Scribe Notes – Planar Graphs and Space Complexity

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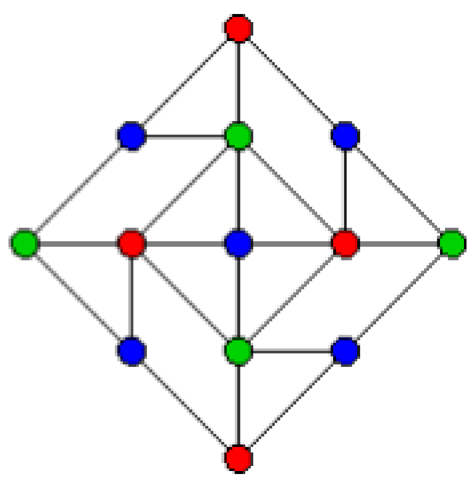
Last Time: 3-COL is NP-complete

Today: We want to show 3-COL $\leq_p$ PLANAR-3-COL

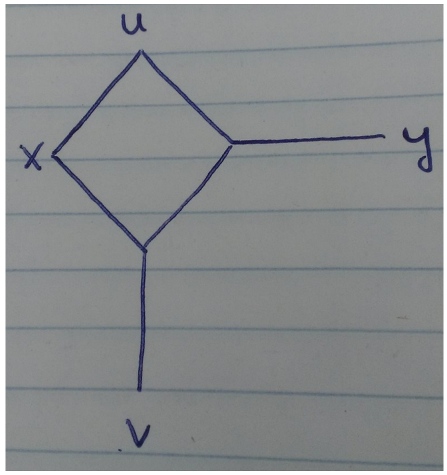
To show 3-COL $\leq_p$ PLANAR-3-COL, we need to take any three-colorable graph G and turn it into a planar graph $\hat{\text{G}}$ that is also three-colorable.

Our first step is to take any three colorable graph, and place all the vertices of the graph on the perimeter of a circle. If any of the edges intersect each other, then the graph needs to be made planar.

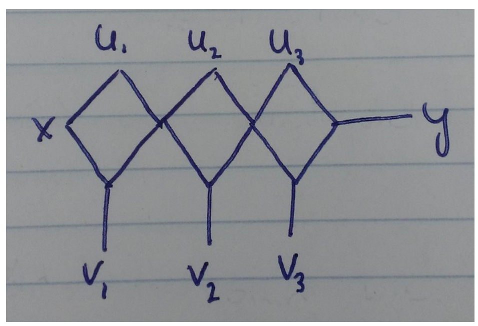
We can remove these intersections using the following gadget:



This gadget makes any intersection and more importantly preserves the following property: Every valid three-coloring has opposite corners of the same color, and any such coloring of the corners can be extended to a valid three-coloring of the gadget. If we have four vertices in a graph, u, v, x, and y, and x $\rightarrow$ y intersects u $\rightarrow$ v, we can resolve that intersection by placing a gadget at the intersection. In our gadget, we would have x be the far left vertex of the gadget, and u be the top vertex. The right vertex of the gadget would point at node y and the bottom vertex of the gadget would point at node v. By arranging our gadget in that way we can maintain the three-coloring of the graph while also making an intersection planar. This gadget placement is shown in the image below.



So now that we know how to manage a single intersection, what happens when we have multiple intersections? If we have multiple intersections, we can chain crossings together. This preserves the three color and planar properties. Below is an example of gadget chaining.



An important thing to note is that given the graph with vertices on the perimeter of a circle, we must apply the gadget to the intersection closest to the edge of the circle first, and subsequently work inwards.

Lemma: 3-COL $\leq_p$ 4-COL $\leq_p$ 5-COL ... etc.

- Simply connect each node in an N-colorable graph to another node. This node must be of an (N+1)th color.

Yes or No Question: 4-COL $\leq_p$ 3-COL?

- Yes: 4-COL $\leq_p$ SAT $\leq_p$ 3-SAT $\leq_p$ 3-COL

Space Complexity

For a deterministic TM M, its space complexity is the function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) =$ max number of tape cells M uses on any input of length n.

For a non-deterministic TM N, $f(n) =$ max number of tape cells N uses on any computation path on any input of length n.

Definitions:

SPACE($f(n)$) = {all languages that can be decided by a deterministic TM of space complexity $f(n)$}

NPACE($f(n)$) = {the above, but with a non-deterministic TM}

PSPACE = $\cup$ SPACE(n^k)

NPSPACE = $\cup$ NSPACE(n^k)

Savitch's Theorem: For any $f(n) \geq n$, NSPACE($f(n)$) $\subseteq$ SPACE($f(n)^2$)

To prove this theorem, we construct a TM CANYIELD(C_1, C_2, t), where C_1 and C_2 are states in a non-deterministic turing machine, and t is a number of time steps. CANYIELD tests whether C_2 can be reached from C_1 in at most t steps. For any given computation path in a non-deterministic TM, a space complexity of $O(f(n))$ implies a time complexity $\leq C^{f(n)}$ where C is a constant.

We run CANYIELD($C_{start}, C_{accept}, C^{f(n)}$)

Example: CANYIELD(C_1, C_2, t) =

1. "If $t = 0$, ACCEPT only if $C_1 = C_2$
2. If $t = 1$, ACCEPT if $C_1 \to C_2$ in 1 step
3. Else for all configurations $C_m \leftarrow O(f(n))$ space
   - call CANYIELD($C_1, C_m, t/2$)
   - call CANYIELD($C_m, C_2, t/2$)
   - if both accept, ACCEPT
4. REJECT"

We will have $O(f(n))$ stack entries, with each entry at $log(C^{f(n)}) \to O(f(n))$ for each entry. This equates to $O(f(n)^2)$ overall. Thus there is a polynomial relationship between NSPACE($f(n)$) and SPACE($f(n)$).