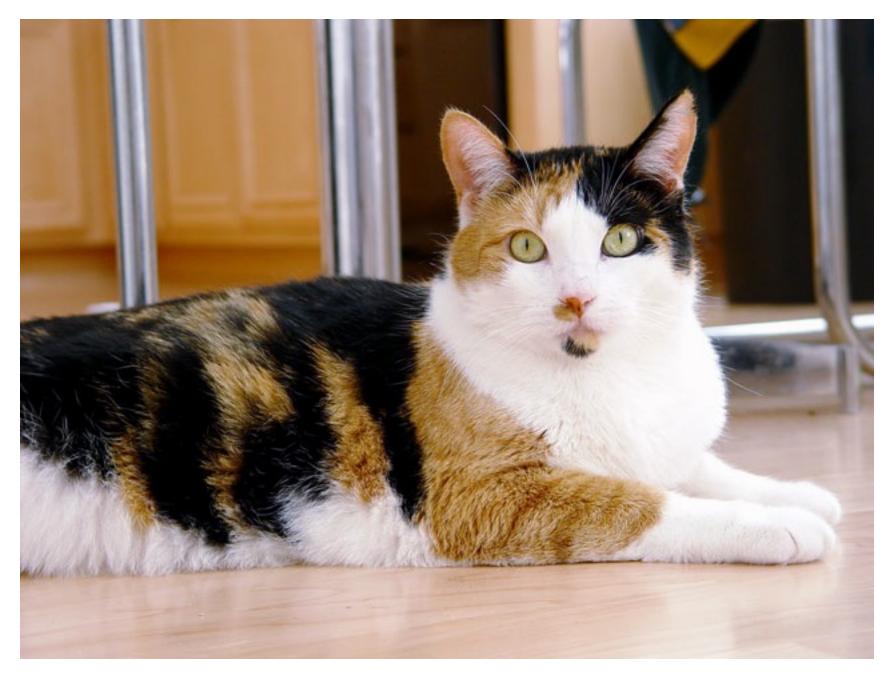
#### **CSE 427**

## Markov Models and Hidden Markov Models



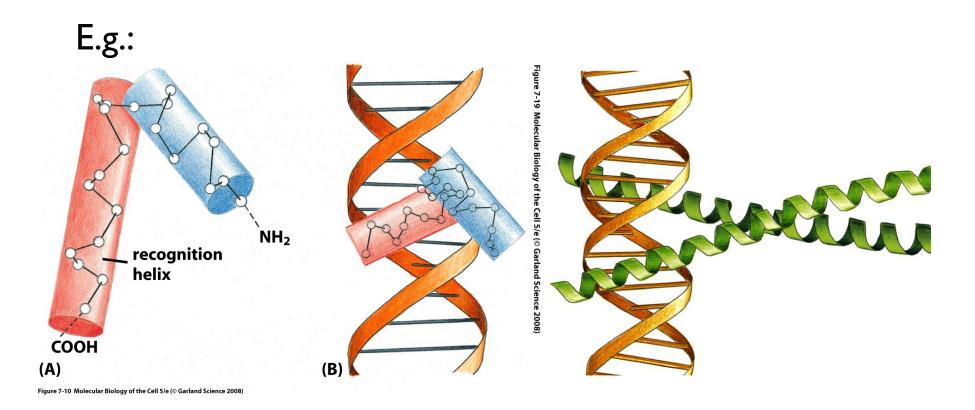
## Dosage Compensation and X-Inactivation

2 copies (mom/dad) of each chromosome I-22
Mostly, both copies of each gene are expressed
E.g., A B O blood group defined by 2 alleles of I gene
Women (XX) get double dose of X genes (vs XY)?
So, early in embryogenesis:

- One X randomly inactivated in each cell How?
- Choice maintained in daughter cells

Calico: a major coat color gene is on X

#### Reminder: Proteins "Read" DNA



# Down in the Groove

Different patterns of hydrophobic methyls, potential H bonds, etc. at edges of different base pairs. They're accessible, esp. in major groove

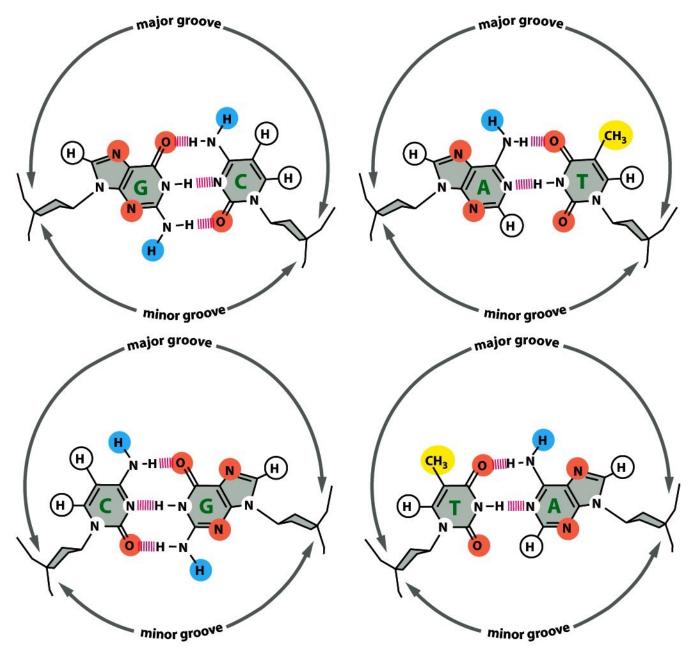


Figure 7-7 Molecular Biology of the Cell 5/e (© Garland Science 2008)

### **DNA** Methylation

CpG - 2 adjacent nts, same strand (not Watson-Crick pair; "p" mnemonic for the phosphodiester bond of the DNA backbone)

C of CpG is often (70-80%) methylated in mammals i.e., CH<sub>3</sub> group added (both strands)

cytosine

## Same Pairing

Methyl-C alters major groove profile (∴ TF binding), but not base-pairing, transcription or replication

