Lecture 7

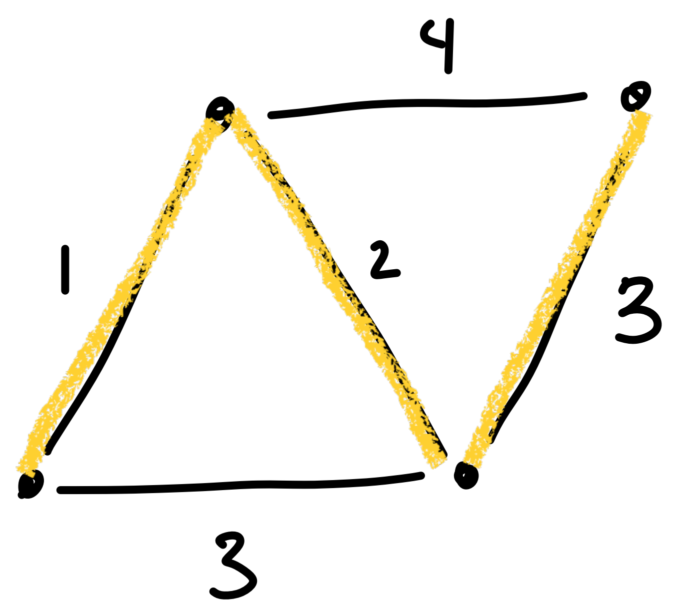
Minimum spanning trees

Chinmay Nirkhe | CSE 421 Winter 2026

Minimum spanning trees/forests

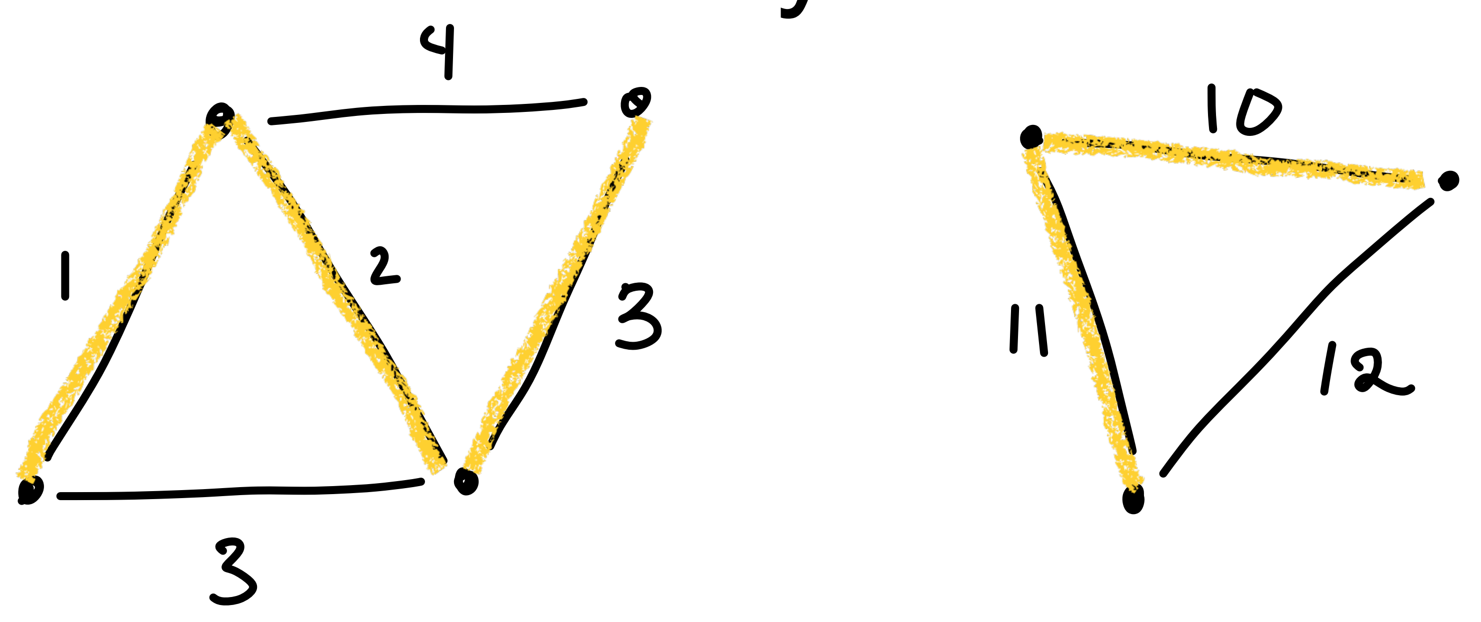
negative weights allowed

- **Input:** connected $G = (V, E)$, edge weights $w : E \to \mathbb{R}$
- **Output:** A tree $T = (V, E')$ such that every vertex is connected and $\sum_{e \in E'} w(e)$ is minimized. Called a **minimum spanning tree**.



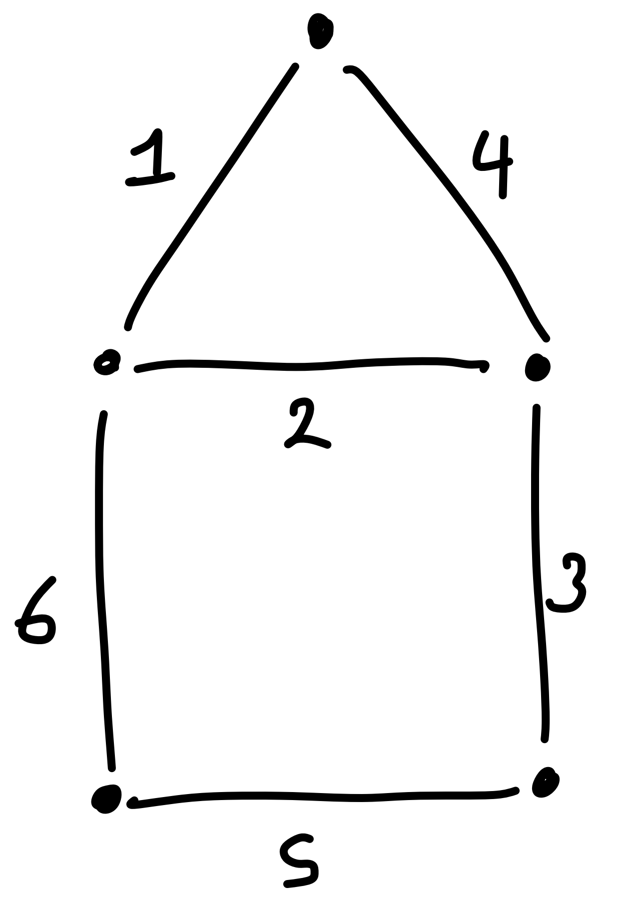
Minimum spanning trees/forests

- **Input:** $G = (V, E)$, edge weights $w : E \to \mathbb{R}$
- **Output:** A forest $F = (V, E')$ with a minimum spanning tree per connected component of G. Called a **minimum spanning forest** (or a minimum spanning tree).
- Equivalently, a subgraph F of minimal total weight such that u, v are connected in F iff they are connected in G.



Prim's algorithm
High level

- Dijkstra's creates a spanning tree as it unfolds.
  - However, Dijkstra's optimizes for a shortest-path tree from a root s.
  - Whereas, we want to optimize for a minimum weight tree (root indep.).



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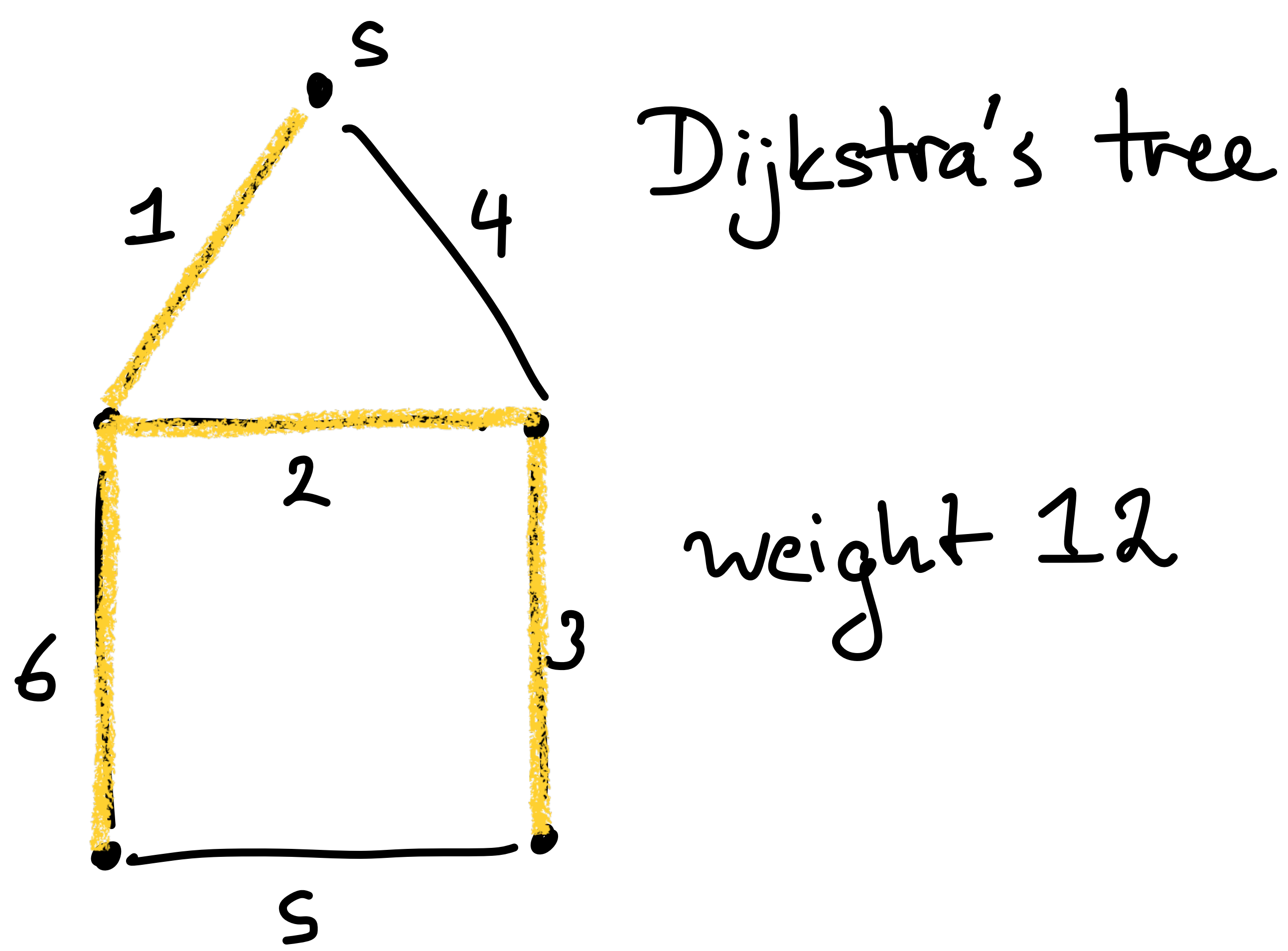

