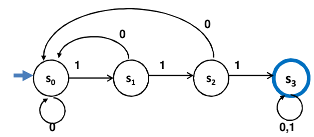
NFAs, Regular Expressions, and Equivalence with DFAs

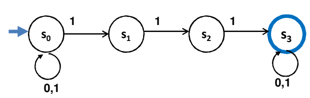
DFAs

Lemma: The language recognized by a DFA is the set of strings x that label some path from its start state to one of its final states



Nondeterministic Finite Automaton (NFA)

- Graph with start state, final states, edges labeled by symbols (like DFA) but
  - Not required to have exactly 1 edge out of each state labeled by each symbol - can have 0 or >1
  - Also can have edges labeled by empty string **ε**
- Definition: The language recognized by an NFA is the set of strings x that label some path from its start state to one of its final states



Three ways of thinking about NFAs

- Outside observer: Is there a path labeled by x from the start state to some final state?
- Perfect guesser: The NFA has input x and whenever there is a choice of what to do it magically guesses a good one (if one exists)
- Parallel exploration: The NFA computation runs all possible computations on x step-by-step at the same time in parallel

Design an NFA to recognize the set of binary strings that contain 111 or have an even # of 1's

NFAs and Regular Expressions

Theorem: For any set of strings (language) A described by a regular expression, there is an NFA that recognizes A.

Proof idea: Structural induction based on the recursive definition of regular expressions...

Note: One can also find a regular expression to describe the language recognized by any NFA but we won't prove that fact

Regular expressions over Σ

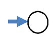
- Basis:
  - $\varnothing$, **ε** are regular expressions
  - ***a*** is a regular expression for any $a \in \Sigma$
- Recursive step:
  - If **A** and **B** are regular expressions then so are:
    - (**A** ∪ **B**)
    - (**AB**)
    - **A***

Basis

- Case $\varnothing$:
- Case **ε**:
- Case ***a***:

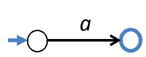
Basis

- Case $\varnothing$:



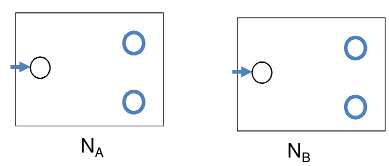
- Case **ε**:

- Case ***a***:



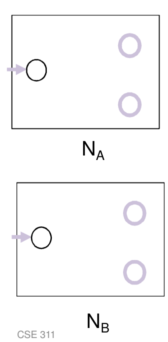
Inductive Hypothesis

- Suppose that for some regular expressions **A** and **B** there exist NFAs N_A and N_B such that N_A recognizes the language given by **A** and N_B recognizes the language given by **B**



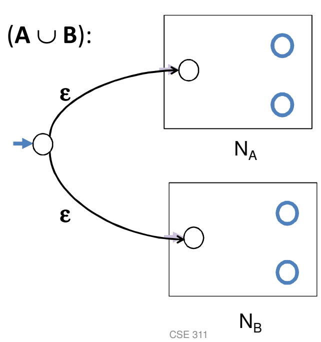
Inductive Step

- Case (**A** ∪ **B**):



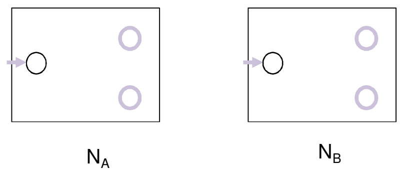
Inductive Step

- Case (**A** ∪ **B**):



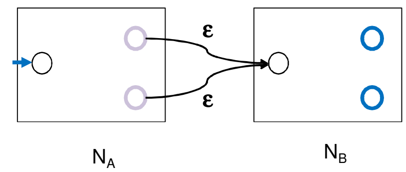
Inductive Step

- Case (**AB**):



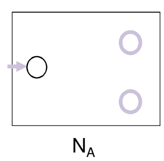
Inductive Step

- Case (**AB**):



Inductive Step

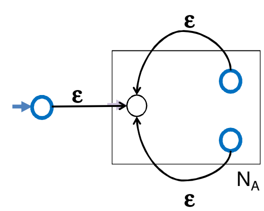
- Case **A***



□

Inductive Step

- Case **A***



□

NFAs and DFAs

Every DFA **is** an NFA

– DFAs have requirements that NFAs don't have

Can NFAs recognize more languages? No!

