of the n cities exactly once, starting and ending in the same city, so that the total cost *of the gasoline used* is at most K? In set notation,

$$TSP = \{\langle C, K \rangle \mid \text{with cost matrix } C \text{ there is a salesperson's tour of total cost } \leq K\}.$$

Use the fact that the directed Hamiltonian cycle problem, $DHAMCYCLE$, is NP-complete to prove that TSP is NP-complete.
Hint: choose the cost c_{ij} to depend on whether or not the edge (i, j) is in the graph G.

7. Prove that the language

$$L = \{\langle M, a, b \rangle \mid \text{there is some } x \in \{0,1\}^* \text{ such that } M \text{ runs for } > a \cdot |x|^2 + b \text{ steps on input } x\}$$

is Turing-recognizable.