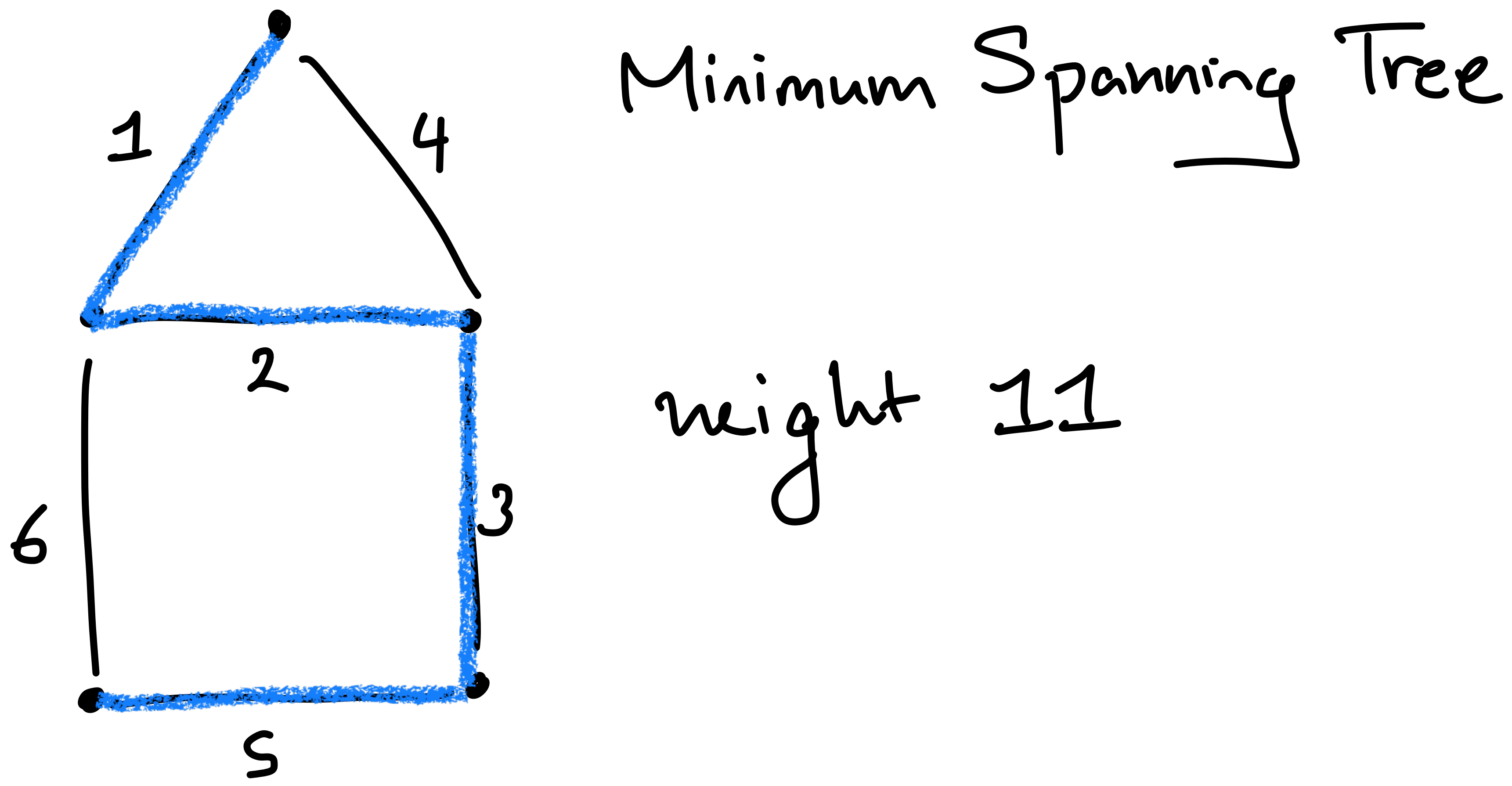

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- Pick a starting vertex $s \in V$. Let $S \leftarrow \{s\}$.
- While S doesn't equal V
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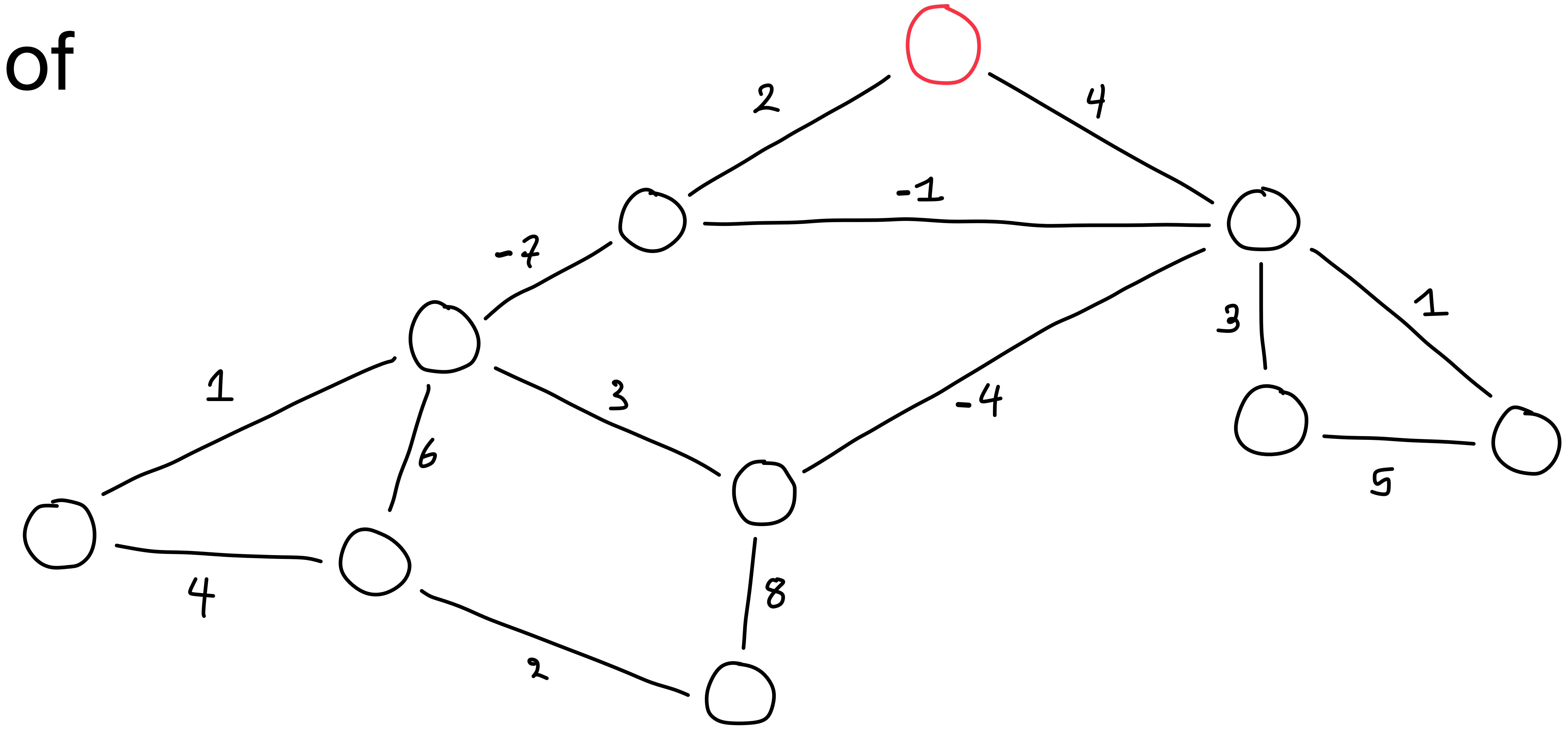

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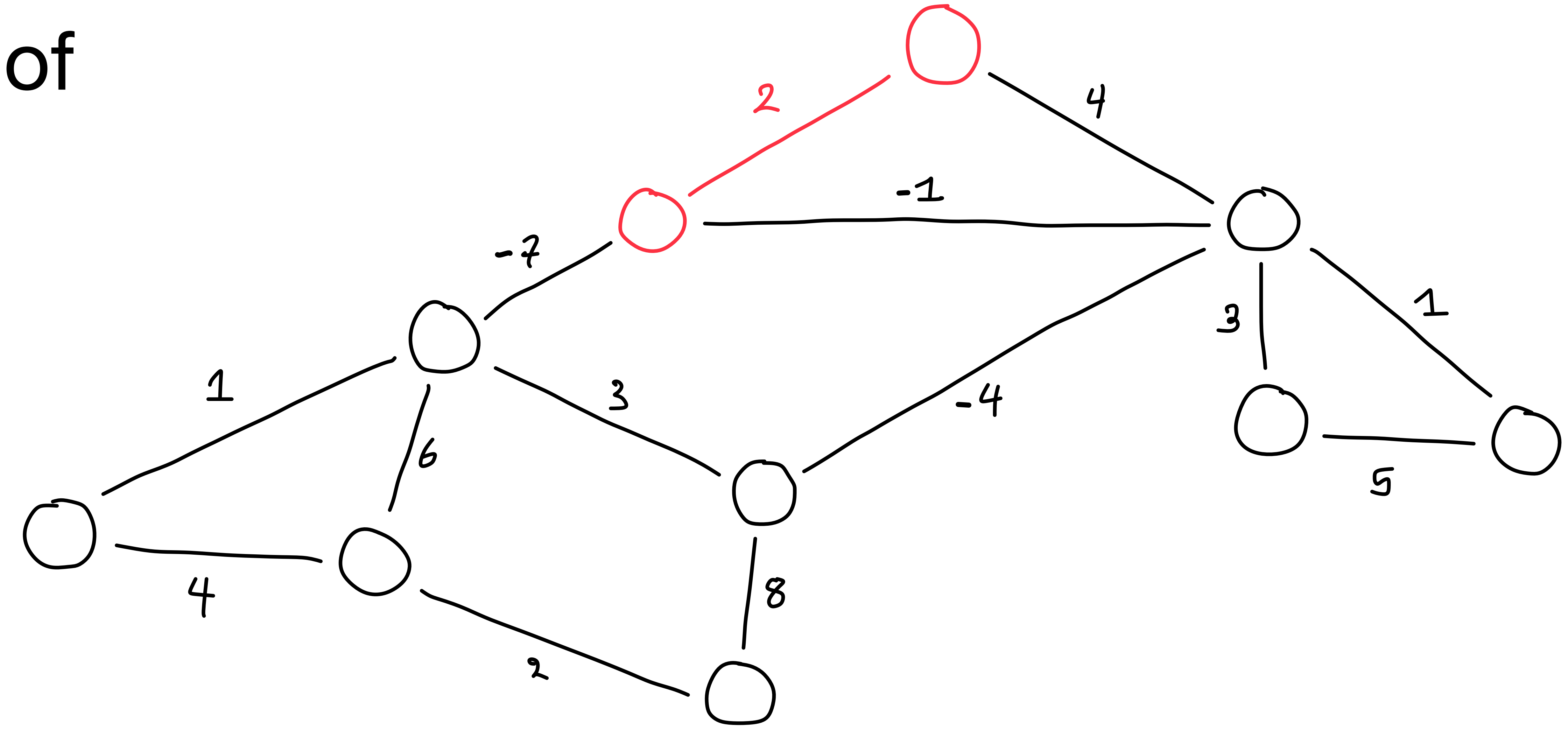

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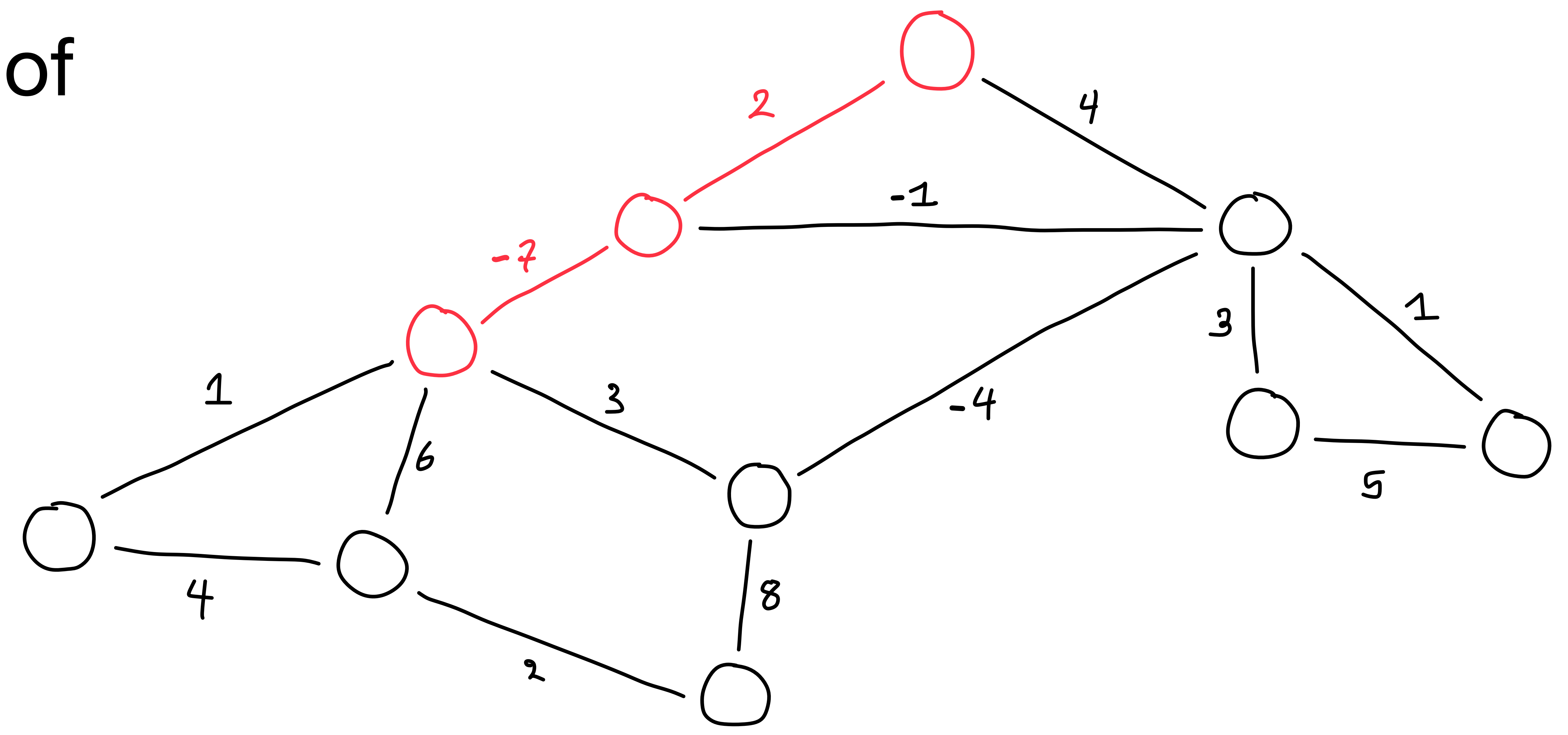

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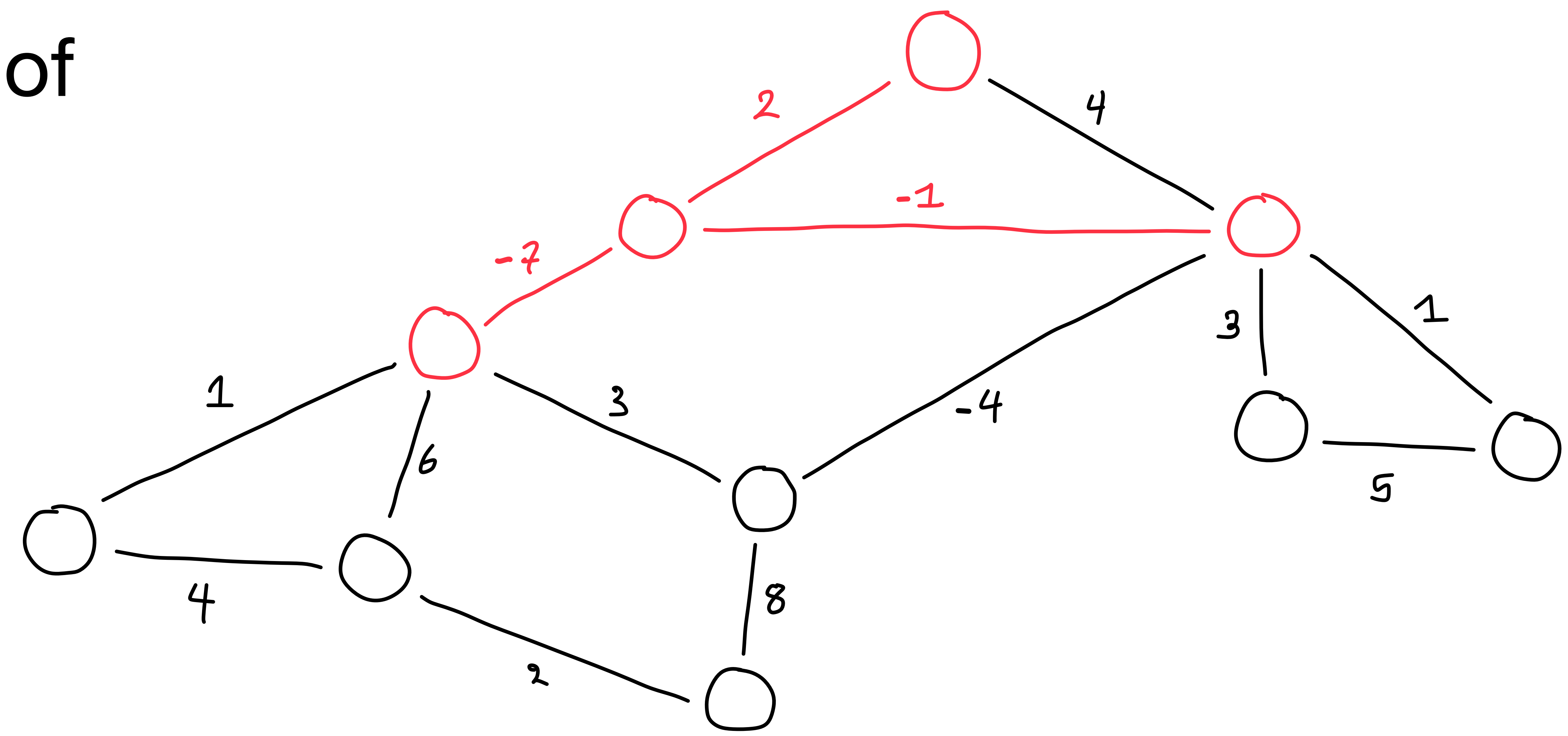

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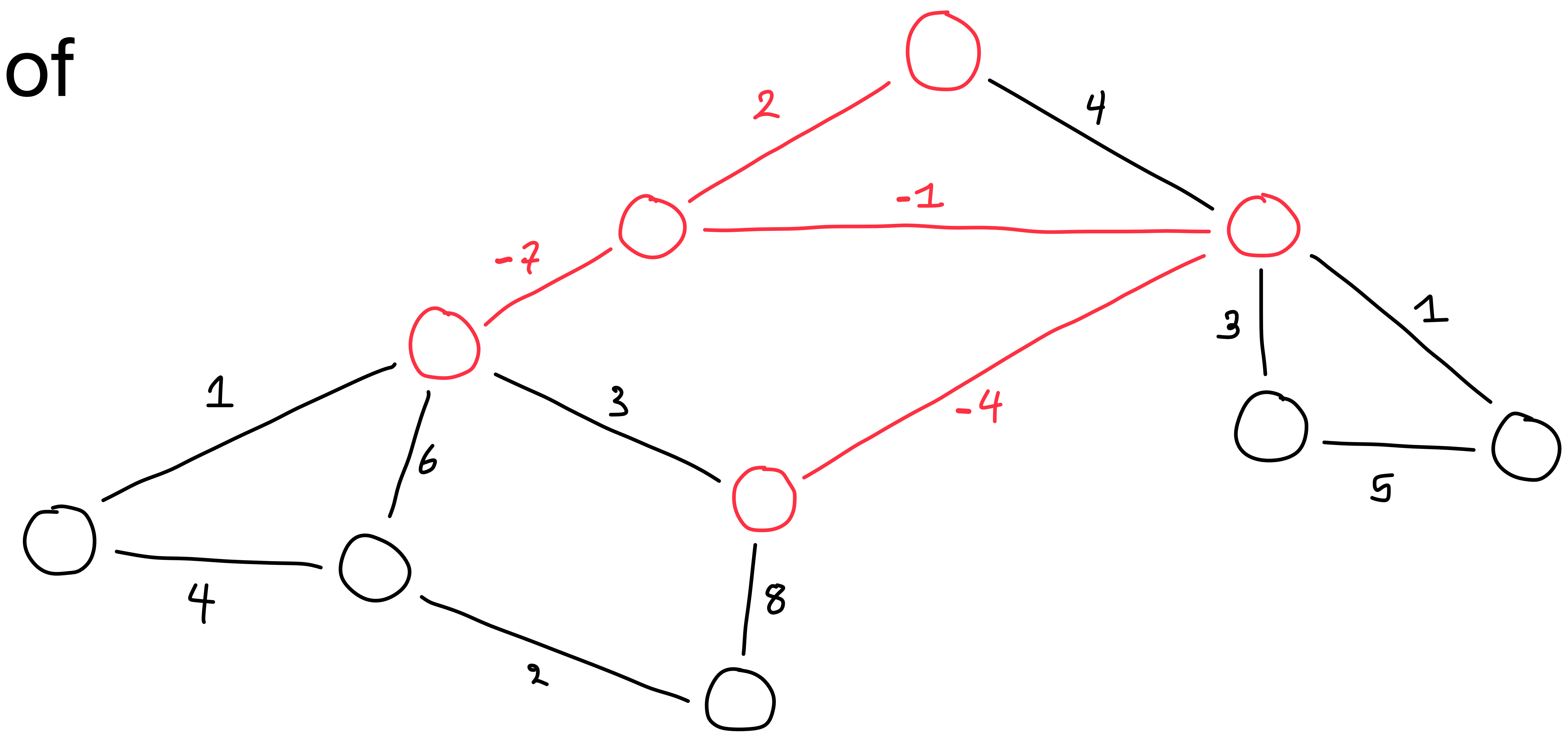