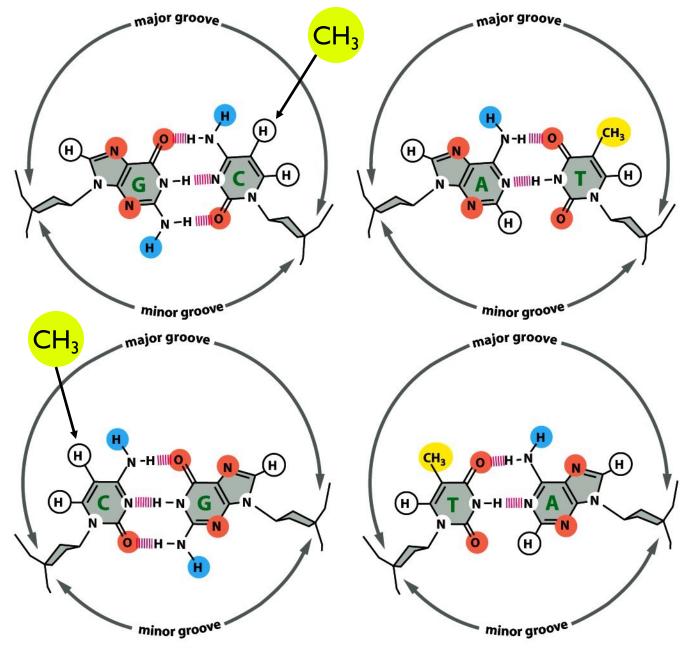


Figure 7-7 Molecular Biology of the Cell 5/e (© Garland Science 2008)

#### Calico Cats



http://upload.wikimedia.org/wikipedia/commons/b/ba/Calico cat

Calico cat story: patchwork coat-color in some female cats partially explained by X-inactivation

And heavily methylating the inactive X is part of the mechanism of X-inactivation

And methylation is broadly important for other reasons, and sculpts the genome...

### DNA Methylation—Why

In vertebrates, it generally silences transcription

(Epigenetics) X-inactivation, imprinting, repression of mobile elements, cancers, aging, and developmental differentiation

E.g., if a stem cell divides, one daughter fated to be liver, other kidney, need to

NH<sub>2</sub>
N
N
N
H

cytosine

- (a) Turn off liver genes in kidney & vice versa, and
- (b) Remember that through subsequent cell divisions

#### How? One way:

- (a) Methylate genes, esp. promoters, to silence them
- (b) After ÷, DNA methyltransferases convert hemi- to fully-methylated (not trivial: deleting methyltransferase is embrionic-lethal in mice)

Major exception: promoters of "housekeeping" genes

### "CpG Islands"

Methyl-C mutates to T relatively easily Net: CpG is less common than expected genome-wide: f(CpG) < f(C)\*f(G)

BUT in some regions (e.g. active promoters), CpGs remain unmethylated, so CpG → TpG less likely there: makes "CpG Islands"; they often mark gene-rich regions

cytosine

thymine

### CpG Islands

#### CpG Islands

More CpG than elsewhere (say, CpG/GpC>50%)

More C & G than elsewhere, too (say, C+G>50%)

Typical length: few 100 to few 1000 bp

#### Questions

Is a short sequence (say, 200 bp) a CpG island or not? Given long sequence (say, 10-100kb), find CpG islands?

## Markov & Hidden Markov Models

References (see also online reading page):

Eddy, "What is a hidden Markov model?" Nature Biotechnology, 22, #10 (2004) 1315-6.

Durbin, Eddy, Krogh and Mitchison, "Biological Sequence Analysis", Cambridge, 1998 (esp. chs 3, 5)

Rabiner, "A Tutorial on Hidden Markov Models and Selected Application in Speech Recognition," Proceedings of the IEEE, v 77 #2,Feb 1989, 257-286

### Independence

A key issue: Previous models we've talked about assume *independence* of nucleotides in different positions - definitely unrealistic.

Markov models allow us to relax that assumption.

#### Markov Chains

A sequence  $x_1, x_2, \ldots$  of random variables is a k-th order Markov chain if, for all i, i<sup>th</sup> value is independent of all but the previous k values:

$$P(x_i \mid \underbrace{x_1, x_2, \dots, x_{i-1}}_{\text{i-l}}) = P(x_i \mid \underbrace{x_{i-k}, x_{i-k+1}, \dots, x_{i-1}}_{\text{k typically } \ll \text{i-l}})$$

Example I: Uniform random ACGT

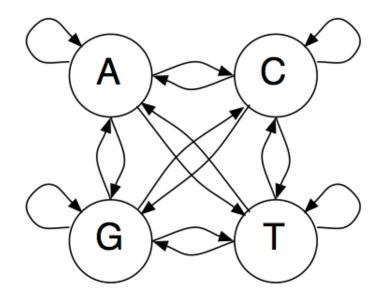
Example 2: Weight matrix model

Example 3: ACGT, but \( \primeter \text{Pr(G following C)} \)

} 0th
order

} Ist
order

#### A Markov Model (1st order)

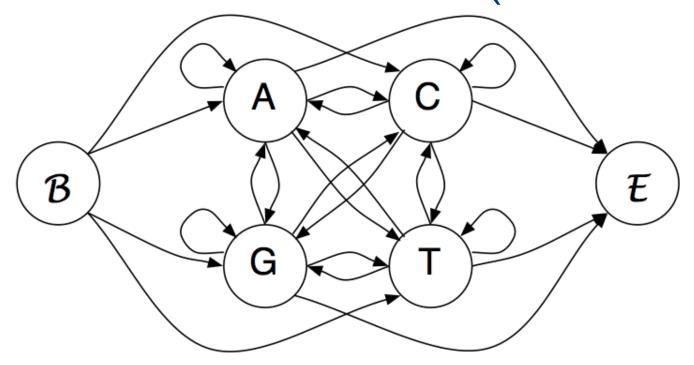


States: A,C,G,T

Emissions: corresponding letter

Transitions:  $a_{st} = P(x_i = t \mid x_{i-1} = s)$  — Ist order

#### A Markov Model (1st order)



States: A,C,G,T

Emissions: corresponding letter

Transitions:  $a_{st} = P(x_i = t \mid x_{i-1} = s)$ 

Begin/End states

## Pr of emitting sequence x

$$\begin{array}{lll} x & = & x_1 \; x_2 \; \dots \; x_n \\ P(x) & = & P(x_1, x_2, \dots, x_n) > \lim_{\substack{\text{of Probability} \\ \text{c'chain}}} \\ & = & P(x_1) \cdot P(x_2 \mid x_1) \cdots P(x_n \mid x_{n-1}, \dots, x_1) \\ & = & P(x_1) \cdot P(x_2 \mid x_1) \cdots P(x_n \mid x_{n-1}) > \lim_{\substack{\text{if NST, NC} \\ \text{order}}} \\ & = & P(x_1) \prod_{i=1}^{n-1} a_{x_i, x_{i+1}} \\ & = & \prod_{i=0}^{n-1} a_{x_i, x_{i+1}} \quad \text{(with Begin state)} \end{array}$$

## Training

Max likelihood estimates for transition probabilities are just the frequencies of transitions when emitting the training sequences

