Lecture 7

Reduction

"A is reducible to B" means I could solve A *if* I had a subroutine for B

Ex:

Finding the max element in a list is reducible to sorting

pf: sort the list in increasing order, take the last element

(A big hammer for a small problem, but never mind...)

The Halting Problem

$HALT_{TM}$ = { <M,w> | TM M halts on input w }

Theorem: The halting problem is undecidable

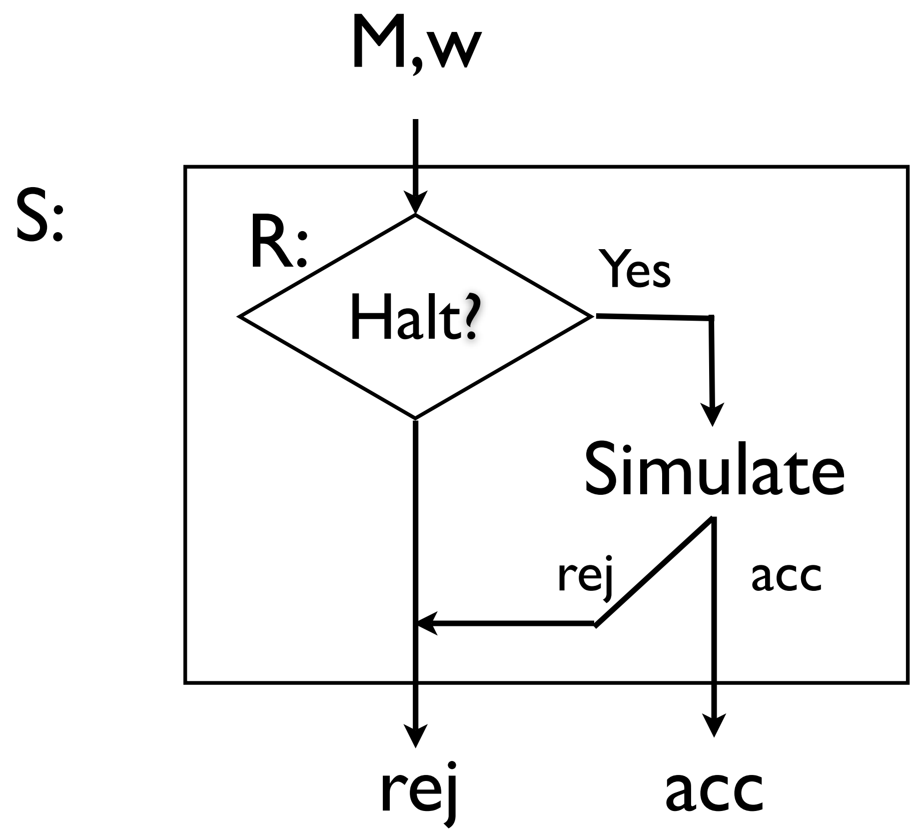
Proof:

A = A_{TM}, B = $HALT_{TM}$ Suppose I can reduce A to B. We already know A is undecidable, so must be that B is, too.

Suppose TM R decides $HALT_{TM}$. Consider S:

> On input <M,w>, run R on it. If it rejects, halt & reject; if it accepts, run M on w; accept/reject as it does.

Then S decides A_{TM}, which is impossible. R can't exist.