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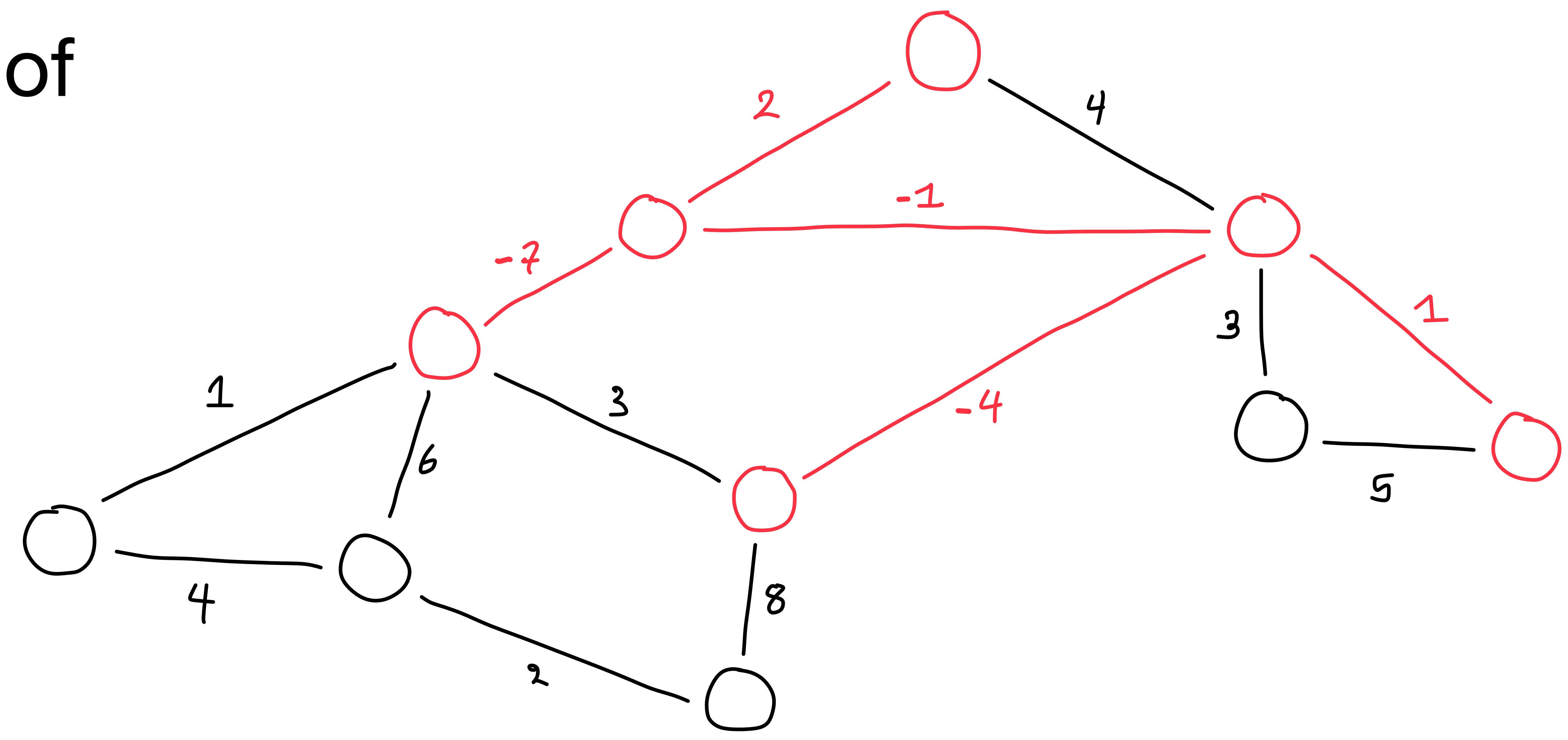

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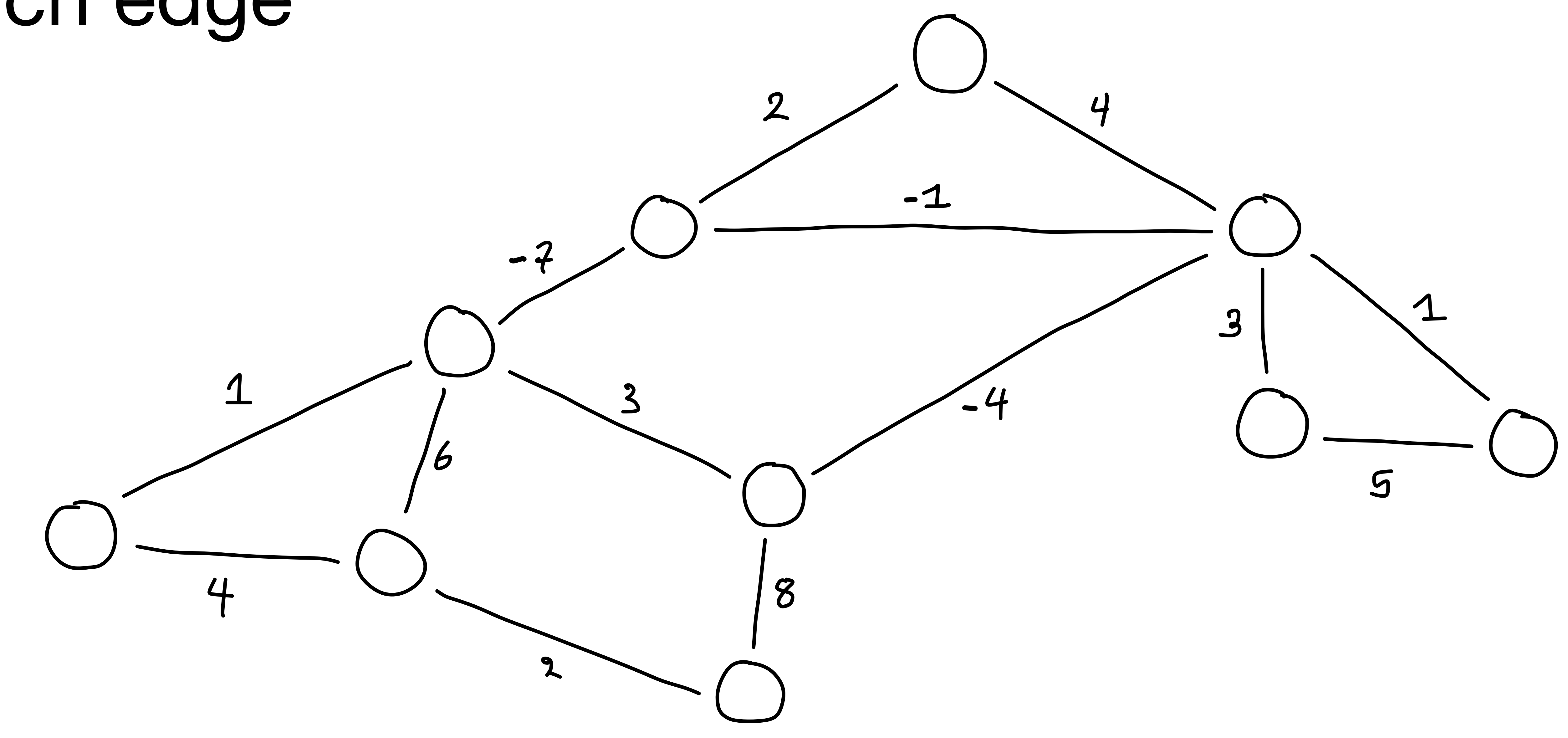

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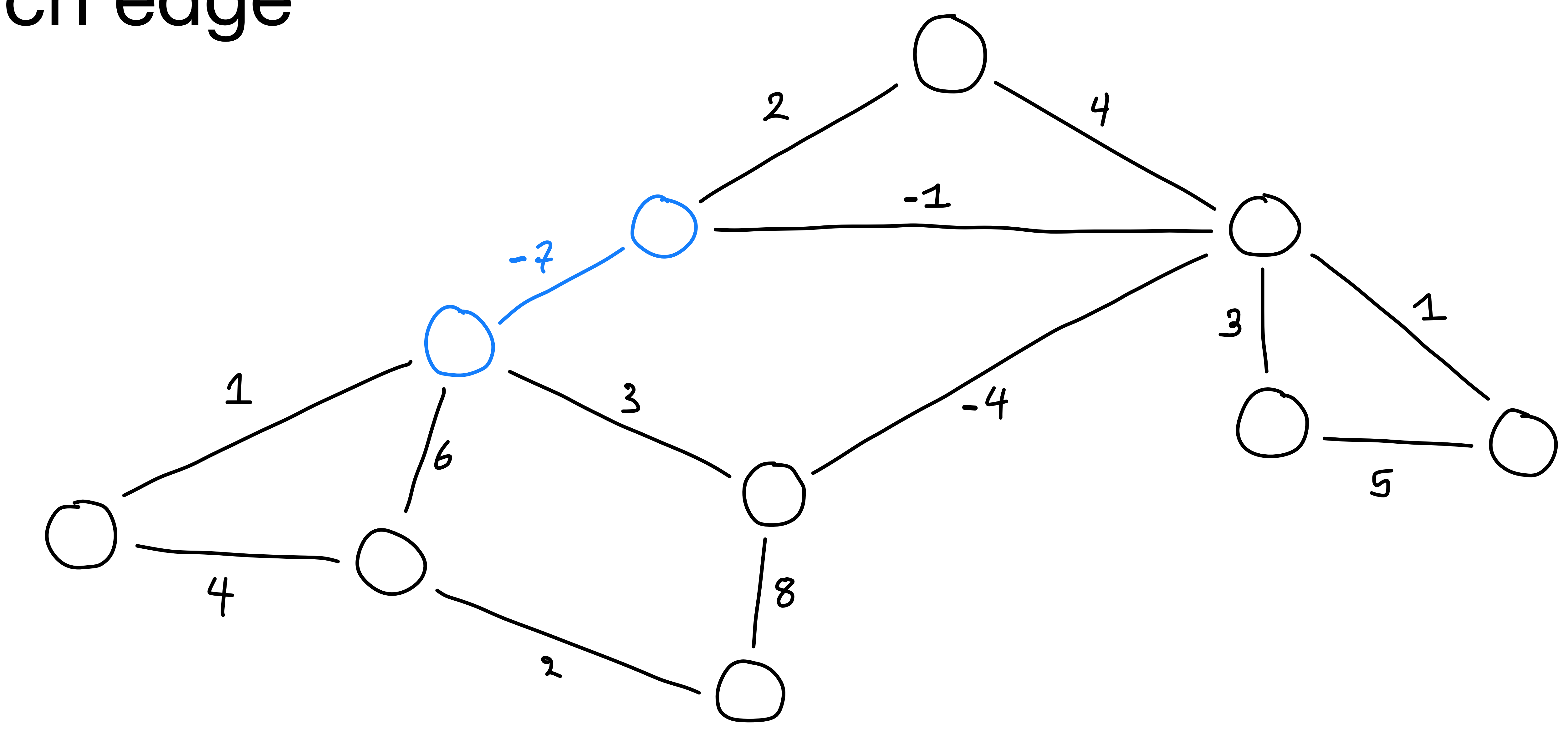

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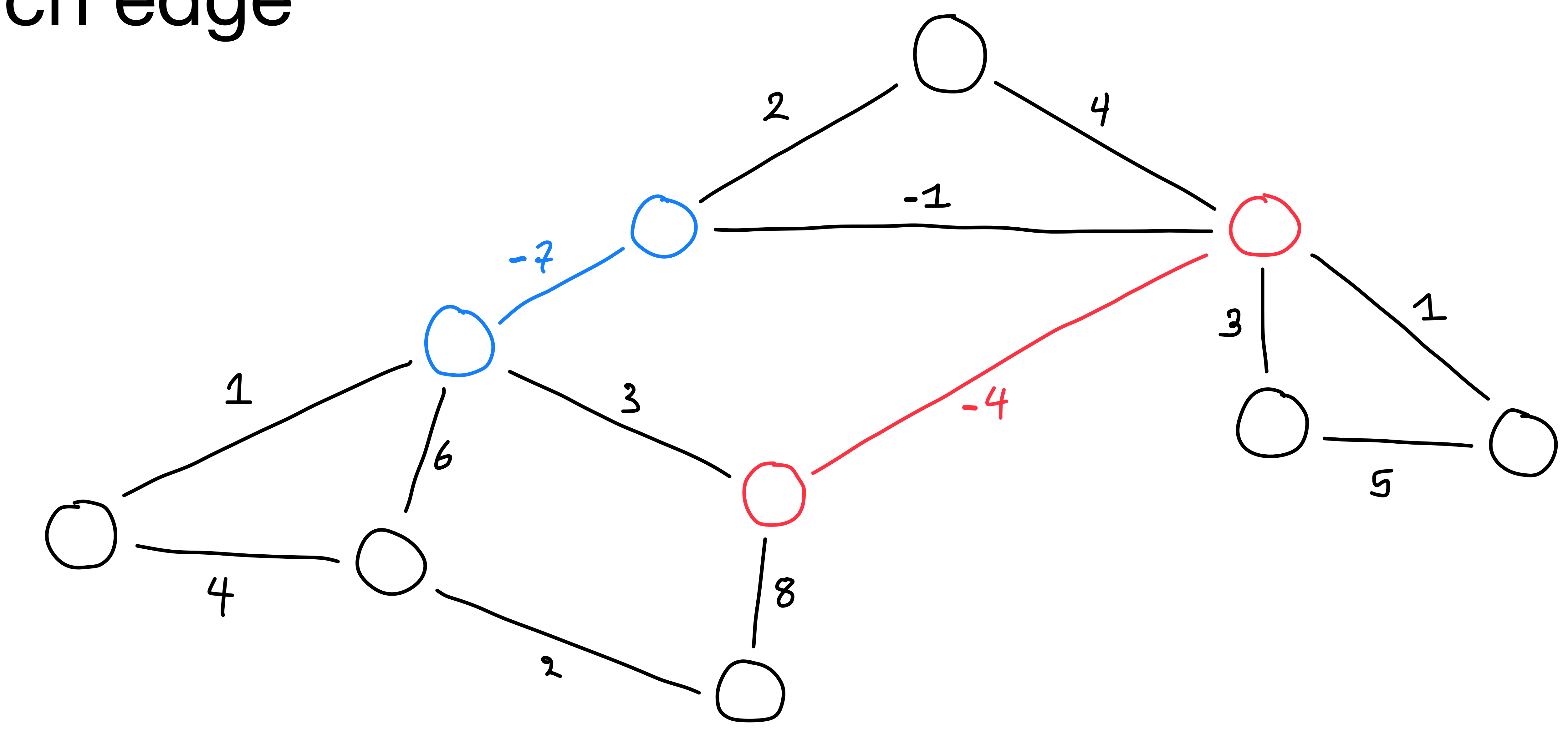