E.g., from 48 CpG islands in 60k bp:

+	A	C	G	т	_	A	C	G	Т
A	0.180	0.274	0.426	0.120	A	0.300	0.205	0.285	0.210
C	0.171	0.368	0.274	0.188	C	0.322	0.298	0.078	0.302
G	0.161	0.339	0.375	0.125	G	0.248	0.246	0.298	0.208
т	0.079	0.355	0.384	0.182	т	0.177	0.239	0.292	0.292
								_	DEIA

#### Discrimination/Classification

Log likelihood ratio of CpG model vs background model

$$S(x) = \log \frac{P(x|+\text{model})}{P(x|-\text{model})} = \sum_{i=1}^{L} \log \frac{a_{x_{i-1},x_i}^+}{a_{x_{i-1},x_i}^-} = \sum_{i=1}^{L} \log \beta_{x_{i-1},x_i}$$
 | Prev slide |

### CpG Island Scores

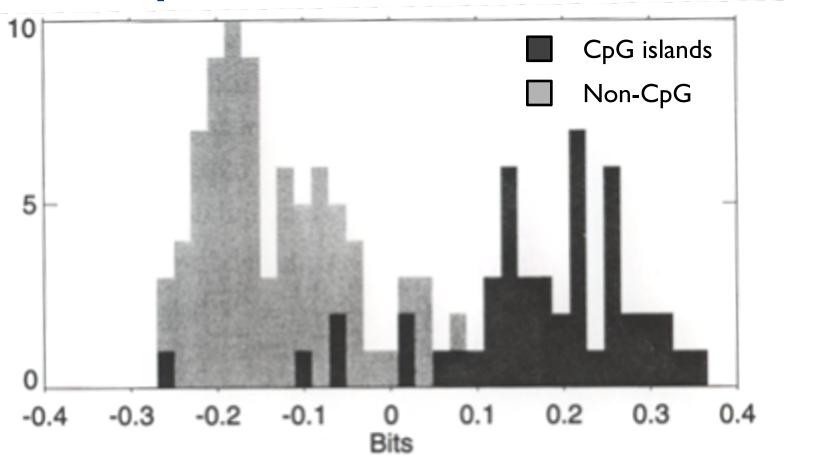
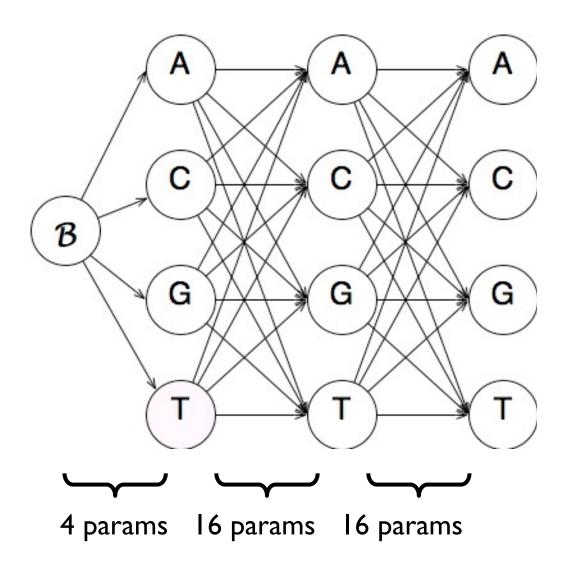


Figure 3.2 Histogram of length-normalized scores.

#### Aside: Ist Order "WMM"



#### Questions

Q1: Given a *short* sequence, is it more likely from feature model or background model? Above

Q2: Given a *long* sequence, where are the features in it (if any)

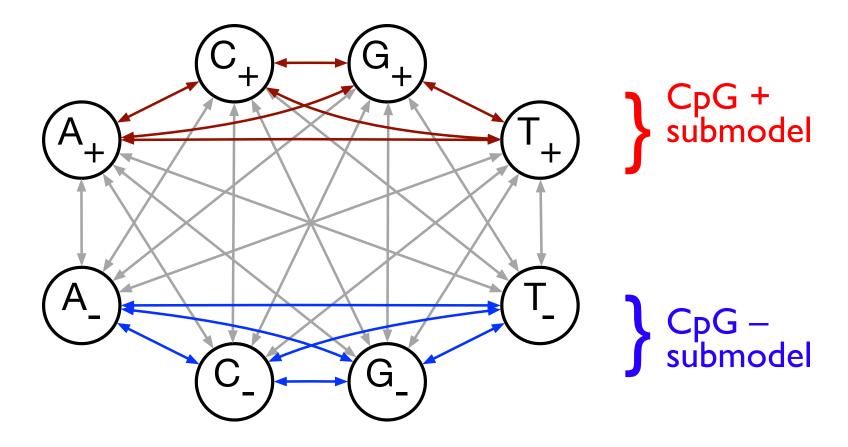
Approach I: score 100 bp (e.g.) windows

Pro: simple

Con: arbitrary, fixed length, inflexible

Approach 2: combine +/- models.

#### Combined Model



Emphasis is "Which (hidden) state?" not "Which model?"

#### Hidden Markov Models

(HMMs; Claude Shannon, 1948)

States:  $1, 2, 3, \ldots$ 

Paths: sequences of states  $\pi = (\pi_1, \pi_2, ...)$ 

Transitions:  $a_{k,l} = P(\pi_i = l \mid \pi_{i-1} = k)$ 

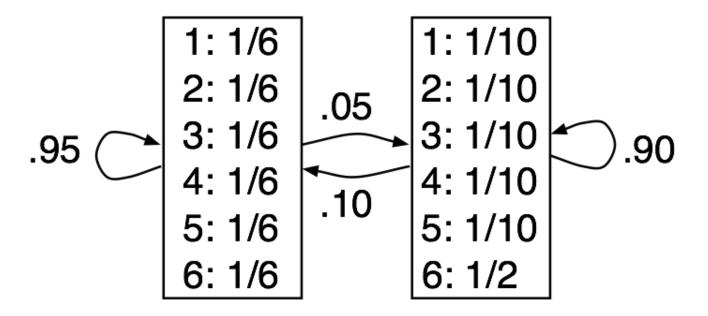
Emissions:  $e_k(b) = P(x_i = b \mid \pi_i = k)$ 

Observed data: emission sequence

Hidden data: state/transition sequence

## The Occasionally Dishonest Casino

1 fair die, 1 "loaded" die, occasionally swapped



Rolls 31511624644664424531132163116415213362514454363165662656666 Die Rolls 651166453132651245636664631636663162326455236266666625151631 Die Rolls 222555441666566563564324364131513465146353411126414626253356 Die Rolls 366163666466232534413661661163252562462255265252266435353336 Die Rolls 2331216253644144323351632436336655624666626326666612355245242 Die 

#### Figure 3.5

Rolls: Visible data-300 rolls of a die as described above.

