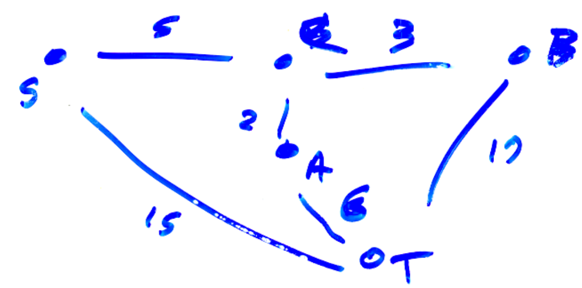
All-Pairs Shortest Paths and Transitive Closure

The Floyd-Warshall Algorithm

All pairs Shortest Paths (ch 26)



$G = (V, E)$ (Directed or undirected)

$w : E \to \mathbb{R}$ weights, lengths, costs

$w \geq 0$

Problem

For all pairs of vertices (i, j) what is (length of) shortest path from i to j.

Idea:

non stops are better

K " . . . . $K+1$

All-Pairs Shortest Paths

Try #1: Add one edge at a time

$G^{(k)} = (V, E^{(k)})$, $E^{(k)} = 1^{st}$ k edges

Induction Hypothesis:

$D^{(k)}_{ij}$ = length of shortest i,j path in $G^{(k)}$

Suppose $(k+1)^{st}$ edge is (x,y)

Then claim

$$\boxed{D^{(k+1)}_{ij} = \min\left(D^{(k)}_{ij},\ D^{(k)}_{ix} + w(x,y) + D^{(k)}_{y,j}\right)}$$

Time: n^2 per edge, $\rightarrow n^2 \cdot e$, $\rightarrow O(n^4)$

Try #2: Add 1 vertex at a time

$V^{(k)} = \{1, 2, \ldots, k\}$

$E^{(k)} = E \cap V^{(k)} \times V^{(k)}$

$G^{(k)} = (V^{(k)}, E^{(k)})$

I. H.:

$D_{ij}^{(k)}$ = min $i \to j$ path in $G^{(k)}$

$\forall i \le k \quad d_{i,k+1} = \min_{1 \le x \le k} D_{ix}^{(k)} + w(x, k+1)$

$\forall j \le k \quad d_{k+1,j} = \min_{1 \le x \le k} w(k+1, x) + D_{xj}^{(k)}$

$d_{i,j} = D_{ij}^{(k)} \quad 1 \le i, j \le k$

$D_{ij}^{(k+1)} = \min(d_{ij},\ d_{i\,k+1} + d_{k+1,j})$

Time $O(n^3)$

Try #3: Add intermediate vertices one at a time

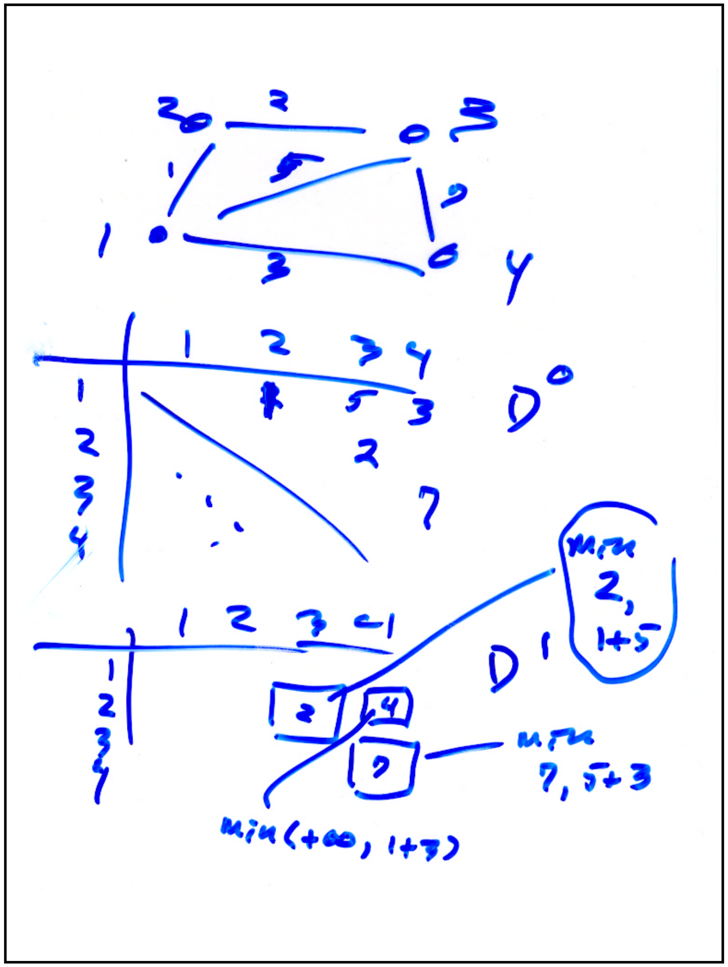
Floyd–Warshall

$G = (V, E)$

$D_{ij}^{(k)}$ = shortest path from i to j none of whose intermediate vertices are $> k$.

$$D_{ij}^{(k+1)} = \min\left(D_{ij}^{k},\ D_{i\,k+1}^{k} + D_{k+1\,j}^{k}\right)$$

$O(n^3)$



Transitive Closure

Given digraph $G=(V,E)$

For all pairs $i,j \in V$, is there a path from i to j?

$$D_{ij}^{(k)} = \begin{cases} 1 & \text{if } \exists \text{ path } i \text{ to } j \text{, with no intermediate vertex} > k. \\ 0 & \text{o/w.} \end{cases}$$

$$D_{ij}^{(k+1)} = D_{ij}^{(k)} \vee \left(D_{i,k+1}^{(k)} \wedge D_{k+1,j}^{(k)}\right)$$

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For k=0,1,...
  for i:=1,2,...
    if D_{i,k+1} then
      for j=1..n
        D_ij := D_ij v D_{k+1,j}
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OR rows i & $k+1$ →

Time: n^3/word length.