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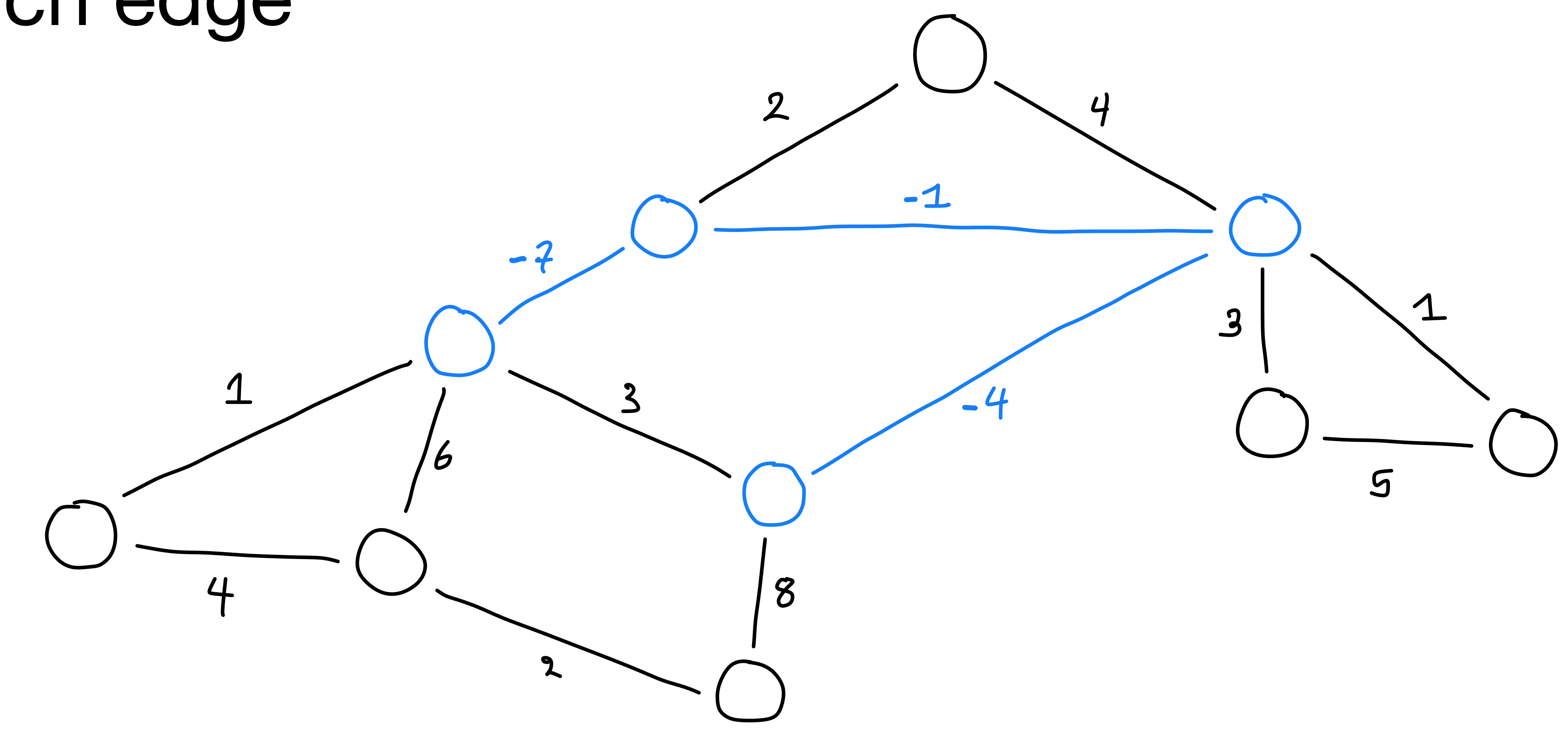

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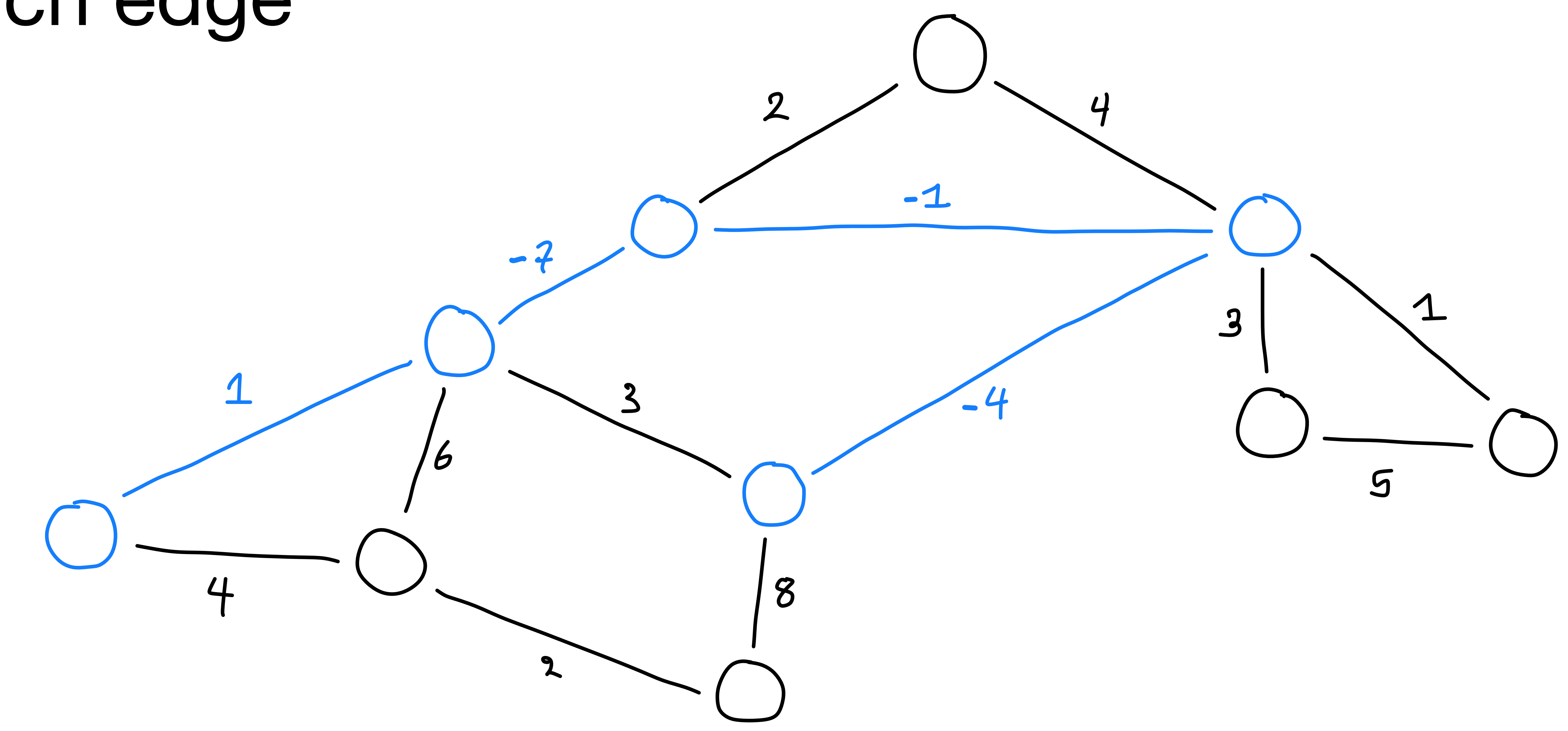

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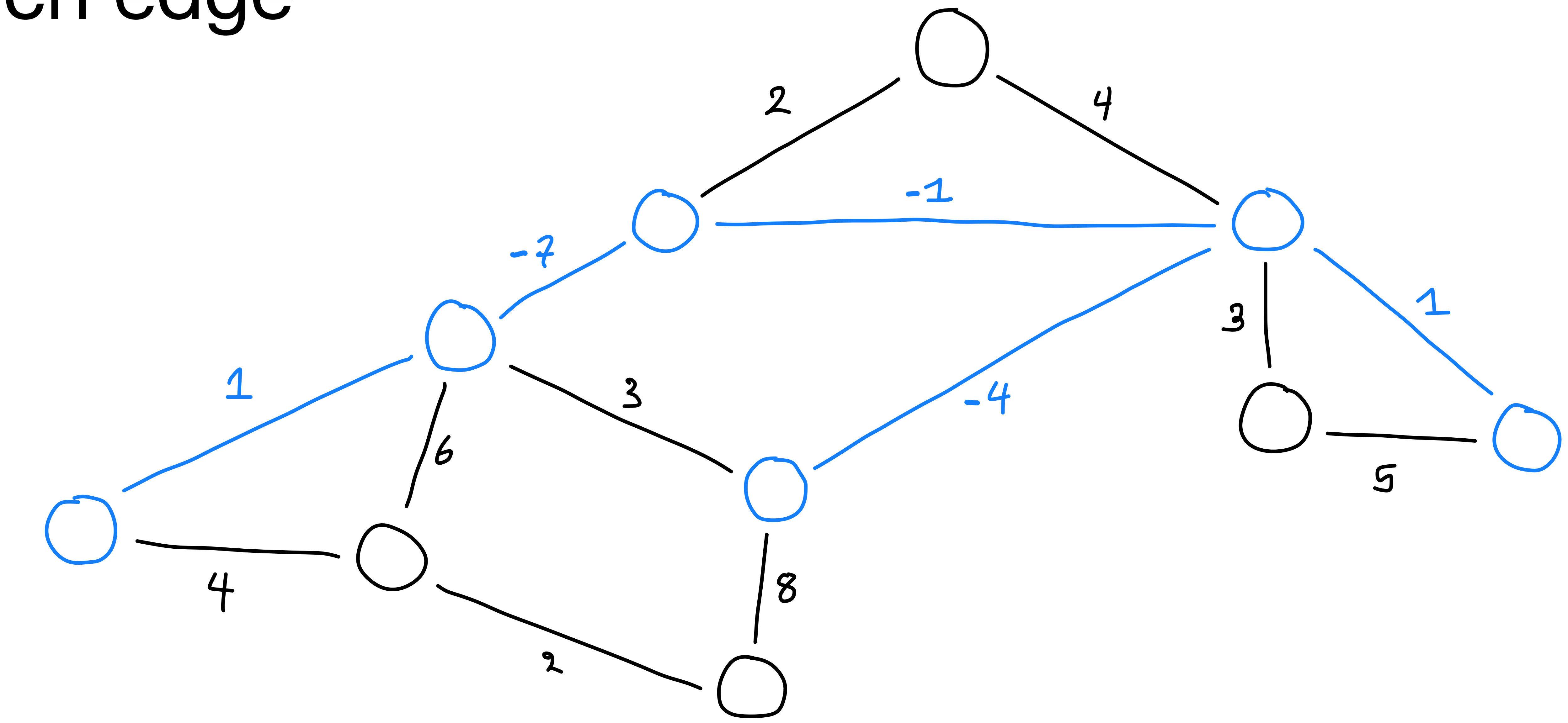

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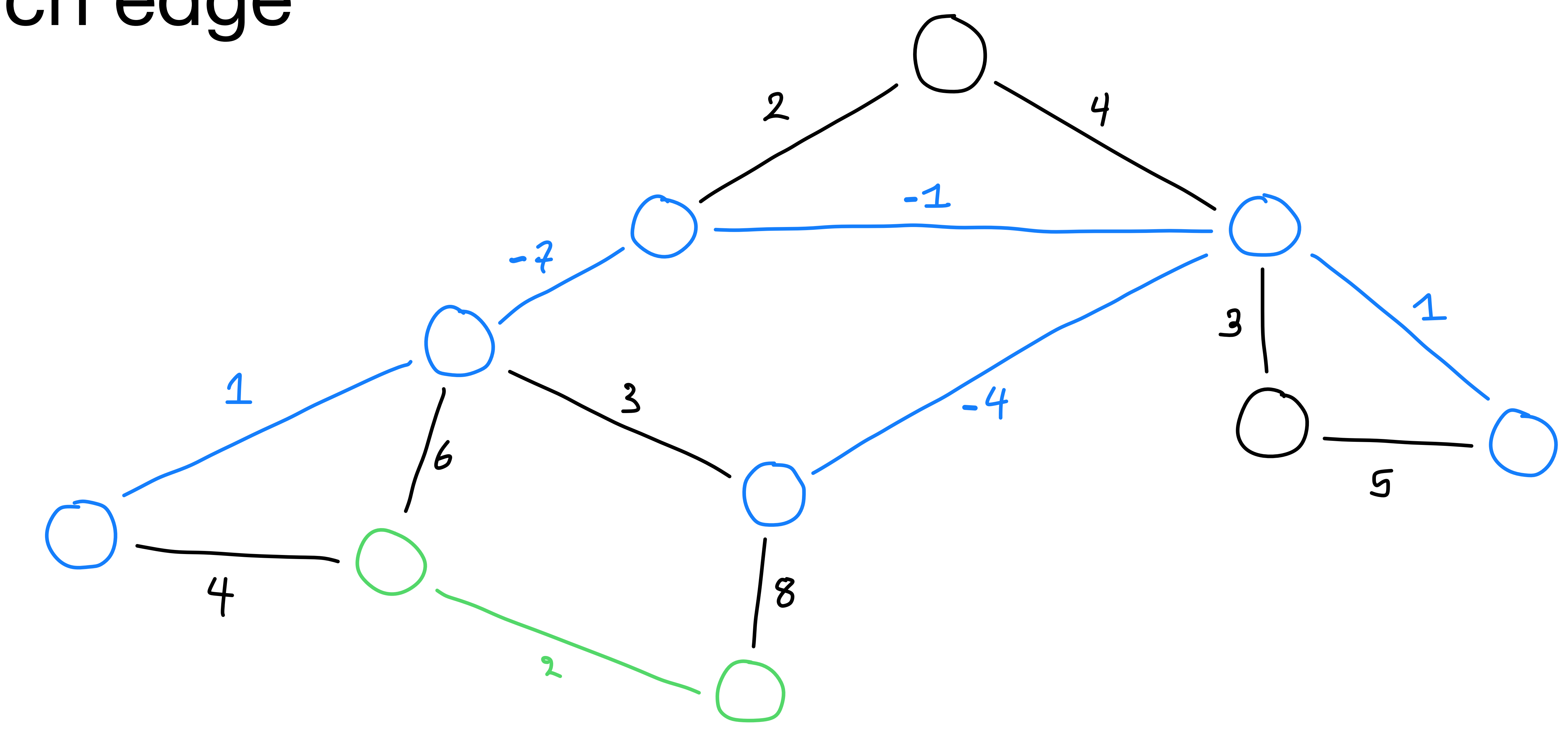

