CSE 431: Theory of Computation (Spring 2008) **Student: You**
Problem Set #4 **Due: Thursday, May 1st**

Reading assignment: Chapter 5 of Sipser.

Instructions: Same as homework #1.

This problem set has **four** regular problems worth 10 points each, and one extra credit problem. Please be as careful as possible in your arguments and your answers.

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1. Show that there is an undecidable language $L \subseteq \{1\}^*$.

2. (Reduction of search problems to decision problems) Let $f : \Sigma_0^* \to \Sigma_0^*$ be an arbitrary function. Define a related language $L_f \subseteq \Sigma^*$ and describe a Turing machine to compute f using a machine that decides L_f. Also show how to decide L_f using a machine that computes f. The alphabet Σ *does not* have to be the same as Σ_0.

3. Tell whether the following languages are (a) decidable, (b) recognizable but not decidable, (c) co-recognizable but not decidable, or (d) neither recognizable or co-recognizable. Justify your answer.

   (a) $\{\langle M \rangle : \text{TM } M \text{ halts within 2008 steps on some input}\}$.

   (b) $\{\langle M \rangle : \text{TM } M \text{ halts within 2008 steps on every input}\}$.

4. Let $\mathcal{P}$ be a collection of Turing-recognizable languages. Suppose that there exists an infinite language $L \in \mathcal{P}$ such that no finite subset of L belongs to $\mathcal{P}$. In this case, prove that the language

$$\mathcal{P}_{\text{TM}} = \{\langle M \rangle : M \text{ is a TM and } L(M) \in \mathcal{P}\}$$

   is *not* Turing-recognizable.

5. **Extra credit.** Define a relation $R \subseteq (\Sigma^*)^k$ to be decidable if the language

$$L_R = \{\langle x_1, x_2, \ldots, x_k \rangle : (x_1, x_2, \ldots, x_k) \in R\}$$

   is decidable. Define Σ_k for $k \geq 0$ to be the class of all languages L for which there is a decidable $(k+1)$-ary relation R such that

$$L = \{x : \exists x_1 \forall x_2 \exists x_3 \cdots Q_k x_k \; R(x_1, x_2, \ldots, x_k, x)\},$$

   where the quantifier Q_k is $\exists$ if k is odd and $\forall$ if k is even. We define

$$\Pi_k = \text{co}\Sigma_k = \{L : \bar{L} \in \Sigma_k\}.$$

   In this notation, Σ_0 is the set of decidable languages, and Σ_1 is the set of Turing-recognizable languages. Finally, define

$$\text{ALL}_{\text{TM}} = \{\langle M \rangle : M \text{ is a TM and } L(M) = \Sigma^*\}.$$

(a) Prove that $\mathrm{ALL}_{\mathrm{TM}}$ is Π_2-complete, in the sense that (i) it belongs to Π_2 and (ii) every language $A \in \Pi_2$ mapping reduces to $\mathrm{ALL}_{\mathrm{TM}}$.

(b) You know how to prove that $\mathrm{ALL}_{\mathrm{TM}} \notin \Pi_1$ (by proving that $A_{\mathrm{TM}} \leq_m \mathrm{ALL}_{\mathrm{TM}}$). You don't have to do this. Instead, use part (a) to show that $\mathrm{ALL}_{\mathrm{TM}} \notin \Sigma_1$, i.e. $\mathrm{ALL}_{\mathrm{TM}}$ is not Turing-recognizable.