CSE 431: Introduction to Theory of Computation Spring 2005

PROBLEM SET 4

Due Friday, April 29, 2005, in class

Reading assignment: Sipser's book, Chapter 5.

Instructions: Same as for Problem set 1.

Each question is worth 10 points. Please be as clear and concise as possible in your arguments and answers. The optional problem is for extra credit.

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1. (Reduction of search to decision) Let $f : \Sigma^* \to \Sigma^*$ be an arbitrary function. You are asked to (i) defined a related language $L_f \subseteq \Gamma^*$, and (ii) describe a Turing machine to compute f using a Turing machine that decides L_f, and vice versa. Here Γ is an alphabet that may or may not be the same as Σ.

2. Tell whether the following languages are (a) decidable, (b) Turing-recognizable but not decidable, (c) co-Turing-recognizable but not decidable, or (d) neither Turing-recognizable nor co-Turing-recognizable. Justify your answer.

   (a) $\{\langle M \rangle \mid \text{TM } M \text{ halts within 2005 steps on some input}\}$.

   (b) $\{\langle M \rangle \mid \text{TM } M \text{ halts within 2005 steps on all inputs}\}$.

3. Prove that telling if the intersection of two context-free languages is empty is undecidable. Formally, prove that the language

$$\text{DISJOINT}_{\text{CFG}} = \{\langle G_1, G_2 \rangle \mid G_1, G_2 \text{ are context-free grammars and } L(G_1) \cap L(G_2) = \emptyset\}$$

   is undecidable.
   (Hint: Use the method of computation histories that we used to show that ALL_{CFG} is undecidable. Specifically, give two grammars the intersection of whose languages equals the set of accepting computation histories of TM M on input w.)

4. Let $\mathcal{P}$ be a property (subset) of Turing-recognizable languages. Suppose that there exists an infinite language $L \in \mathcal{P}$ such that no finite subset of L belongs to $\mathcal{P}$. Then, prove that the language

$$\mathcal{P}_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \in \mathcal{P}\}$$

   is not *Turing-recognizable.*

5. * **(Optional Problem)** Define a relation $R \subseteq (\Sigma^*)^k$ to be decidable if the language

$$L_R = \{\langle x_1, x_2, \ldots, x_k \rangle \mid (x_1, x_2, \ldots, x_k) \in R\}$$

   is decidable. Define Σ_k, for $k \geq 0$, to be the class of all languages L for which there is a decidable $(k+1)$-ary relation R such that

$$L = \{x \mid \exists x_1 \forall x_2 \cdots Q_k x_k \; R(x_1, x_2, \ldots, x_k, x)\} \; ,$$

where the quantifier Q_k is $\exists$ if k is odd and $\forall$ if k is even. We define $\Pi_k = \mathrm{co}\Sigma_k$, i.e. Π_k is the set of all complements of languages in Σ_k.

In this notation, Σ_0 is the set of decidable languages, and Σ_1 is the set of Turing-recognizable languages. Now to your exercises:

(a) Prove that ALL_{TM} is "complete" for Π_2 in the sense that (i) it belongs to Π_2 and (ii) every language $A \in \Pi_2$ mapping reduces to ALL_{TM}.

(b) * In class, we showed that $ALL_{\mathrm{TM}} \notin \Pi_1$ (by proving $A_{\mathrm{TM}} \leq_m ALL_{\mathrm{TM}}$). Use part (a) above to show that $ALL_{\mathrm{TM}} \notin \Sigma_1$, i.e., ALL_{TM} is not Turing-recognizable.