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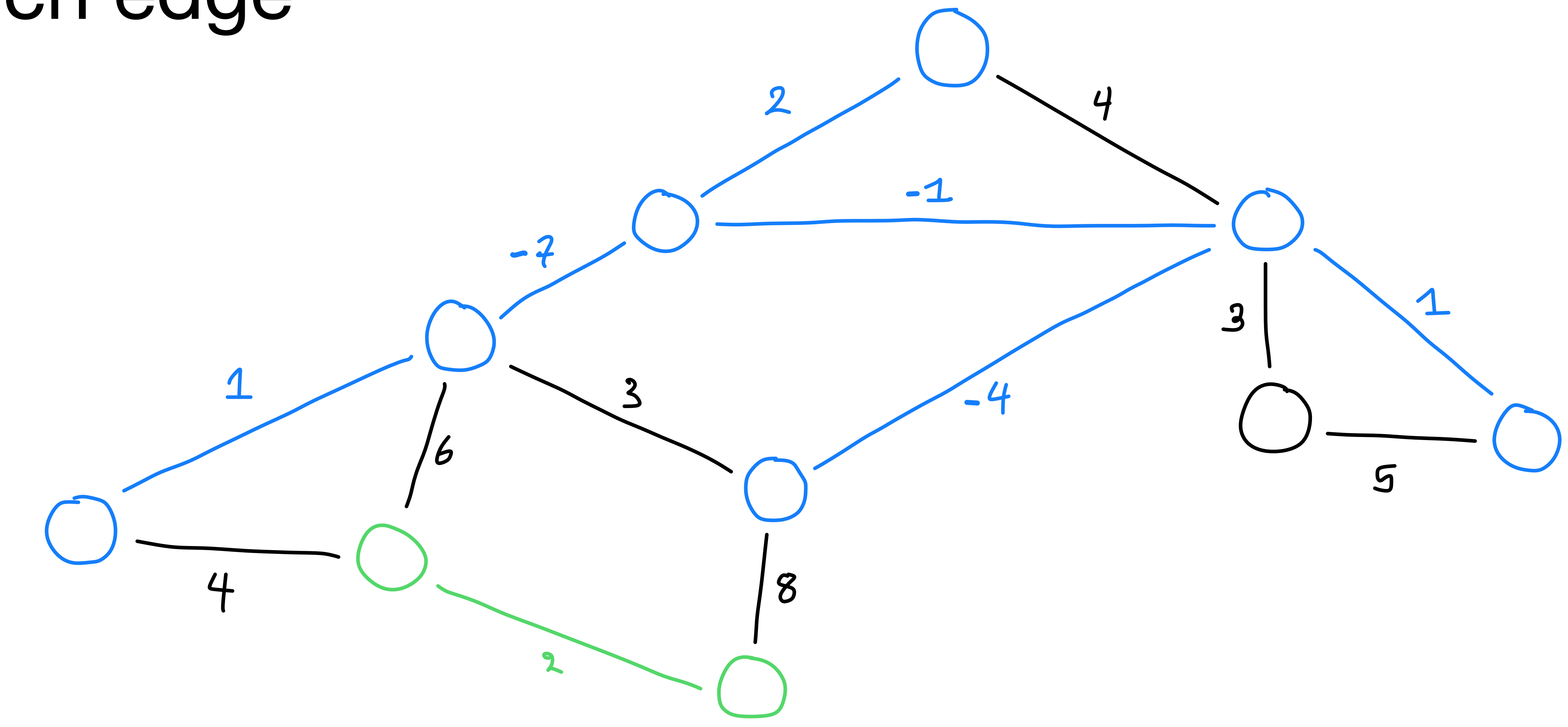

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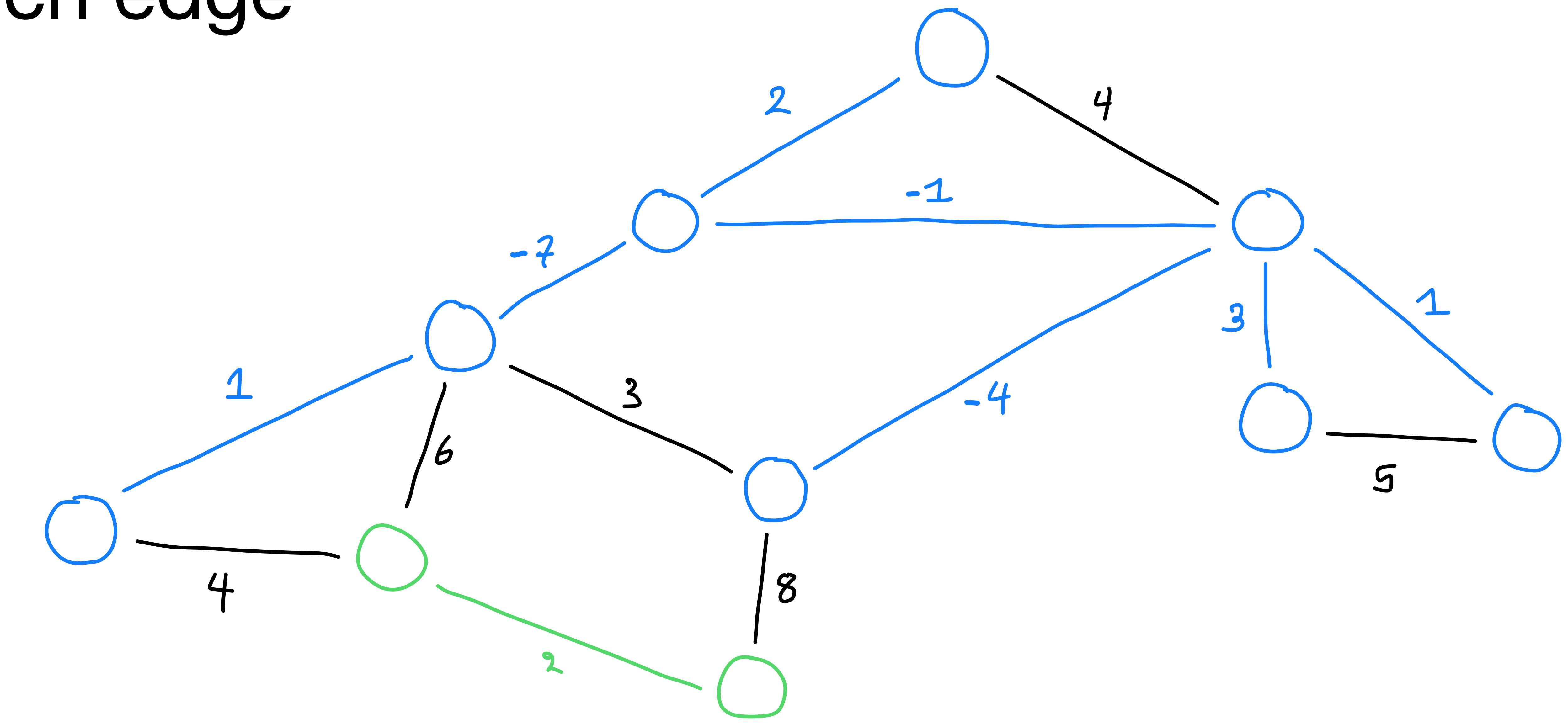

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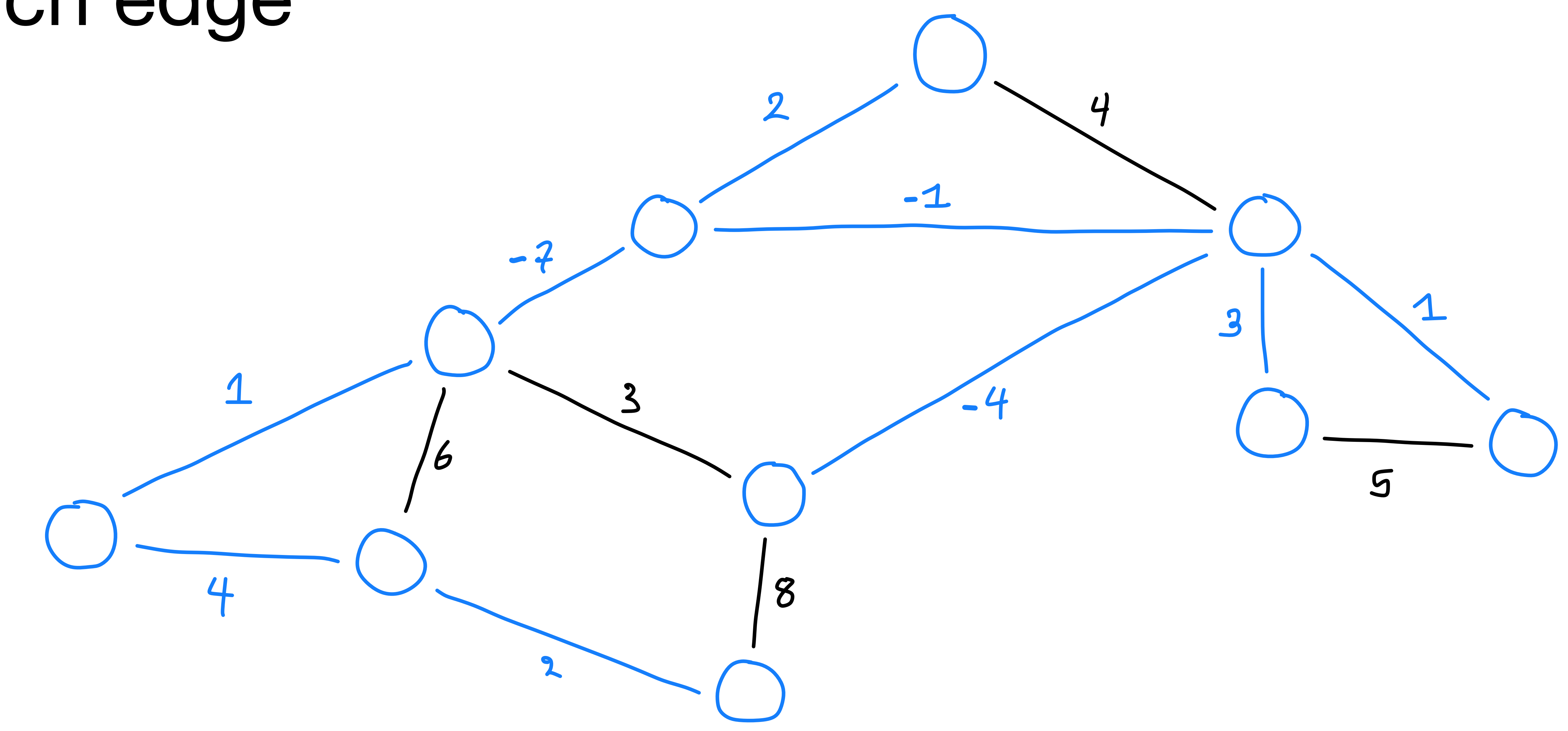
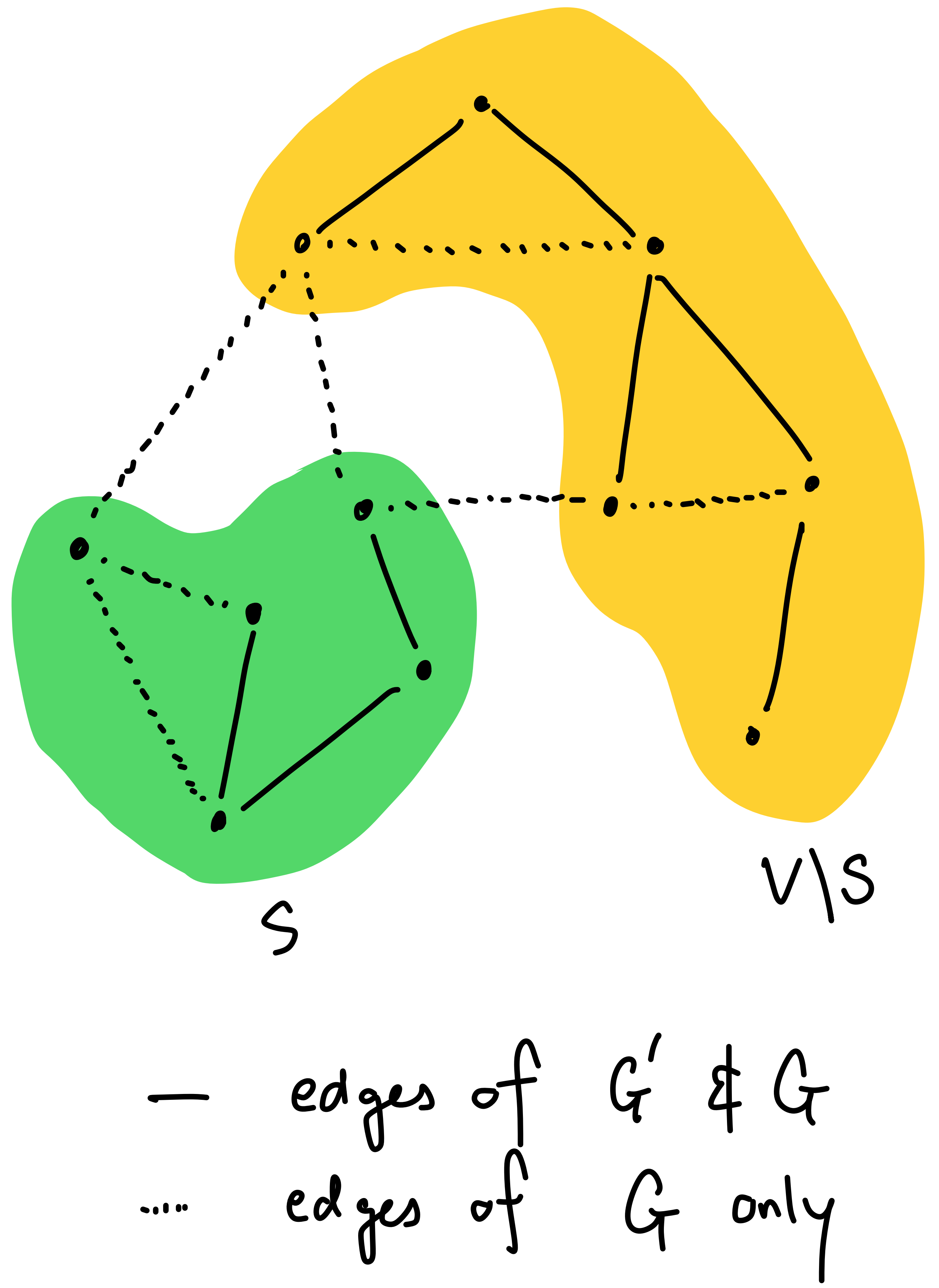


A unified argument for proving correctness

Of both Prim's and Kruskal's algorithm

- A **partition/cut** of the vertices is a split into two pieces S and $V \backslash S$.
- The cut is denoted as $(S, V \backslash S)$.
- An edge **crosses** the cut if $e = (u, v)$ and $u \in S$ and $v \in V \backslash S$.
- We say a subgraph $G' \subseteq G$ **respects** the cut $(S, V \backslash S)$ iff no edge of G' crosses the cut.