Die: Hidden data—which die was actually used for that roll (F = fair, L = loaded).

Viterbi: the prediction by the Viterbi algorithm is shown.

### Inferring hidden stuff

Joint prob of a given path  $\pi$  & emission sequence x:

$$P(x,\pi) = a_{0,\pi_1} \prod_{i=1}^{n} e_{\pi_i}(x_i) \cdot a_{\pi_i,\pi_{i+1}}$$

But  $\pi$  is hidden; what to do? Some alternatives:

Most probable single path

$$\pi^* = \arg\max_{\pi} P(x,\pi)$$

Sequence of most probable states

$$\hat{\pi}_i = \arg\max_k P(\pi_i = k \mid x)$$

Etc.

#### Notation:

 $\frac{\text{max}_{x}F(x)}{\text{arg max}_{x}F(x)}$  = the maximum y-value attained by F()

# The Viterbi Algorithm: The most probable path

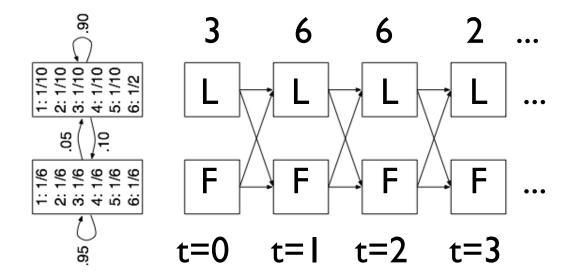
Viterbi finds:  $\pi^* = \arg \max_{\pi} P(x, \pi)$ 

Possibly there are 10<sup>99</sup> paths of prob 10<sup>-99</sup> (If so, non-Viterbi approaches may be preferable.)

More commonly, one path (+ slight variants) dominate others; Viterbi finds that

Key problem: exponentially many paths  $\pi$ 

## Unrolling an HMM



Conceptually, sometimes convenient Note exponentially many paths

#### Viterbi

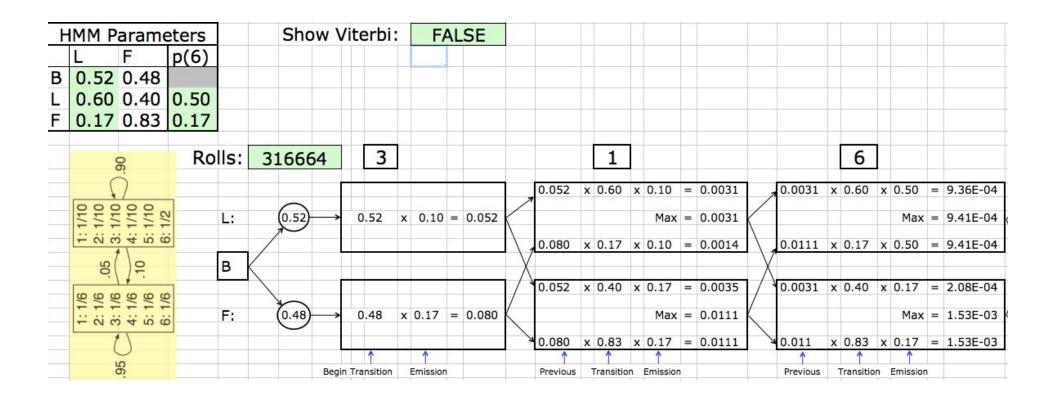
 $v_l(i) =$  probability of the most probable path emitting  $x_1, x_2, \ldots, x_i$  and ending in state  $\ell$ 

#### Initialize:

$$v_l(0) = \left\{ egin{array}{lll} 1 & ext{if } l = B ext{egin state} & \longrightarrow & 1 & \cdots & 1 & 1 & 1 \\ 0 & ext{otherwise} & & & 2 & \cdots & 2 & 2 & 2 \end{array} \right.$$

#### General case:

#### **HMM Casino Example**



(Excel spreadsheet on web; download & play...)

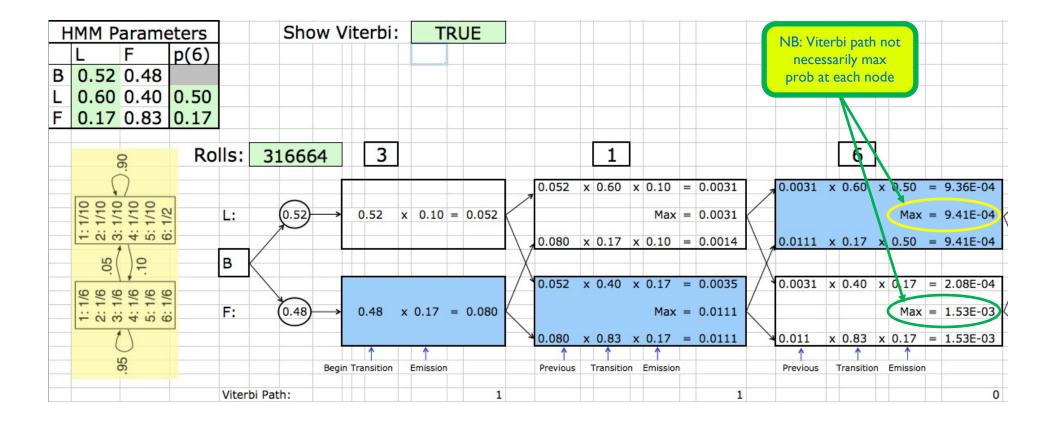
#### Viterbi Traceback

Above finds *probability* of best path

To find the path itself, trace *backward* to the state *k* attaining the max at each stage

$$v_l(i+1) = e_l(x_{i+1}) \cdot \max_k(v_k(i) a_{k,l})$$

#### **HMM Casino Example**



(Excel spreadsheet on web; download & play...)

Rolls 31511624644664424531132163116415213362514454363165662656666 Die Rolls 651166453132651245636664631636663162326455236266666625151631 Die Rolls 222555441666566563564324364131513465146353411126414626253356 Die Rolls 366163666466232534413661661163252562462255265252266435353336 Die Rolls 2331216253644144323351632436336655624666626326666612355245242 Die 

#### Figure 3.5

Rolls: Visible data-300 rolls of a die as described above.

