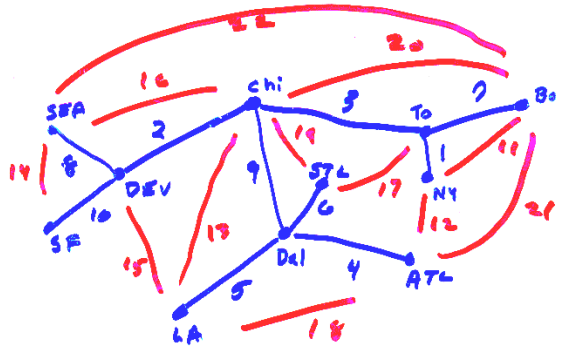
# Minimum Spanning Trees

Kruskal's Algorithm:

An Example of the Greedy Method

## Minimum Cost Spanning Tree




Given: Undirected graph $G = (V, E)$
Edge cost/weight $c(e) \in \mathbb{R}$

Find: Subset $T \subseteq E$
s.t. $\sum_{e \in T} c(e) = c(T)$
is minimal
subject to: $(V, T)$ connected

**FACT:**

Adding an edge to a tree creates a cycle.

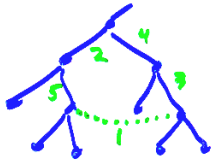
And deleting any cycle edge leaves a tree

**Cor 1:**

Solution to MST problem is a Tree.

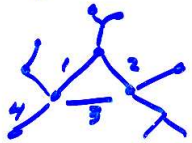
**Cor 2:**

Cheapest edge in E is in T.




Exercise: show 2nd-cheapest $\in T$, too.

3rd cheapest? Maybe not:




An Algorithm: Kruskal's Algorithm

```
Sort edges.
T ← ∅
While |T| < n-1 do
    e ← next cheapest edge
    if T∪{e} acyclic
       Then T ← T∪{e}
       else discard e.
```

## Theorem: It works!

Proof. By contradiction. Suppose greedy tree is *not* min.

- Suppose greedy picks edges $g_1, g_2 \cdots$

  Suppose $g_1 \le g_2 \le g_3 \le \cdots$ (really $c(g_1) \le c(g_2) \le \cdots$)

- Let $e_1, e_2 \cdots$ be edges in some MST $T$

  suppose $e_1 \le e_2 \le \cdots$

- Let $k$ be *least* index s.t. edges $g_k$ and $e_k$ are not identical

- Suppose $e_1, e_2 \cdots$ *maximizes* $k$ among all MST's.

- Claim: $T \cup \{g_k\}$ has a unique cycle & it (the cycle) contains at least one edge $f$ s.t.

  (1) $f \notin \{g_1, g_2, \cdots\}$ and

  (2) $c(f) \ge c(g_k)$

- Claim: $T - \{f\} \cup \{g_k\}$

  (1) is a spanning tree

  (2) costs no more than $T$

  (3) has larger "$k$"

  A contradiction. QED

*Proof: $\{g_1 \cdots\}$ is a tree, so contains no cycle, so some cycle edge must be $\notin \{g_1 \cdots\}$

←Proof: Let $f$ be any cycle edge $\notin \{g_1, \cdots\}$. If $c(f) < c(g_k)$, greedy alg. will consider $f$ before $g_k$. But then it would have selected it, since $\{g_1, g_2, \cdots, g_{k-1}\} \cup \{f\} \subseteq T$, hence acyclic. This is a contradiction, so $c(f) \ge c(g_k)$