Recurrences

Divide & Conquer

Given problem running time of size n

$\Theta(1)$ if n small – solve directly

if n large

$f(n)$ – divide into a subproblems of size n/a

– solve each

– combine to solve full problem.

$$T(n) = \begin{cases} \Theta(1) & n \text{ small} \\ bT(n/a) + f(n) & n \text{ large.} \end{cases}$$

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eg T cubic, $a = 2$

$T(n/2) \leq \frac{1}{8} T(n)$

afford 7 subproblems and still be fast

[if $f(n)$ not too big]

$a = b = 2$

if T linear

$T(n/2) \simeq \frac{1}{2} T(n)$

if T super linear

$T(n/2) << \frac{1}{2} T(n)$

Example : Merge Sort

Problem: Sort $A[1], A[2], \ldots, A[n]$

Method:

MSort (A, p, q) : Sort $A[p] \ldots A[q]$

if ($p \geq q$) return ;

MSORT (A, p, $\lfloor \frac{p+q}{2} \rfloor$)

MSORT (A, $\lfloor \frac{p+q}{2} \rfloor + 1$, q)

MERGE (A, p, q)

Analysis:

$n = 2^k$

count only comparisons

$$T(n) = \begin{cases} 0 & n = 1 \\ 2T(n/2) + \underbrace{n-1}_{\text{merge}} & n > 1 \end{cases}$$

Solving Recurrences: Substitution

$$T(n) = \begin{cases} 0 & n=1 \\ 2T(n/2) + n-1 & n>1 \end{cases}$$

claim: $T(n) = n \lg n - n + 1$

Proof (induction on n):

basis ($n=1$):

$1 \cdot \lg 1 + 1 - 1 = 0$

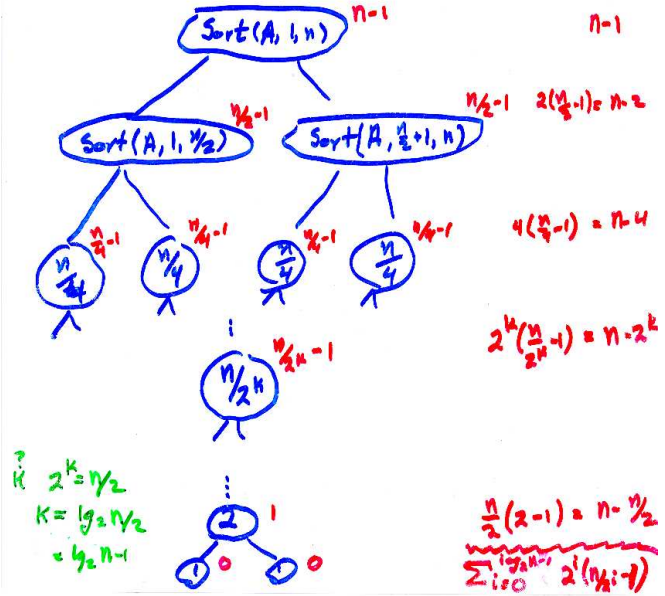
induction ($n>1$):

Assume $T(j) = j \lg j - j + 1$ for $j < n$

$$\begin{aligned} T(n) &= 2T(n/2) + n - 1 \\ &= 2\left(\frac{n}{2}\lg\frac{n}{2} - \frac{n}{2} + 1\right) + n - 1 \\ &= (n \lg \tfrac{n}{2} - n + 2) + n - 1 \\ &= n \lg \tfrac{n}{2} + 1 \\ &= n(\lg n - 1) + 1 = n \lg n - n + 1 \end{aligned}$$

Solving Recurrences: RecursionTree

$$T(n) = \begin{cases} 0 & n=1 \\ 2T(n/2) + n-1 & n>1 \end{cases}$$



Recursion Tree (cont.)

$$\sum_{i=0}^{\lg n-1} 2^i (n/2^i - 1)$$

$$= \sum_{i=0}^{\lg n-1} (n - 2^i)$$

$$= n \lg n - \sum_{i=0}^{\lg n-1} 2^i$$

$$= n \lg n - \frac{2^{\lg n} - 1}{2-1}$$

$$= n \lg n - n + 1$$

$2^{\lg n} = n$

$\sum_{i=0}^{k} x^i = \frac{x^{k+1}-1}{x-1}$

Iteration

$$T(n) = \begin{cases} 1 & n = 1 \\ 4T(n/2) + cn^2 & n > 1 \end{cases}$$

$$T(n) = cn^2 + 4T(n/2)$$

$$= cn^2 + 4\left(c(\tfrac{n}{2})^2 + 4T(\tfrac{n}{4})\right)$$

$$= cn^2(1 + \tfrac{4}{4}) + 4^2 T(\tfrac{n}{2^2})$$

$$= cn^2\left(1 + \tfrac{4}{4} + (\tfrac{4}{4})^2\right) + 4^3 T(\tfrac{n}{2^3})$$

$$= cn^2 \sum_{i=0}^{k-1} (\tfrac{4}{4})^i + 4^k T(\tfrac{n}{2^k})$$

$$= cn^2 \sum_{i=0}^{\lg n - 1} (\tfrac{4}{4})^i + 4^{\lg n} T(1)$$

$4^{\lg n} = n^{\lg 4}$

$O(n^{\lg 4}) = O(n^2)$

Theorem 4.1 (Master theorem)
Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n) ,$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ can be bounded asymptotically as follows.

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■

A Pitfall

Claim: $\sum_{k=1}^{n} k = O(n)$

FALSE!

"Proof":

Basis: $\sum_{k=1}^{1} k = 1 = O(1)$

Ind: $\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1)$

$= O(n) + (n+1)$

$= O(n+1)$

Right approach:

prove $\exists$ a single constant c s.t. $\sum_{k=1}^{n} k \leq cn$

(will fail):

$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1)$

$\leq cn + n + 1 = (c+1)n + 1$