M,w
S:
R:
Halt?
Yes
Simulate
rej
acc
rej
acc

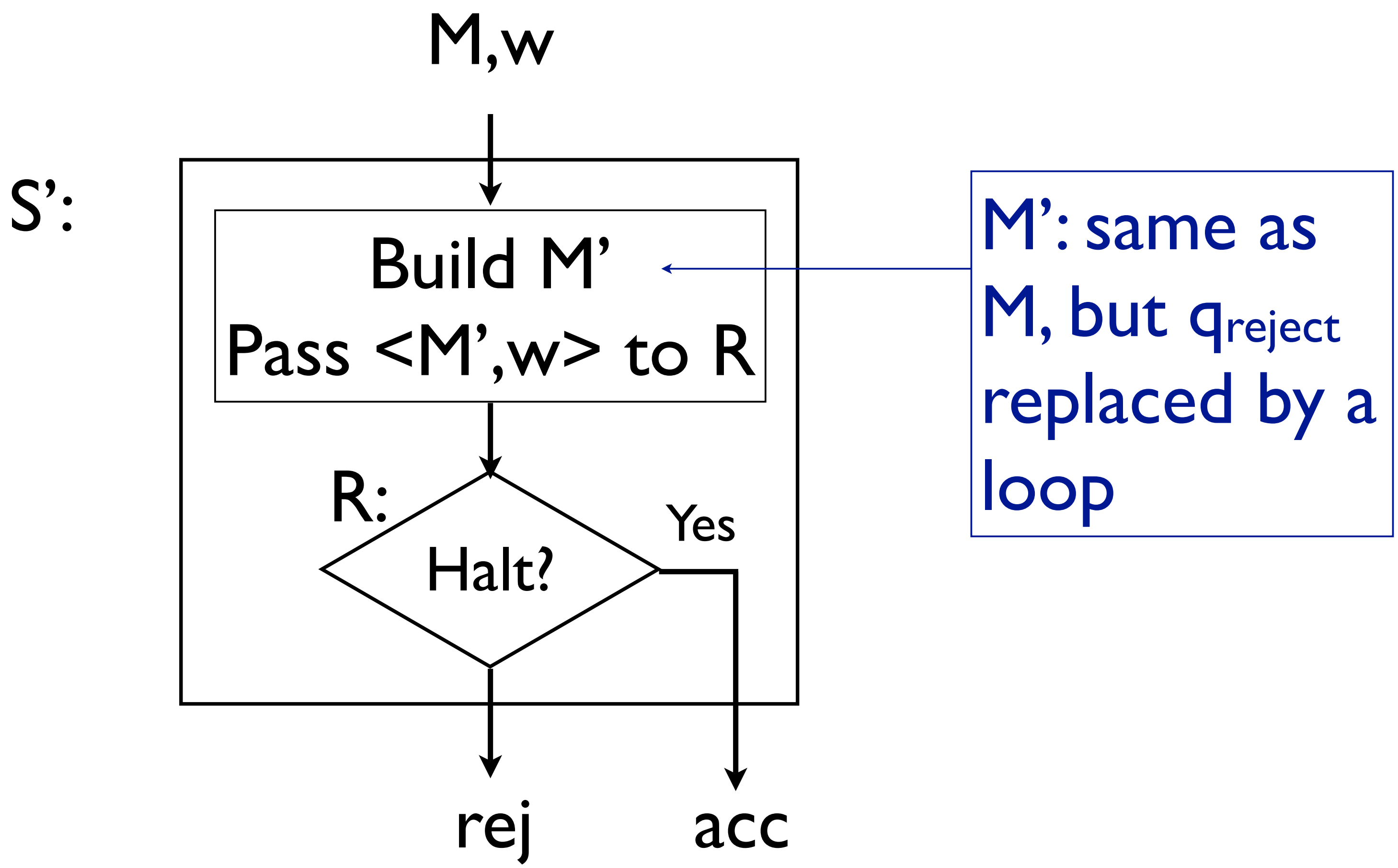
Another Way

Rather than running R on <M,w>, and manipulating that answer, manipulate the input to build a new M' so that R's answer about <M',w> *directly* answers the question of interest.

Specifically, build M' as a clone of M, but modified so that if M halts-and-rejects, M' instead rejects by looping.

Then halt/not-halt for M' == accept/reject for M

Again, this reduces A_{TM} to $HALT_{TM}$


M,w
S':
Build M'
Pass <M',w> to R
R:
Halt?
Yes
rej
acc
M': same as M, but q_{reject} replaced by a loop

Reduction

Notation (not in book, but common):

$A \leq_T B$ means "A is Turing Reducible to B"

I.e., if I had a TM deciding B, I could use it as a subroutine to solve A

Facts:

$A \leq_T B$ & B decidable implies A decidable (definition)

$A \leq_T B$ & A *un*decidable implies B undecidable (contrapositive)

$A \leq_T B$ & $B \leq_T C$ implies $A \leq_T C$

EMPTY_{TM} is undecidable

$\text{EMPTY}_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM s.t. } L(M) = \varnothing \}$

$\langle M, w \rangle \in A_{TM}$?

iff M accepts w ?

$L(M') = \emptyset$

M': erase input

write w

Run M

$$L(M') = \begin{cases} \emptyset & \text{if } M \text{ rejects } w \\ \Sigma^* & \text{if } M \text{ accept } w \end{cases}$$

$A_{TM} \leq_T \text{EMPTY}_{TM}$

$\text{REGULAR}_{\text{TM}}$ is undecidable

$\text{REGULAR}_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM s.t. } L(M) \text{ is regular} \}$

$A_{TM} \leq_T Reg_{TM}$

Given: $\langle M, w \rangle$ build M'
That

$\{0^n 1^n \mid n \geq 0\}$

M' on input x:
if $x \in \{0^n 1^n \mid n \geq 0\}$ accept
if not erase tape
write w,
simulate M on w