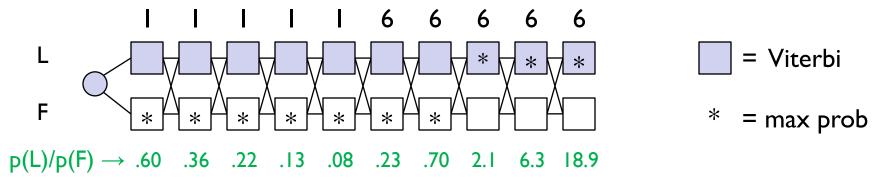
Die: Hidden data—which die was actually used for that roll (F = fair, L = loaded).

Viterbi: the prediction by the Viterbi algorithm is shown.

## Most probable path ≠ Sequence of most probable states

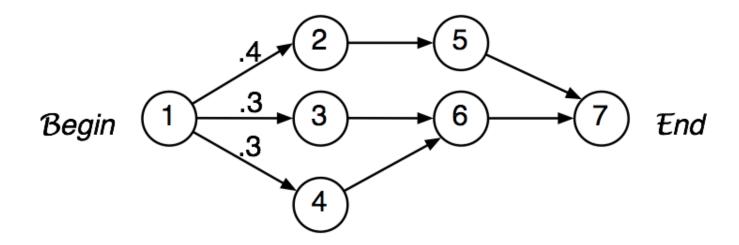
Another example, based on casino dice again:

Suppose p(fair  $\leftrightarrow$  loaded) transitions are  $10^{-99}$  and roll sequence is IIIII66...666; then fair state is more likely all through I's & into the run of 6's, but eventually loaded wins, and the improbable  $F \rightarrow L$  transitions make Viterbi = all L.



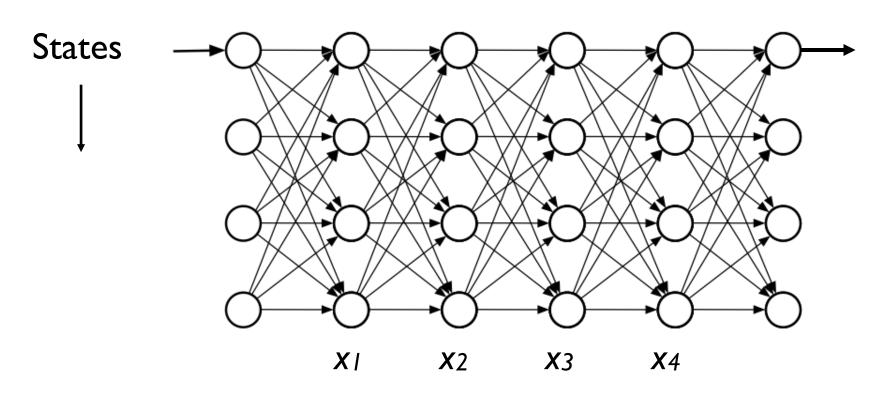
#### Is Viterbi "best"?

Viterbi finds  $\pi^* = \arg \max_{\pi} P(x, \pi)$ 



Most probable (Viterbi) path goes through 5, but most probable state at 2nd step is 6 (l.e., Viterbi is not the only interesting answer.)

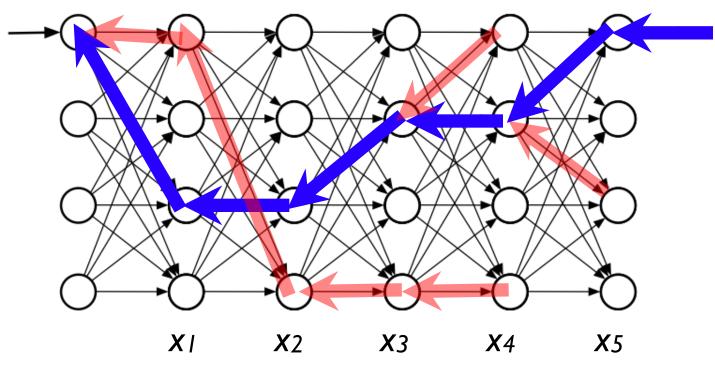
# An HMM (unrolled)



Emissions/sequence positions \_\_\_\_\_

### Viterbi: best path to each





Viterbi score:

$$v_l(i+1) = e_l(x_{i+1}) \cdot \max_k(v_k(i) \, a_{k,l})$$

Viterbi path<sup>R</sup>:

$$back_l(i+1) = \arg\max_k(v_k(i) \, a_{k,l})$$

### Another Q: What's P(x)?

Given an HMM and a sequence x, Viterbi finds the single path  $\pi$  having the greatest probability of emitting x (and implicitly finds that probability  $P(x, \pi)$ )

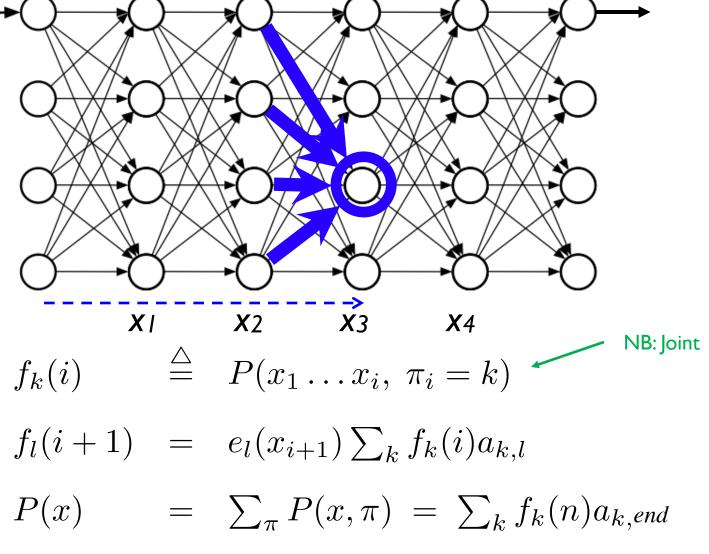
What if I don't care about  $\pi$ ? E.g., what is the probability P(x) of emitting x, on some path?

Of course,  $P(x) = \sum_{\pi} P(x, \pi)$ , i.e. sum over all paths, but exponentially many, so nontrivial ...

