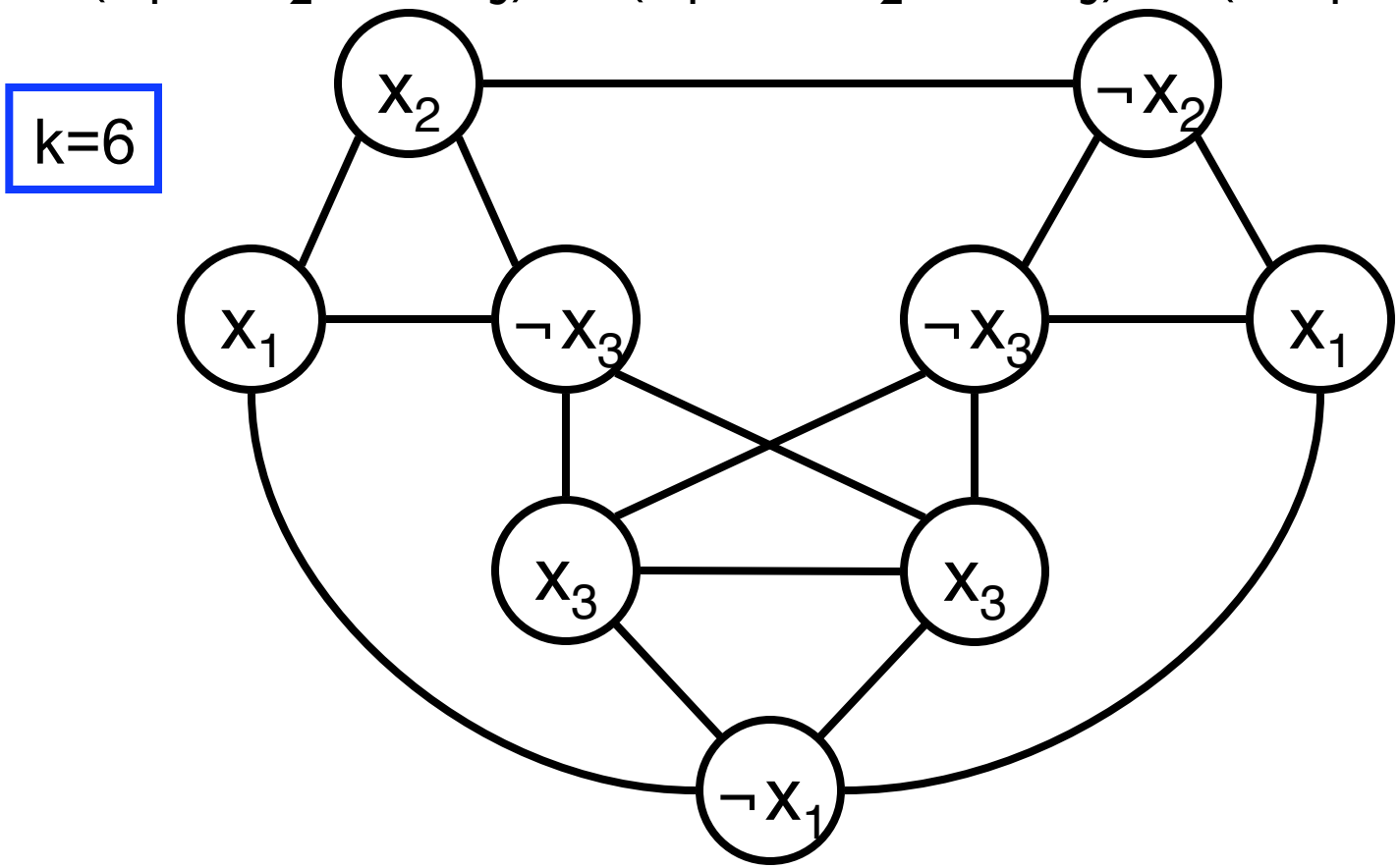
Lecture 22

3SAT $\leq_p$ VertexCover

$(x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3)$



3SAT $\leq_p$ VertexCover

f

3-SAT Instance:

- Variables: $x_1, x_2, \ldots$
- Literals: $y_{i,j}$, $1 \leq i \leq q$, $1 \leq j \leq 3$
- Clauses: $c_i = y_{i1} \vee y_{i2} \vee y_{i3}$, $1 \leq i \leq q$
- Formula: $c = c_1 \wedge c_2 \wedge \ldots \wedge c_q$

