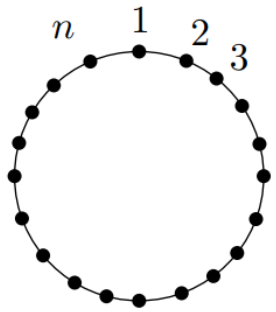
CSE 422 Autumn 26 HW 6

Note: For all problems, you will be expected to explain your reasoning. Providing an expression without reasoning will not receive any partial credit.

Problem 1: The cycle of homework

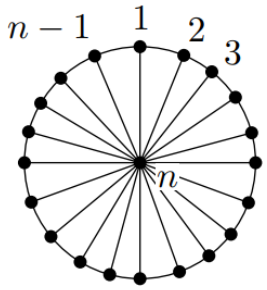
Consider the following cycle graph:



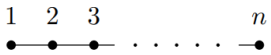
It turns out that the eigenvectors and eigenvalues of the Laplacian of this graph have a fairly nice analytic form. For general $n \geq 2$, derive the form of all of the eigenvectors and eigenvalues of this graph.
Hint: do you remember your trig identities?

Problem 2: Messing with the cycle

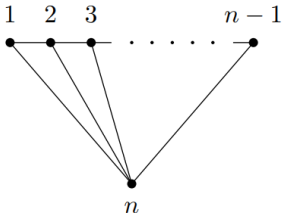
Now, let's consider three modifications to the cycle graph:



A spoke and wheel graph



A line graph

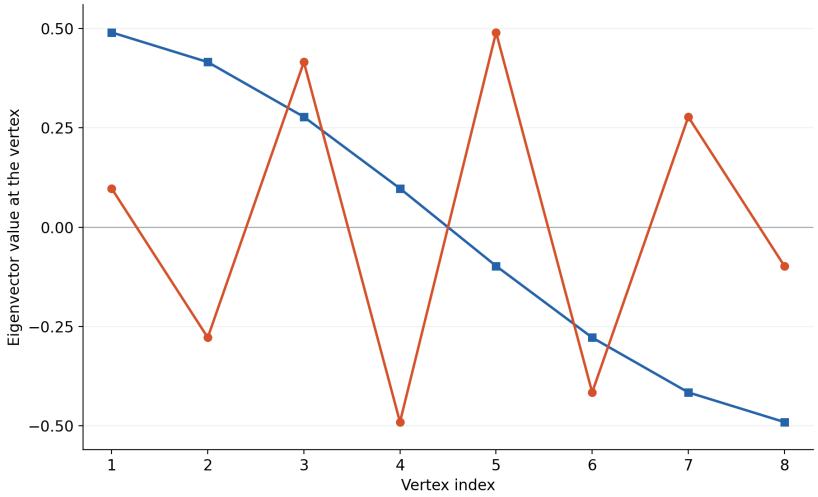


Line graph + a point

(a) [3 points] For $n = 5$, write down the Laplacian of each one of these graphs.

(b) [5 points] How does the smallest nonzero eigenvalue of the line graph on n vertices compare to the smallest nonzero eigenvalue of the cycle graph of the same size? What about the largest nonzero eigenvalue?

(c) [6 points] Our hapless professor strikes again: he has produced a plot of the largest eigenvector, and the smallest eigenvector for the line graph with $n = 8$, but he forgot to label which one is which! See the plot below. Here, the x-axis indicates the i-th index, and the y-axis indicates the value of the eigenvector at that index. Can you recover which curve is which?



(d) [6 points] Show that the largest eigenvalue of the Laplacian of the spoke and wheel graph and of the line plus a point graph is always n. In fact, show that any graph on n vertices containing a vertex adjacent to every other vertex has largest eigenvalue n.

Problem 3: Finishing the proof of Cheeger's inequality

We only sort-of proved the proof of the "hard side" of Cheeger's inequality in class. So, in this problem, you'll finish the proof. Let $G = (V, E)$ be an arbitrary d-regular graph on n vertices. Recall that what we showed in class is that for any vector $x \in \mathbb{R}^n$ with $x_i \in [0, 1]$ for all i, then the SWEEP algorithm outputs a subset $S_t \subseteq V$ so that

$$\frac{|E(S, \overline{S})|}{d|S|} \leq \frac{\sum_{(i,j)\in E} |x_i - x_j|}{d \sum_{i=1}^n x_i} ,$$

where $S_t = \{v \in V : x_v > t\}$. The basic idea will be to apply this bound to the positive part of the second smallest eigenvector. Let $v \in \mathbb{R}^n$ satisfy $\|v\|_2 = 1$, and $v^\top \mathcal{L}_G v = \lambda_2$, and let $u \in \mathbb{R}^n$ be given by $u_i = \max(v_i, 0)$, for all $i = 1, \ldots, n$.

(a) [5 points] Argue that

$$\sum_{(i,j)\in E} (u_i - u_j)^2 \leq 2\lambda_2 \cdot \|u\|_2^2 .$$

(b) [10 points] Now, apply the SWEEP algorithm to the vector $x \in \mathbb{R}^n$ given by $x_i = u_i^2$ for $i = 1, \ldots, n$. Show that this gives a subset $S \subseteq V$ so that

$$\frac{|E(S, \overline{S})|}{d|S|} \leq \sqrt{2\lambda_2} .$$

Hint: it may be useful here to judiciously use the Cauchy-Schwarz inequality, which states that for any vectors $a, b \in \mathbb{R}^n$, we have that $|\langle a, b\rangle| \leq \|a\|_2 \|b\|_2$.

(c) [5 points] We are almost done—but not quite! What is the missing step, and how do we fix the argument to complete the proof?