Answer to this and related Qs is easiest to think about by focusing on intermediate states

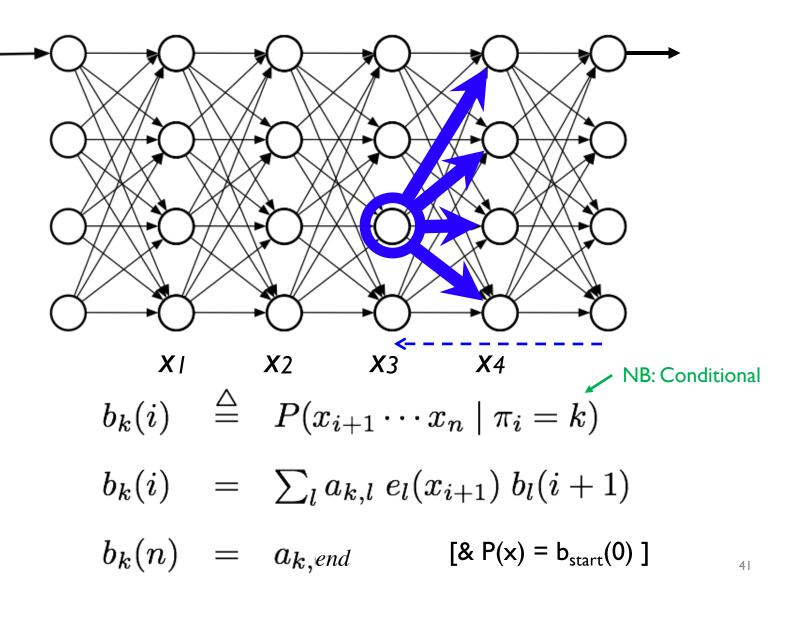
# The Forward Algorithm

For each state/time, want total probability of all paths leading to it, with previous emissions



### The Backward Algorithm

Similar: for each state/time, want total probability of all paths from it, with subsequent emissions, conditional on that state.



# In state k at step i?

$$P(x, \pi_i = k)$$

$$= P(x_1, \dots, x_i, \pi_i = k) \cdot P(x_{i+1}, \dots, x_n \mid x_1, \dots, x_i, \pi_i = k)$$

$$= P(x_1, \dots, x_i, \pi_i = k) \cdot P(x_{i+1}, \dots, x_n \mid \pi_i = k)$$

$$= f_k(i) \cdot b_k(i)$$

$$P(\pi_i = k \mid x) = \frac{P(x, \pi_i = k)}{P(x)} = \frac{f_k(i) \cdot b_k(i)}{P(x)}$$

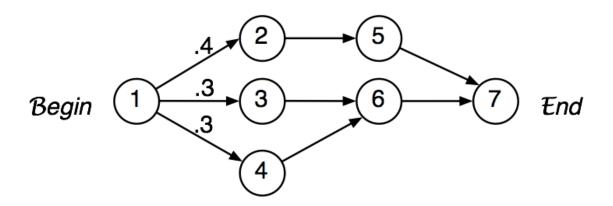
### Posterior Decoding, I

Alternative 1: what's the most likely state at step i?

$$\hat{\pi}_i = \arg\max_k P(\pi_i = k \mid x)$$

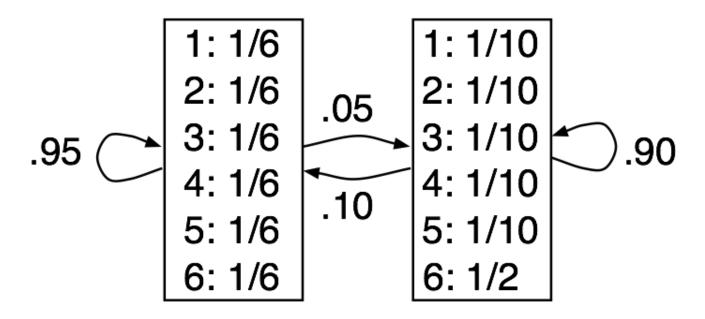
Note: the most likely sequence of states (a path) ≠ the sequence of most likely states.

That may even be an illegal path! (E.g. 1,2,6,7 below)



# The Occasionally Dishonest Casino

1 fair die, 1 "loaded" die, occasionally swapped



Rolls 31511624644664424531132163116415213362514454363165662656666 Die Rolls 651166453132651245636664631636663162326455236266666625151631 Die Rolls 222555441666566563564324364131513465146353411126414626253356 Die Rolls 366163666466232534413661661163252562462255265252266435353336 Die Rolls 2331216253644144323351632436336655624666626326666612355245242 Die 

#### Figure 3.5

Rolls: Visible data-300 rolls of a die as described above.

Die: Hidden data—which die was actually used for that roll (F = fair, L = loaded).

Viterbi: the prediction by the Viterbi algorithm is shown.

# Posterior Decoding

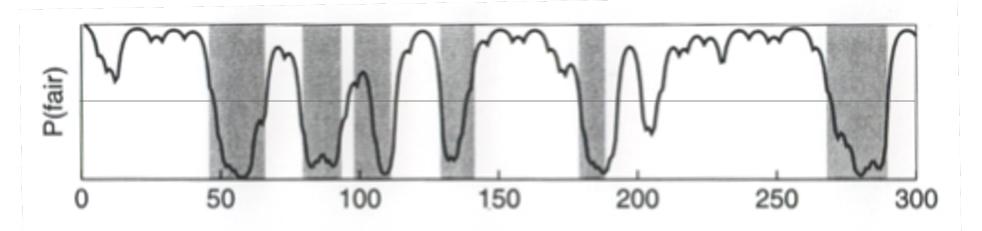


Figure 3.6 The posterior probability of being in the state corresponding to the fair die in the casino example. The x axis shows the number of the roll. The shaded areas show when the roll was generated by the loaded die.

### Posterior Decoding, II

Alternative 1: what's most likely state at step i?

$$\hat{\pi}_i = \arg\max_k P(\pi_i = k \mid x)$$

Alternative 2: given some function g(k) on states, what's its expectation. E.g., what's probability of "+" model in CpG HMM (g(k)=1) iff k is "+" state)?

$$G(i \mid x) = \sum_{k} P(\pi_i = k \mid x) \cdot g(k)$$

# CpG Islands again

Data: 41 human sequences, totaling 60kbp, including 48 CpG islands of about 1kbp each

Viterbi: Post-process:

Found 46 of 48 46/48

plus 121 "false positives" 67 false pos

Posterior Decoding:

same 2 false negatives 46/48

plus 236 false positives 83 false pos

Post-process: merge within 500; discard < 500

# Training

Given model topology & training sequences, learn transition and emission probabilities

If  $\pi$  known, then MLE is just frequency observed in training data

$$a_{k,l} = rac{ ext{count of } k o l ext{ transitions}}{ ext{count of } k o anywhere transitions} \leftarrow e_k(b) = \dots$$

If  $\pi$  hidden, then use EM:

given  $\theta$ , estimate  $\pi$ ; given  $\pi$  estimate  $\theta$ ; repeat

pseudocounts?