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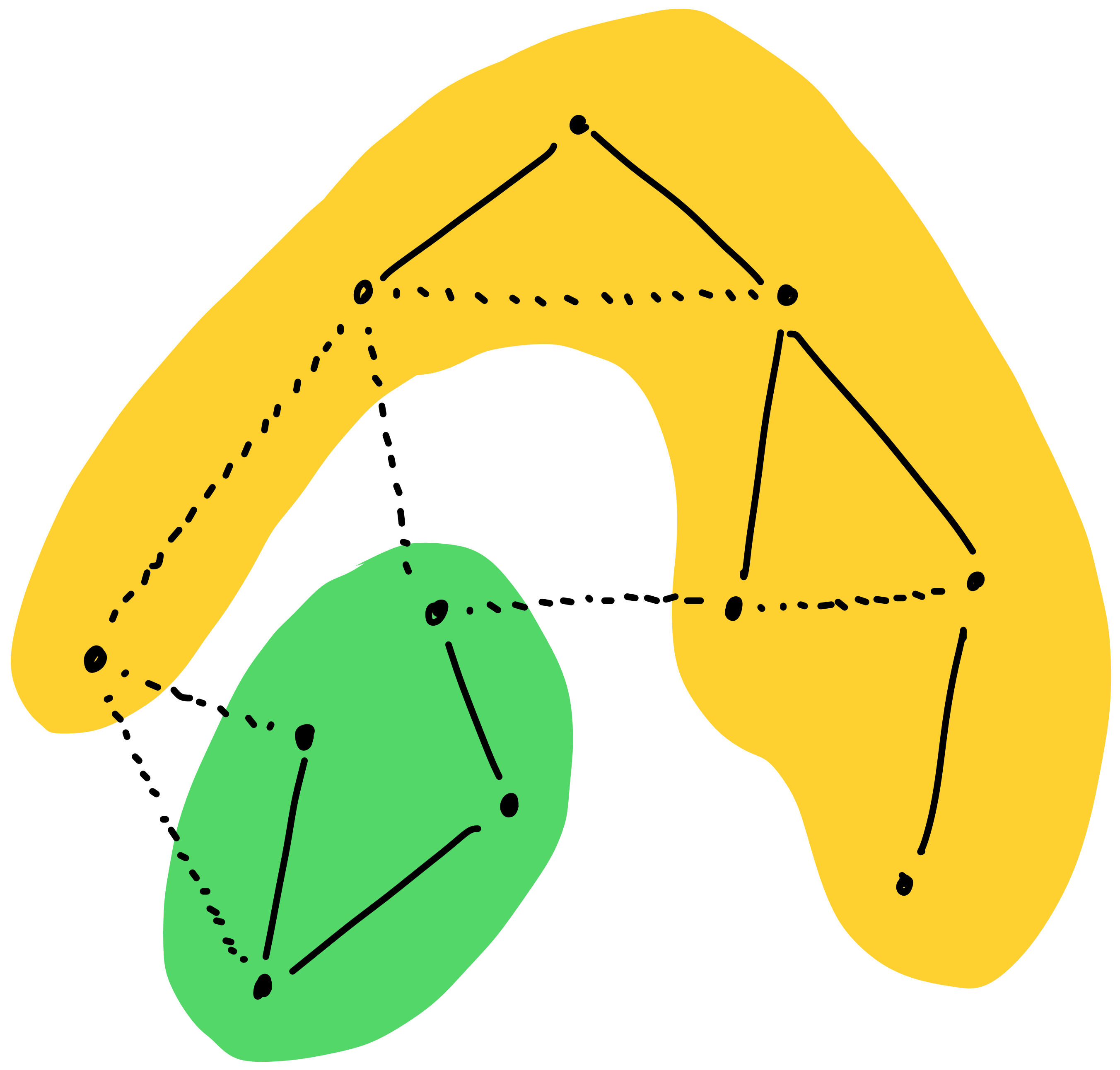
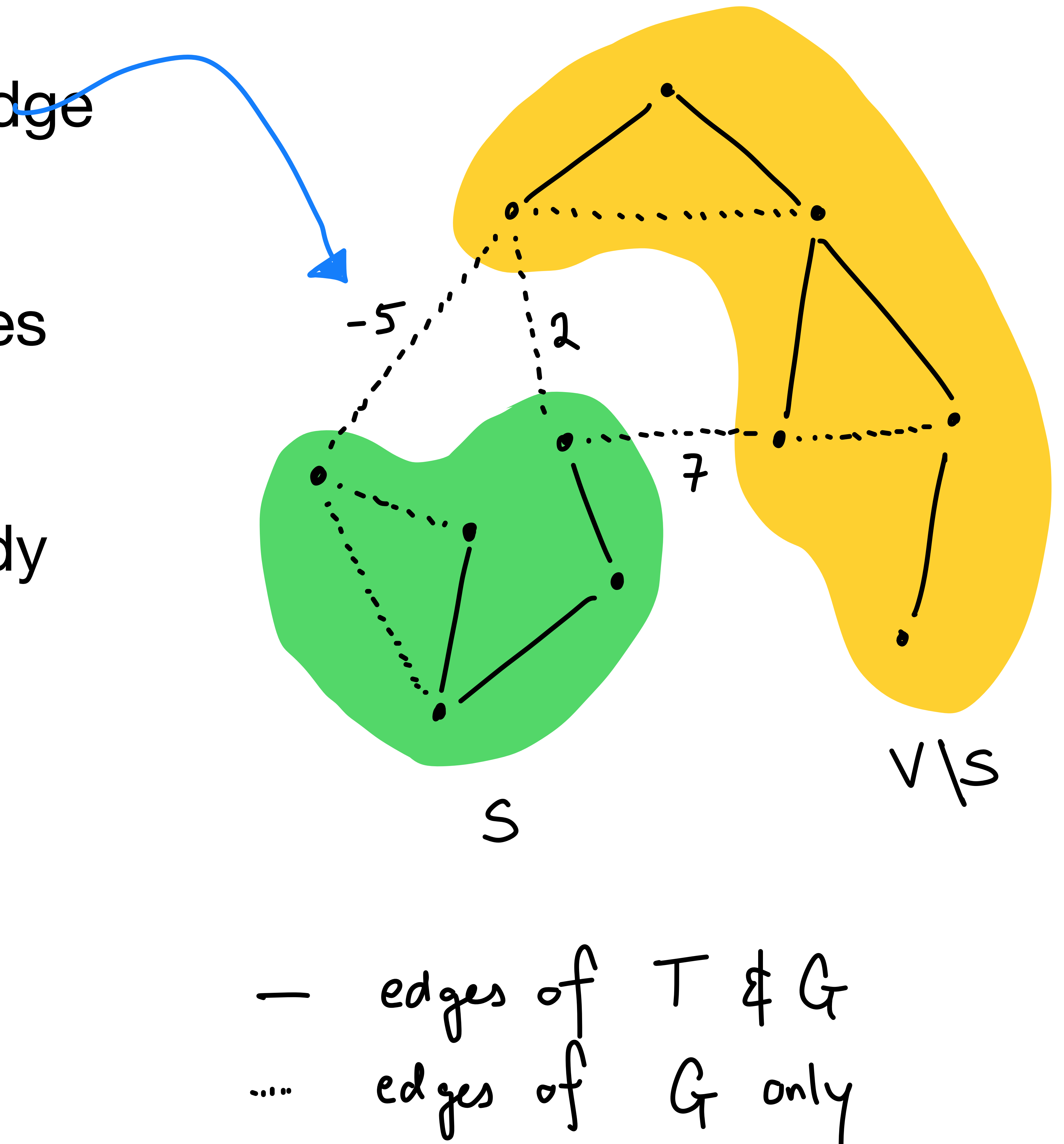


A different partition which also respects the cut.

Arguing correctness of greedy MST algorithms

- **Definition:** An edge e is **safe** for a forest T iff there is *some* cut $(S, V\backslash S)$ respected by T such that e is the **cheapest** edge crossing $(S, V\backslash S)$.
- **Theorem:** Greedy algorithms that *always* choose **safe** edges for the current forest T correctly compute an MST
- **Proof:** By induction. Let e be the **first** edge added by greedy algorithm to forest T that is **not** contained in any MST.
- e (by construction) is the cheapest **safe** edge for *some cut* $(S, V\backslash S)$. It suffices to show there is some MSF which contains $T \cup \{e\}$.



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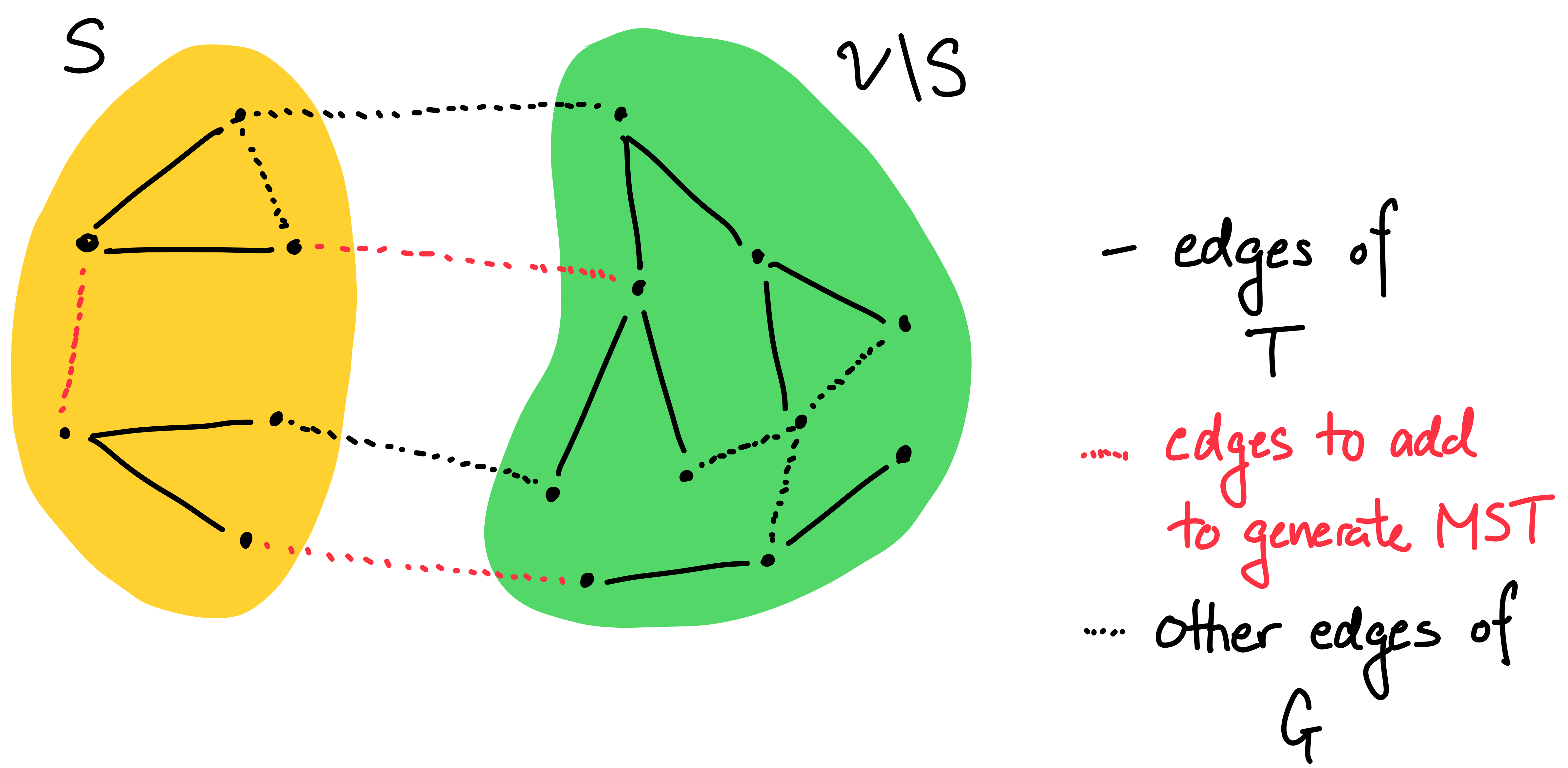

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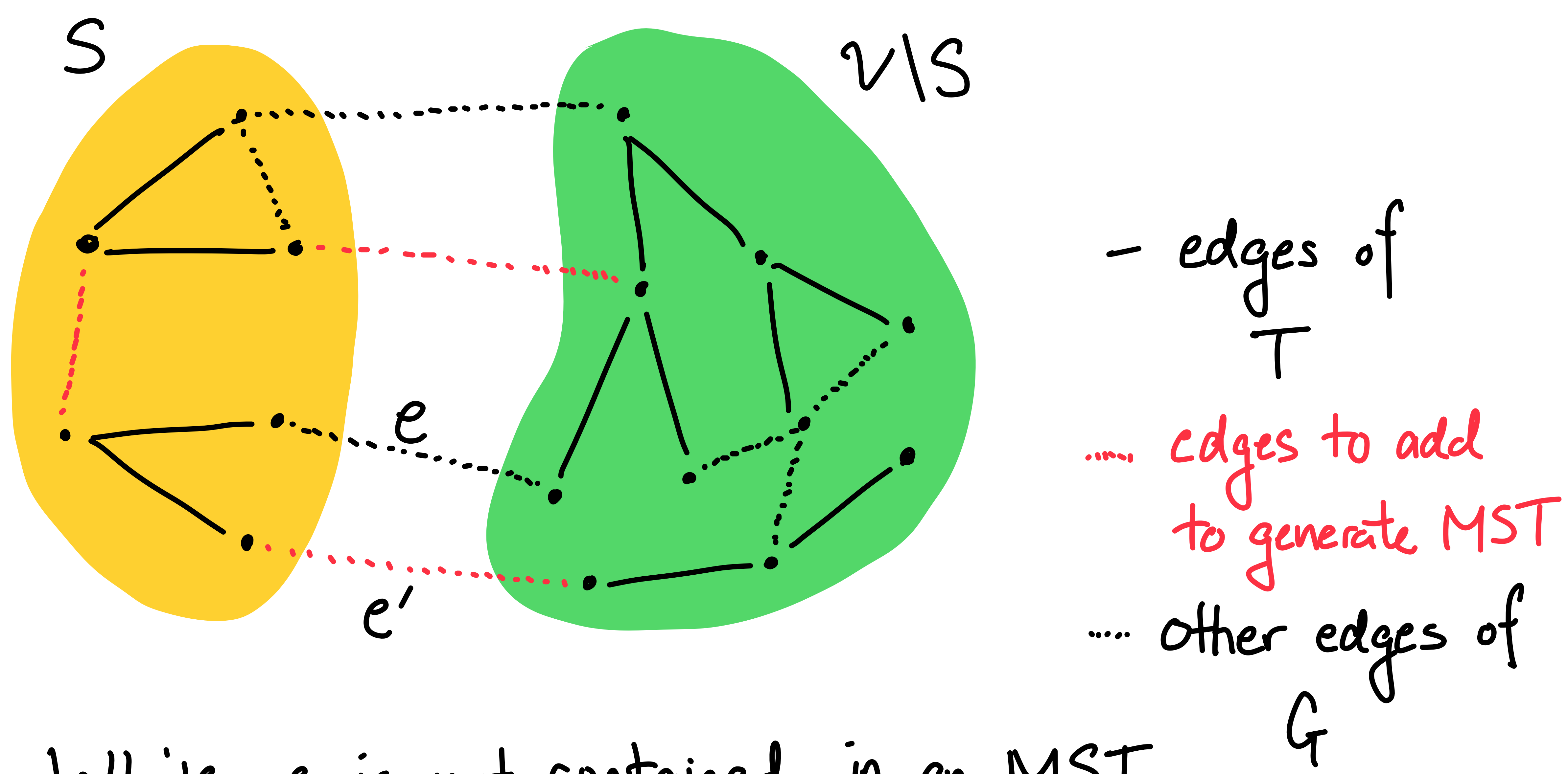


While e is not contained in an MST, some other edge e′ crossing (S,V\S) must. Since $w(e) \leq w(e')$, exchanging e for e′ cannot increase weight of spanning tree.

Applying proof for Prim's and Kruskal's

- **Prim's algorithm**
  - Add cheapest vertex from current tree to the rest
  - S equals the vertices connected by the tree T at that moment.
- **Kruskal's algorithm**
  - Add cheapest vertex connecting two trees T_1 and T_2
  - $S =$ the vertices in T_1 (amongst many possible defs. of S)

Implementation details for Prim's

- We need a data structure to keep track of distance from $u \in V \backslash S$ to S with the ability to quickly calculate the minimal element u.
- **Answer:** Priority queue
- **Initial state:** Q includes all of V with keys equaling ∞ except key of s is 0.
- **Update rule** when processing vertex u that we pop off the priority queue:
    - For each neighbor v, update key to $w(u, v)$ if necessary.

Runtime of Prim’s

- $O(n)$ insertions, $O(n)$ runs of delete-min, and $O(m)$ updates to the key
- Same resultant complexity as Dijkstra’s
  - Array implementation: $O(n^2)$ time
  - Heap implementation: $O(m \log n)$ time

Implementation details for Kruskal's

- Need to add edges of minimal weight but only if they don't form a cycle
- Helpful to first sort all the edges by weight: $O(m \log m) = O(m \log n)$ time
- Iterate through edges in sorted order
  - If the edge connects two trees in the forest, we add. Otherwise skip.
  - Need a data structure to handle this type of query: **Union-Find**
- Total cost of Union-Find is $O(m \cdot \alpha(n))$ with $\alpha(n) \ll \log m$
- Dominant runtime is from sorting for $O(m \log m)$ time.