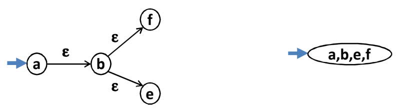
Theorem: For every NFA there is a DFA that recognizes exactly the same language

Conversion of NFAs to a DFAs

- Proof Idea:
  - The DFA keeps track of ALL the states that the part of the input string read so far can reach in the NFA
  - There will be one state in the DFA for each *subset* of states of the NFA that can be reached by some string

Conversion of NFAs to a DFAs

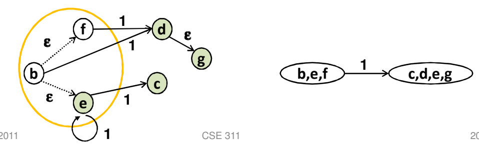
- New start state for DFA
  - The set of all states reachable from the start state of the NFA using only edges labeled λ



NFA DFA

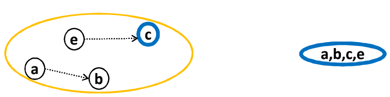
Conversion of NFAs to a DFAs

- For each state of the DFA corresponding to a set S of states of the NFA and each symbol **s**
  - Add an edge labeled **s** to state corresponding to T, the set of states of the NFA reached by
    - starting from some state in S, then
    - following one edge labeled by **s**, and
    - then following some number of edges labeled by ε
  - T will be ∅ if no edges from S labeled **s** exist



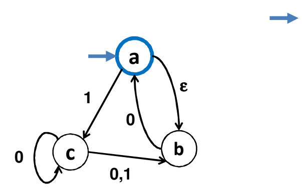
Conversion of NFAs to a DFAs

- Final states for the DFA
  - All states whose set contain some final state of the NFA



NFA DFA

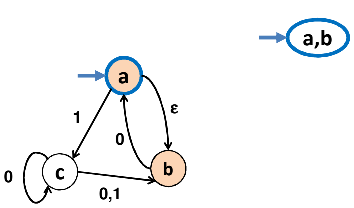
Example: NFA to DFA



NFA

DFA

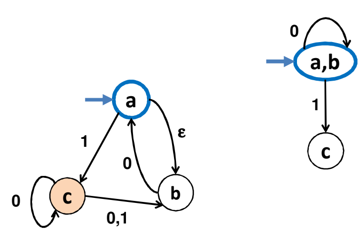
Example: NFA to DFA



NFA

DFA

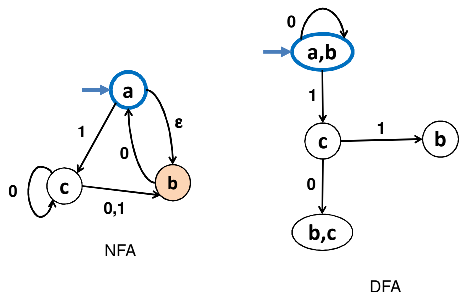
Example: NFA to DFA



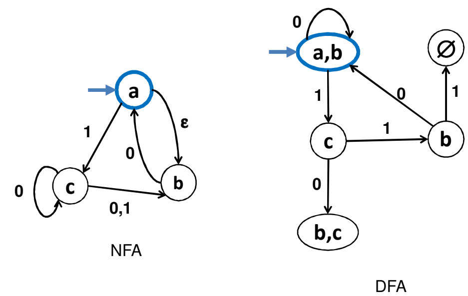
NFA

DFA

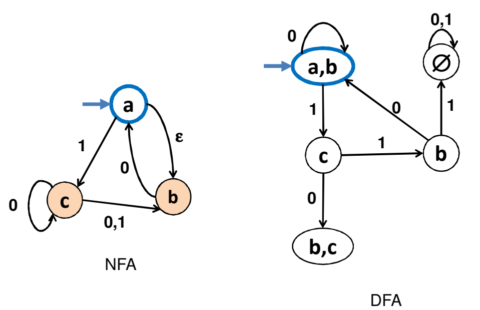
Example: NFA to DFA



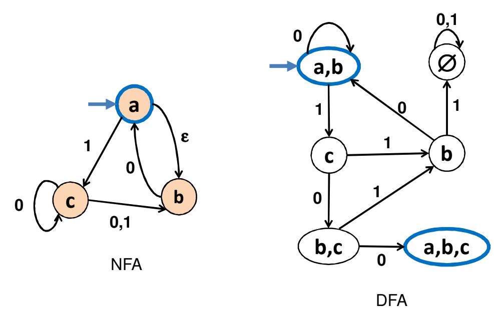
Example: NFA to DFA



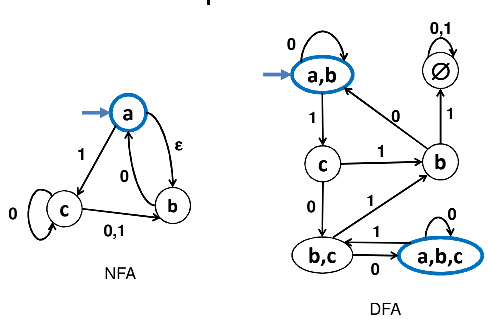
Example: NFA to DFA



Example: NFA to DFA



Example: NFA to DFA



Exponential blow-up in simulating nondeterminism

- In general the DFA might need a state for every subset of states of the NFA
  - Power set of the set of states of the NFA
  - n-state NFA yields DFA with at most 2^n states
  - An example where roughly 2^n is necessary
    - Is the (n-1)st char from the end a 1?
- The famous "P=NP?" question asks whether a similar blow-up is always necessary to get rid of nondeterminism for polynomial-time algorithms