### Viterbi Training

given  $\theta$ , estimate  $\pi$ ; given  $\pi$  estimate  $\theta$ ; repeat

Make initial estimates of parameters  $\theta$  Find Viterbi path  $\pi$  for each training sequence Count transitions/emissions on those paths, getting new  $\theta$  Repeat

Not rigorously optimizing desired likelihood, but still useful & commonly used.

(Arguably good if you're doing Viterbi decoding.)

(And see note about "classification EM," ~#45 in MLE-EM slides.)

### Baum-Welch Training

EM: given  $\theta$ , estimate  $\pi$  ensemble; then re-estimate  $\theta$ 

$$P(\pi_{i} = k, \, \pi_{i+1} = l \mid x, \theta)$$

$$= \frac{f_{k}(i \mid \theta) \, a_{k,l} \, e_{l}(x_{i+1}) \, b_{l}(i+1 \mid \theta)}{P(x \mid \theta)}$$

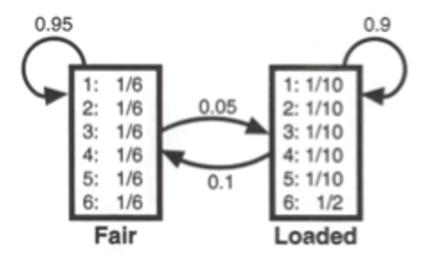
Estimated # of k o l transitions  $\hat{A}_{k,l}$  on set of seqs  $\mathsf{x}^\mathsf{j}$ 

$$=\sum_{\text{training seqs }x^{j}}\sum_{i}P(\pi_{i}=k,\,\pi_{i+1}=l\mid x^{j},\theta)$$

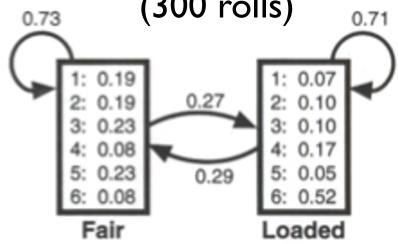
New estimate 
$$\hat{a}_{k,l} = \frac{\hat{A}_{k,l}}{\sum_{l} \hat{A}_{k,l}}$$

Emissions: similar

#### True Model



B-W Learned Model (300 rolls)



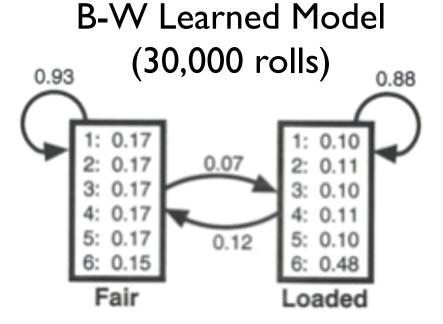
Log-odds (vs all F) per roll

True model 0.101 bits

300-roll est. 0.097 bits

30k-roll est. 0.100 bits

(NB: overestimated)



### HMMs in Action: Pfam

### http://pfam.xfam.org

Proteins fall into families, across & within species

Ex: Globins, Zinc fingers, Leucine zippers, GPCRs, ...

Identifying family very useful: suggests function, etc.

So, search & alignment are both important

Q. Why not just use Blast/Smith-Waterman?

A. There is more info in *multiple* examples (e.g., psiBLAST)

One very successful approach: profile HMMs

```
Helix
                     AAAAAAAAAAAAAA
                                        HBA HUMAN
              -----VLSPADKTNVKAAWGKVGA--HAGEYGAEALERMFLSFPTTKTYFPHF
HBB_HUMAN
               ----VHLTPEEKSAVTALWGKV----NVDEVGGEALGRLLVVYPWTORFFESF
MYG_PHYCA
               ----VLSEGEWQLVLHVWAKVEA--DVAGHGQDILIRLFKSHPETLEKFDRF
GLB3_CHITP -----LSADQISTVQASFDKVKG-----DPVGILYAVFKADPSIMAKFTQF
GLB5_PETMA PIVDTGSVAPLSAAEKTKIRSAWAPVYS--TYETSGVDILVKFFTSTPAAQEFFPKF
LGB2_LUPLU -----GALTESQAALVKSSWEEFNA--NIPKHTHRFFILVLEIAPAAKDLFS-F
GLB1_GLYDI -----GLSAAQRQVIAATWKDIAGADNGAGVGKDCLIKFLSAHPQMAAVFG-F
Consensus
                    Ls.... vaWkv. .
                                            a . L., f . P .
Helix
              DDDDDDDEEEEEEEEEEEEEEE
                                                     FFFFFFFFFFF
HBA HUMAN
          -DLS-----HGSAQVKGHGKKVADALTNAVAHV---D--DMPNALSALSDLHAHKL-
HBB_HUMAN
          GDLSTPDAVMGNPKVKAHGKKVLGAFSDGLAHL---D--NLKGTFATLSELHCDKL-
MYG_PHYCA
          KHLKTEAEMKASEDLKKHGVTVLTALGAILKK----K-GHHEAELKPLAOSHATKH-
GLB3_CHITP_AG-KDLESIKGTAPFETHANRIVGFFSKIIGEL--P---NIEADVNTFVASHKPRG-
GLB5_PETMA KGLTTADQLKKSADVRWHAERIINAVNDAVASM--DDTEKMSMKLRDLSGKHAKSF-
LGB2_LUPLU LK-GTSEVPQNNPELQAHAGKVFKLVYEAAIQLQVTGVVVTDATLKNLGSVHVSKG-
GLB1_GLYDI SG----AS---DPGVAALGAKVLAQIGVAVSHL--GDEGKMVAQMKAVGVRHKGYGN
Consensus
                   ... v...Hg kv. a a....l
                                                 . a 1. 1
Helix
           FFGGGGGGGGGGGGGG
                                    НИНИНИНИНИНИНИНИНИНИНИНИНИ
HBA HUMAN
           -RVDPVNFKLLSHCLLVTLAAHLPAEFTPAVHASLDKFLASVSTVLTSKYR
HBB HUMAN
           HVDPENFRLLGNVLVCVLAHHFGKEFTPPVOAAYOKVVAGVANALAHKYH
           KIPIKYLEFISEAIIHVLHSRHPGDFGADAQGAMNKALELFRKDIAAKYKELGYQG
MYG_PHYCA
           -VTHDOLNNFRAGFVSYMKAHT--DFA-GAEAAWGATLDTFFGMIFSKM
GLB3_CHITP
GLB5_PETMA
           QVDPQYFKVLAAVIADTVAAG------DAGFEKLMSMICILLRSAY
           -VADAHFPVVKEAILKTIKEVVGAKWSEELNSAWTIAYDELAIVIKKEMNDAA
LGB2_LUPLU
GLB1_GLYDI
          KHIKAOYFEPLGASLLSAMEHRIGGKMNAAAKDAWAAAYADISGALISGLOS-
Consensus
                                         aa.
```

Alignment of 7 globins. A-H mark 8 alpha helices. Consensus line: upper case = 6/7, lower = 4/7, dot=3/7. Could we have a profile (aka weight matrix) w/ indels?