Union-find data structure

Also known as disjoint-set data structure

- Stores a collection of **disjoint** (non-overlapping) subsets of $[n]$
- Allowed operations and runtimes
  - $\text{Makeset}(x)$ create a new set with only the element x. Takes $O(1)$ time
  - $\text{Find}(x)$ returns the “name” of the set containing x. Takes $O(\alpha(n))$ time*
  - $\text{Merge}(x, y)$ merges the sets containing x and y. Takes O($\alpha(n)$) time*

Implementation details for Kruskal's

- Kruskal's requires $O(n)$ initalizations, $O(m)$ finds and $O(n)$ merges of sets
- Total *amortized* runtime is $O(m \log n) + O(m\alpha(n)) = O(m \log n)$.
- Data structures matter!
    - Union-find is a data structure optimized for an algorithm like Kruskal's
    - Generically using an array would yield $O(n^2)$ since merge is slow.

The "cut"/"lightest-edge" property

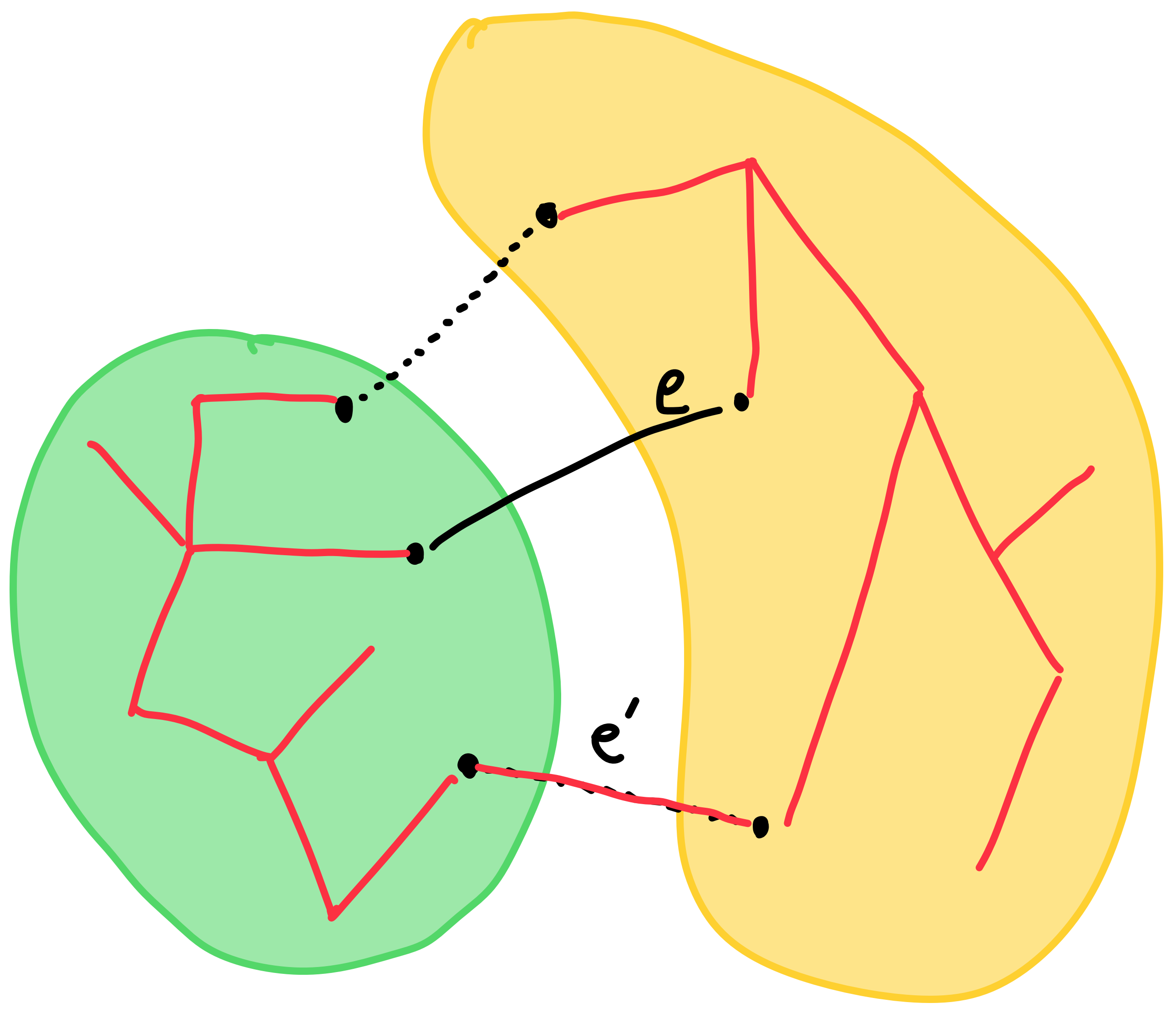
- **Lemma:** For any cut $(S, V\backslash S)$ of G, if e is a **minimum**-weight edge crossing the cut, then there exists an MST that contains e.

- **Proof:**

  - An exchange principle argument. Assume all MSTs do not contain e.

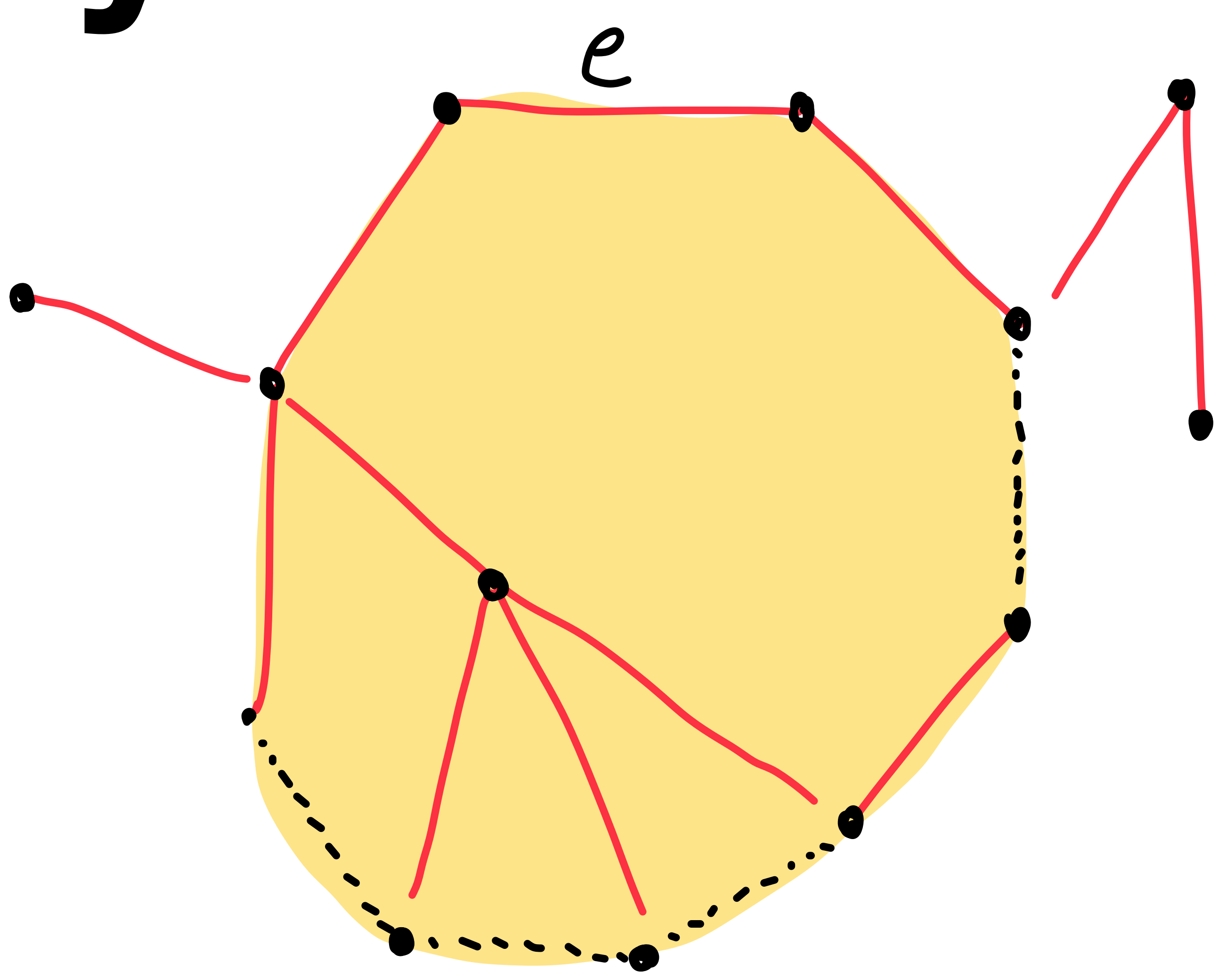
  - Any MST must contain at least one edge e' crossing cut.

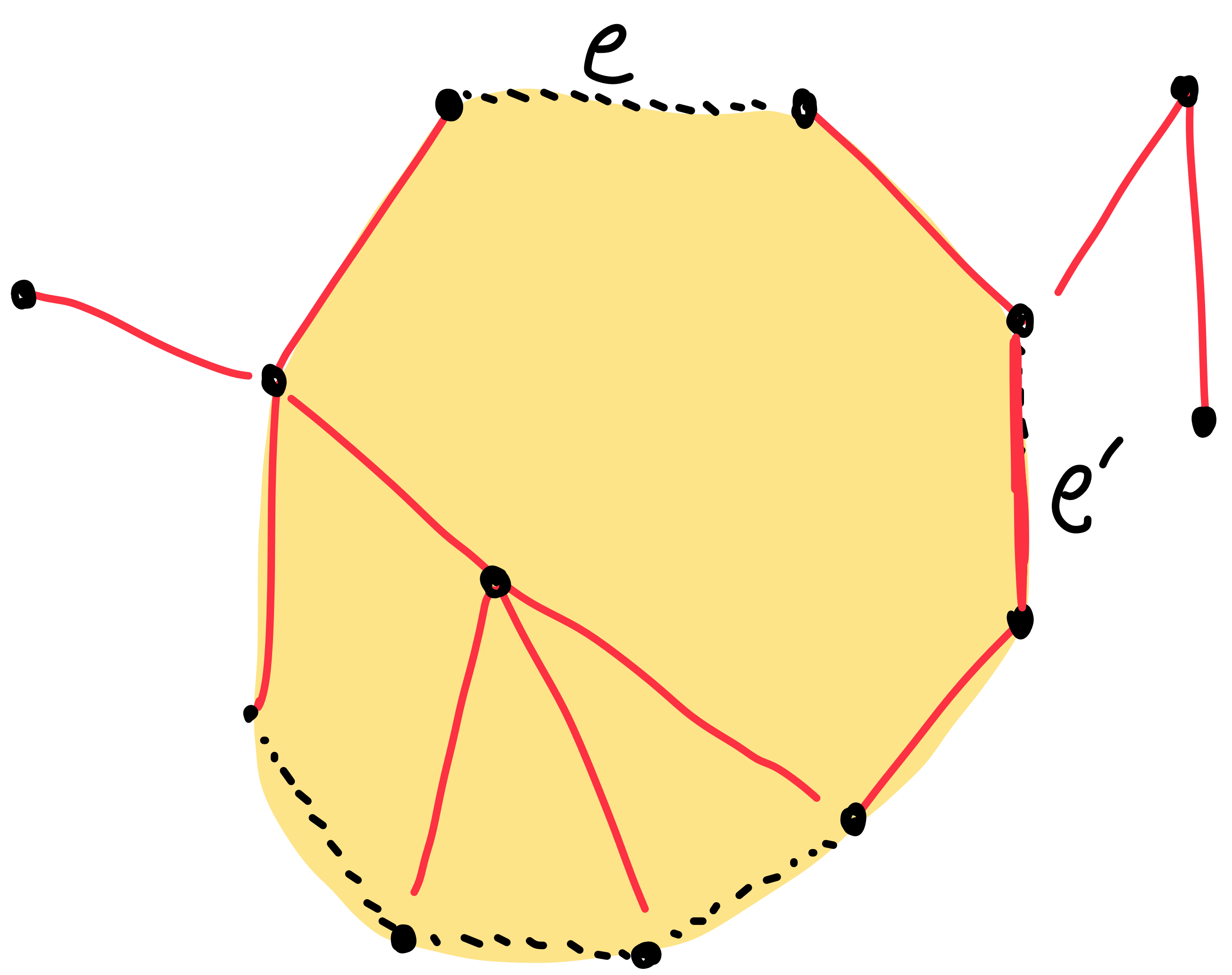
  - Replacing e' with e can only improve or maintain weight. So there exists a MST with e.



The "cycle"/"heavy edge" property

- **Lemma:** For any cycle C of G, if e is **strictly** the heaviest edge of C, then e does not participate in any MST.
- **Proof:**
  - An exchange principle argument. Consider any MST containing e. Now remove e.
  - This disconnects the MST into two trees.
  - Each little tree contains some continuous subset of the vertices of C.
  - Because it's a cycle, there exists edge e' connecting the two trees.
  - Replacing e with e' makes the tree lighter, a contradiction!





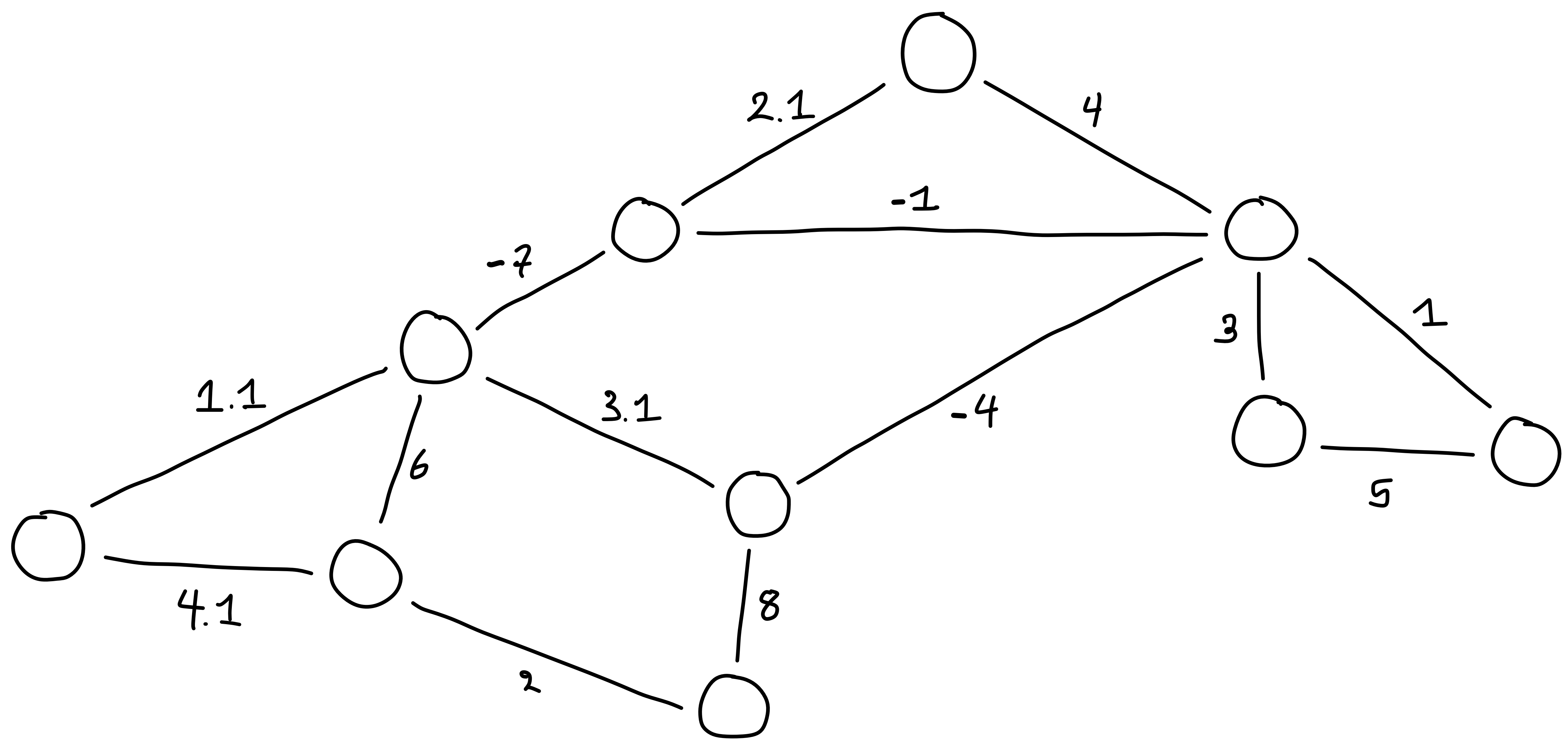
Parallelizing MST finding

Boruvka's algorithm (1927)

- Notice that until the trees in the forest during Kruskal's could grow in parallel until they join together
- Is there an algorithm for parallelizing this growth?
- At each step
  - Each tree chooses its cheapest outgoing edge
  - Two trees in the forest can choose to add the same edge
  - Need a tiebreaker on edge weights (no equal weights) to avoid generating cycles

Boruvka implementation example

Requires unique weights!