=

VertexCover Instance:

- $k = 2q$
- $G = (V, E)$
- $V = \{ [i,j] \mid 1 \leq i \leq q, 1 \leq j \leq 3 \}$
- $E = \{ ([i,j], [k,l]) \mid i = k \text{ or } y_{ij} = \neg y_{kl} \}$

Correctness of "3SAT $\leq_p$ VertexCover"

Summary of reduction function f: Given formula, make graph G with one group per clause, one node per literal. Connect each to all nodes in same group, plus complementary literals (x, ¬x). Output graph G plus integer k = 2 * number of clauses. Note: *f does not know whether formula is satisfiable or not; does not know if G has k-cover; does not try to find satisfying assignment or cover.*

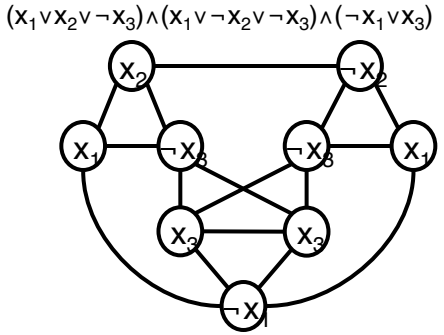
Correctness:

- Show f poly time computable: A key point is that graph size is polynomial in formula size; mapping basically straightforward.
- Show c in 3-SAT iff f(c)=(G,k) in VertexCover:

(⇒) Given an assignment satisfying c, pick one true literal per clause. Add other 2 nodes of each triangle to cover. Show it is a cover: 2 per triangle cover triangle edges; only true literals (but perhaps not all true literals) uncovered, so at least one end of every (x, ¬x) edge is covered.

(⇐) Given a k-vertex cover in G, uncovered labels define a valid (perhaps partial) truth assignment since no (x, ¬x) pair uncovered. It satisfies c since there is one uncovered node in each clause triangle (else some other clause triangle has > 1 uncovered node, hence an uncovered edge.)

Utility of "3SAT $\leq_p$ VertexCover"



Suppose we had a fast algorithm for VertexCover, then we could get a fast algorithm for 3SAT:

Given 3-CNF formula w, build Vertex Cover instance y = f(w) as above, run the fast VC alg on y; say "YES, w is satisfiable" iff VC alg says "YES, y has a vertex cover of the given size"

On the other hand, suppose no fast alg is possible for 3SAT, then we know none is possible for VertexCover either.

"3SAT $\leq_p$ VertexCover" Retrospective

Previous slide: two suppositions

Somewhat clumsy to have to state things that way.

Alternative: abstract out the key elements, give it a name ("polynomial time mapping reduction"), then properties like the above always hold.

Polynomial-Time Reductions

Definition: Let A and B be two problems.

We say that A is *polynomially (mapping) reducible* to B ($A \leq_p B$) if there exists a polynomial-time algorithm f that converts each instance x of problem A to an instance f(x) of B such that:

x is a YES instance of A iff f(x) is a YES instance of B

$$x \in A \quad \Leftrightarrow \quad f(x) \in B$$

Polynomial-Time Reductions (cont.)

Define: $A \leq_p B$ "A is polynomial-time reducible to B", iff there is a polynomial-time computable function f such that: $x \in A \Leftrightarrow f(x) \in B$

Why the notation?

"complexity of A" ≤ "complexity of B" + "complexity of f"

polynomial

(1) $A \leq_p B$ and $B \in P \Rightarrow A \in P$

(2) $A \leq_p B$ and $A \notin P \Rightarrow B \notin P$

(3) $A \leq_p B$ and $B \leq_p C \Rightarrow A \leq_p C$ (transitivity)

Two definitions of "$A \leq_p B$"

Some books use more general defn: "could solve A in poly time, *if* I had a poly time subroutine for B." Cook

Defn on previous slides is special case where you only get to call the subroutine once, and must report its answer. Karp

This special case is used in ~98% of all reductions (And is the only one used in Ch 7, I think.)

NP-Completeness

Definition: Problem B is *NP-hard* if every problem in NP is polynomially reducible to B.

Definition: Problem B is *NP-complete* if:

(1) B belongs to NP, and

(2) B is NP-hard.