### Profile Hmm Structure

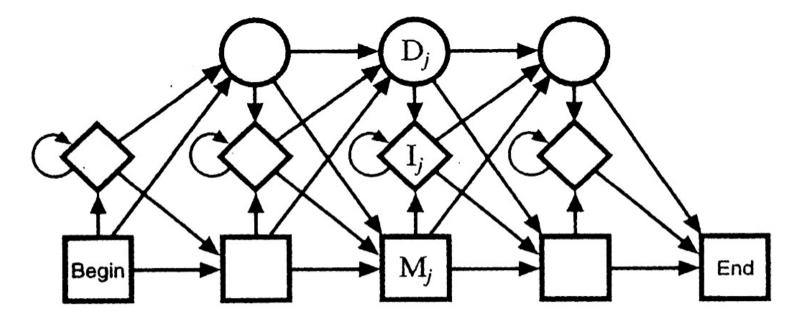


Figure 5.2 The transition structure of a profile HMM.

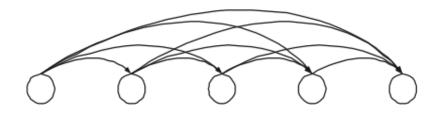
M<sub>j</sub>: Match states (20 emission probabilities)

Ij: Insert states (Background emission probabilities)

Dj: Delete states (silent - no emission)

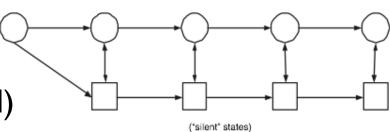
### Silent States

Example: chain of states, can skip some



Problem: many parameters.

A solution: chain of "silent" states; fewer parameters (but less detailed control)



Algorithms: basically the same.

# Using Profile HMM's

#### Search

Forward or Viterbi

Scoring

Log likelihood (length adjusted)

Log odds vs background

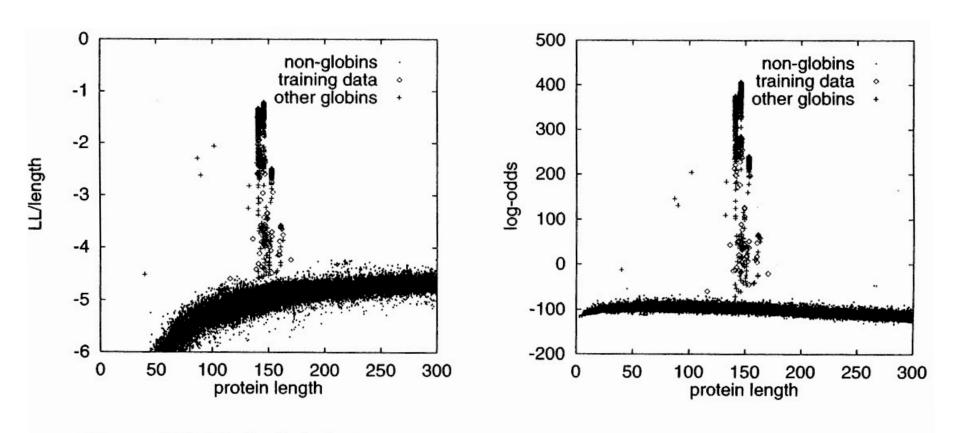
Z scores from either



#### Alignment

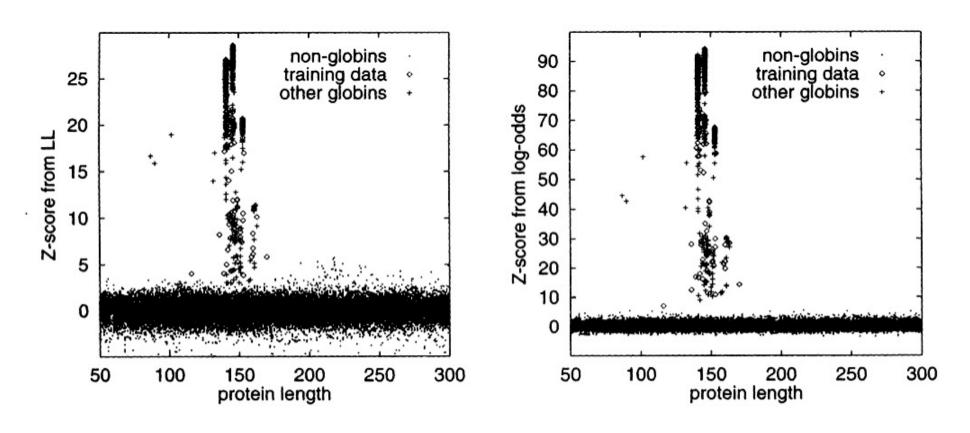
Viterbi

### Likelihood vs Odds Scores



**Figure 5.5** To the left the length-normalized LL score is shown as a function of sequence length. The right plot shows the same for the log-odds score.

### **Z-Scores**



**Figure 5.6** The Z-score calculated from the LL scores (left) and the log-odds (right).

http://xfam.org

# Pfam Model Building

Hand-curated "seed" multiple alignments
Train profile HMM from seed alignment
Hand-chosen score threshold(s)

Automatic classification/alignment of all other protein sequences

% of human proteins containing ≥ I PFAM domain

Version	Date	#Families	Coverage
25.0	3/2011	12273	75
27.0	3/2013	14831	90
31.0	3/2017	16712	
32.0	9/2018	17929	
33.1	5/2020	18259	

# HMM Summary

joint vs conditional probs

```
Inference
  Viterbi – best single path
                                          (max of products)
  Forward – sum over all paths
                                          (sum of products)
  Backward – similar
  Posterior decoding
Model building
  Semi-supervised – typically fix architecture (e.g. profile
     HMM), then learn parameters
   Baum-Welch – training via EM and forward/backward
     (aka the forward/backward algorithm)
  Viterbi training – also "EM-like", but Viterbi-based
```

# HMM Summary (cont.)

```
Search:
Viterbi or forward

Scoring:
Odds ratio to background
Z-score
E-values, etc., too

Excellent tools available (HMMer, Pfam, ...)

Very widely used for bioseq analysis (& elsewhere)
```