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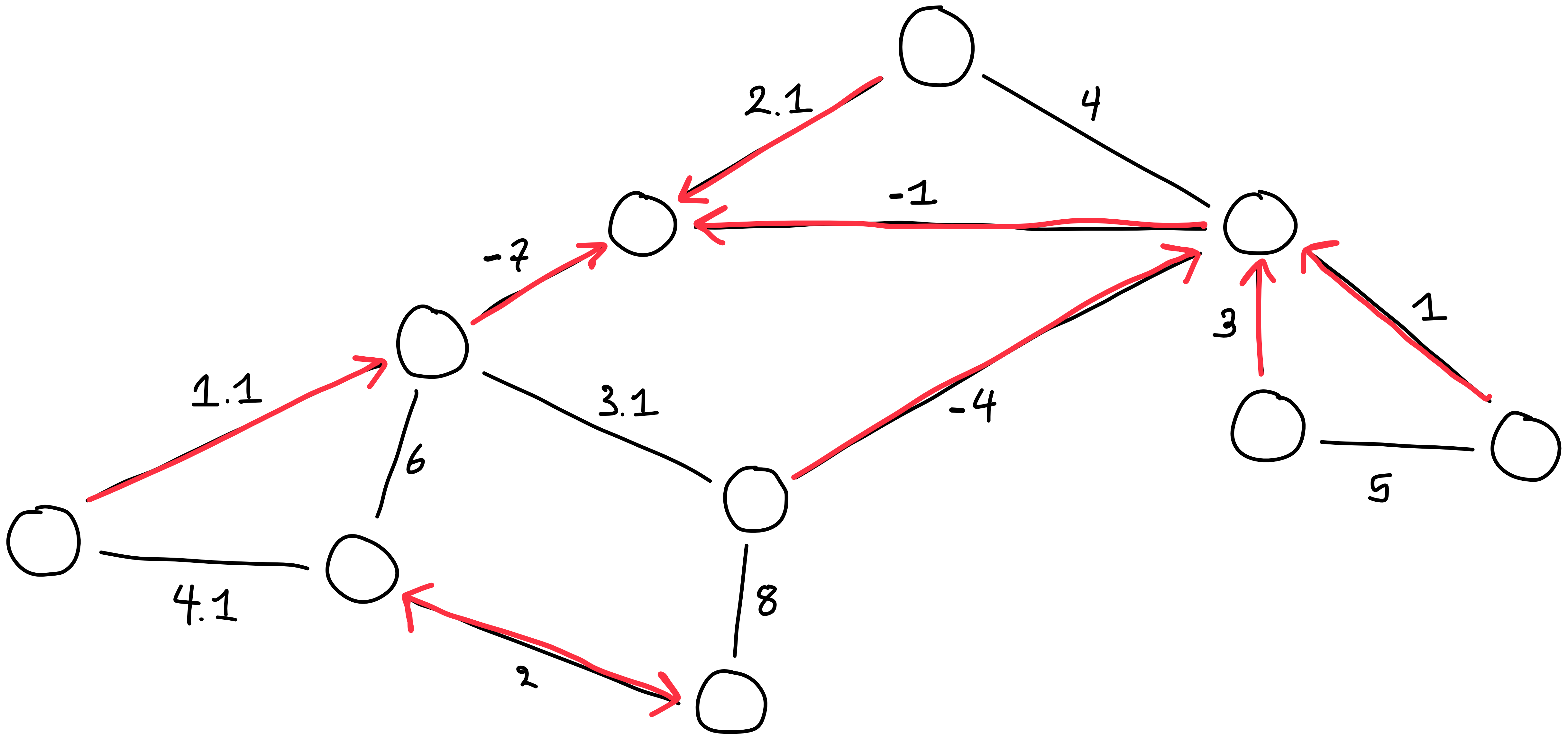

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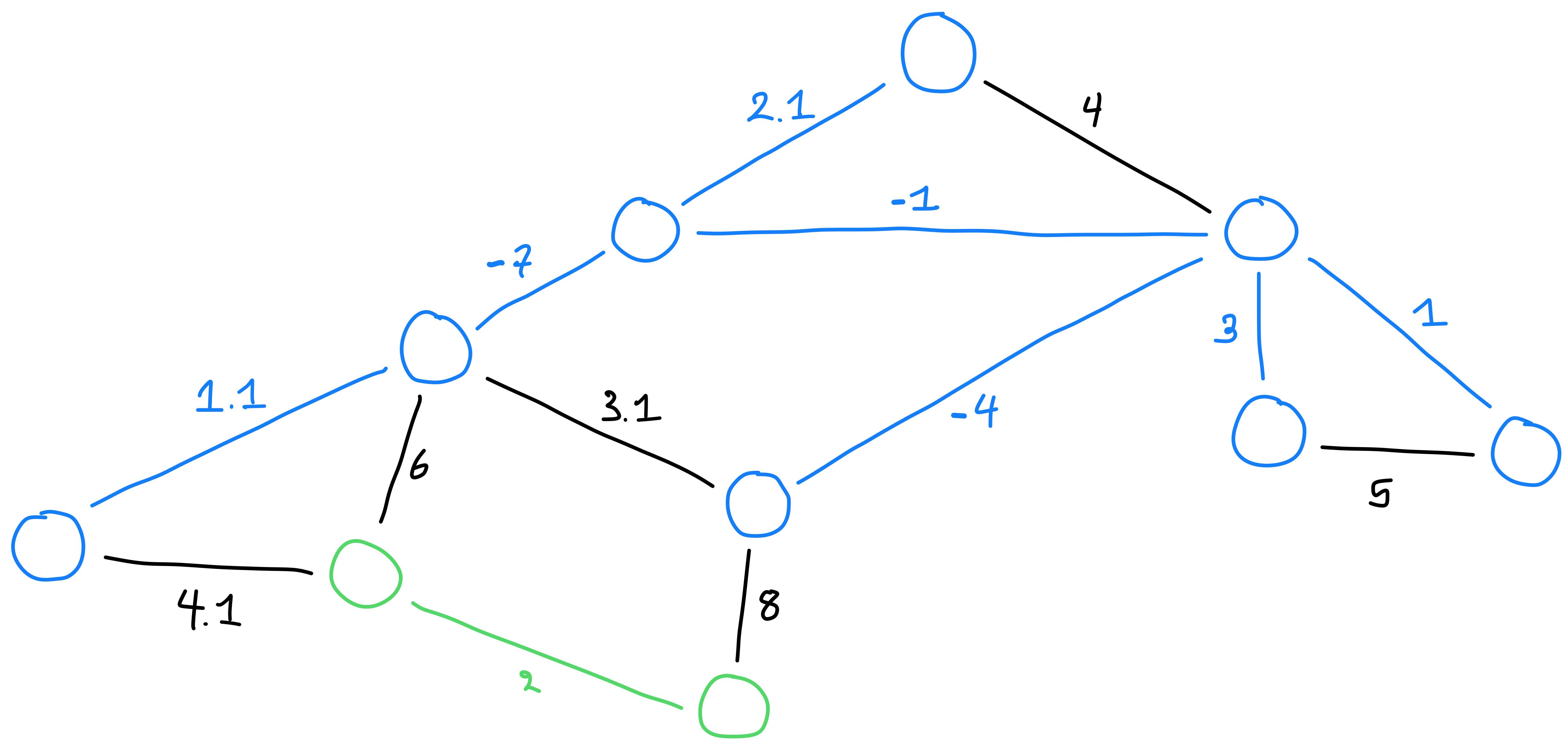

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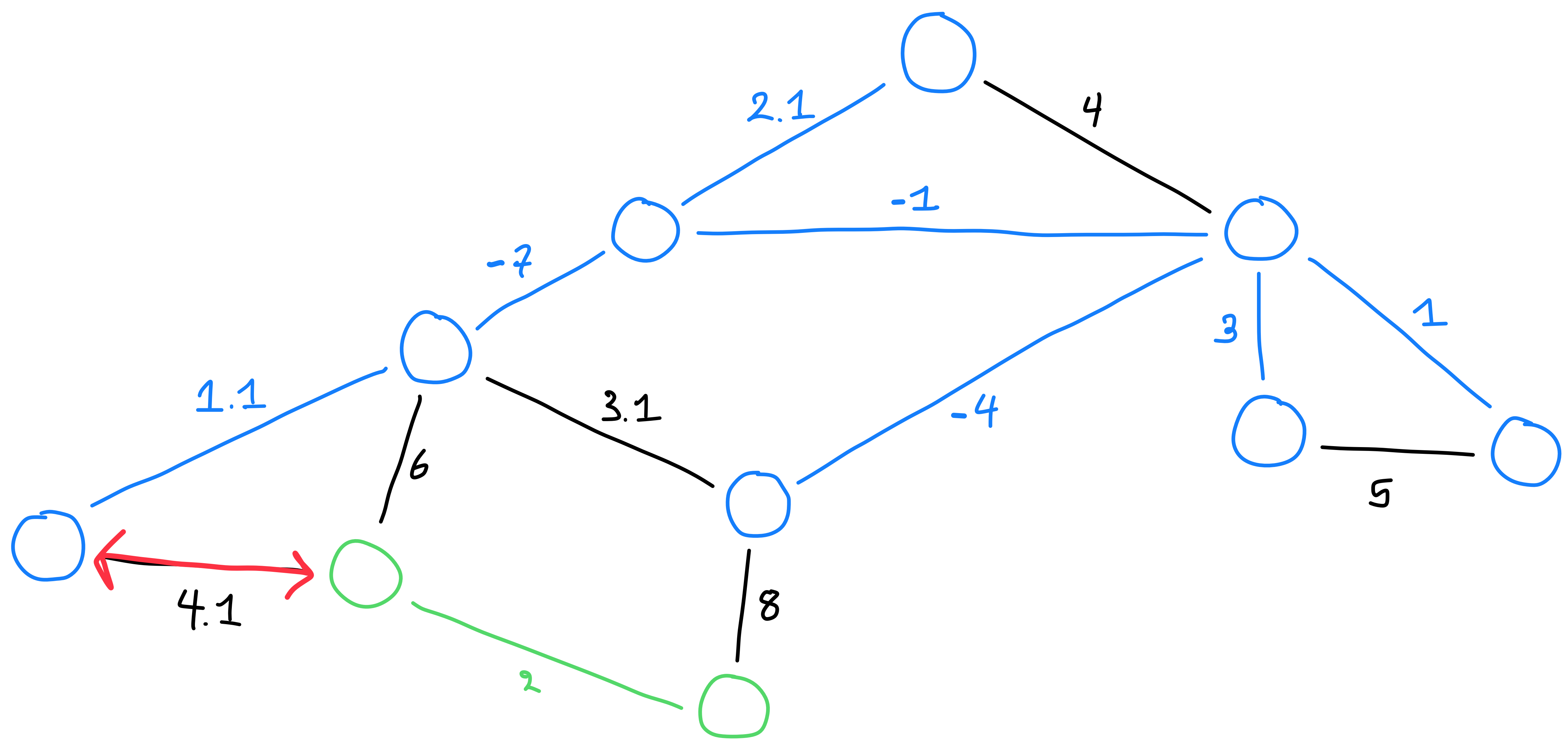

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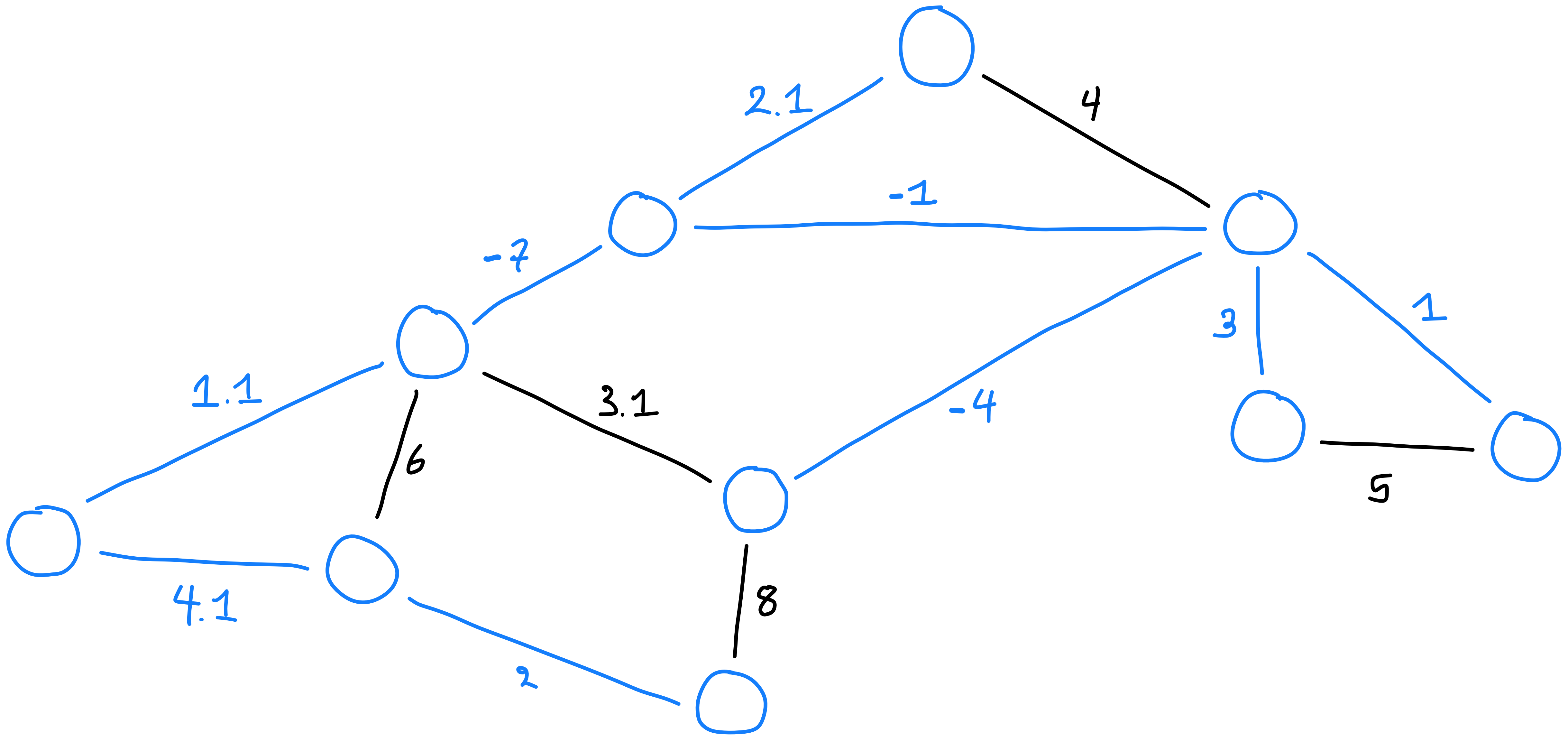


Other MST algorithms

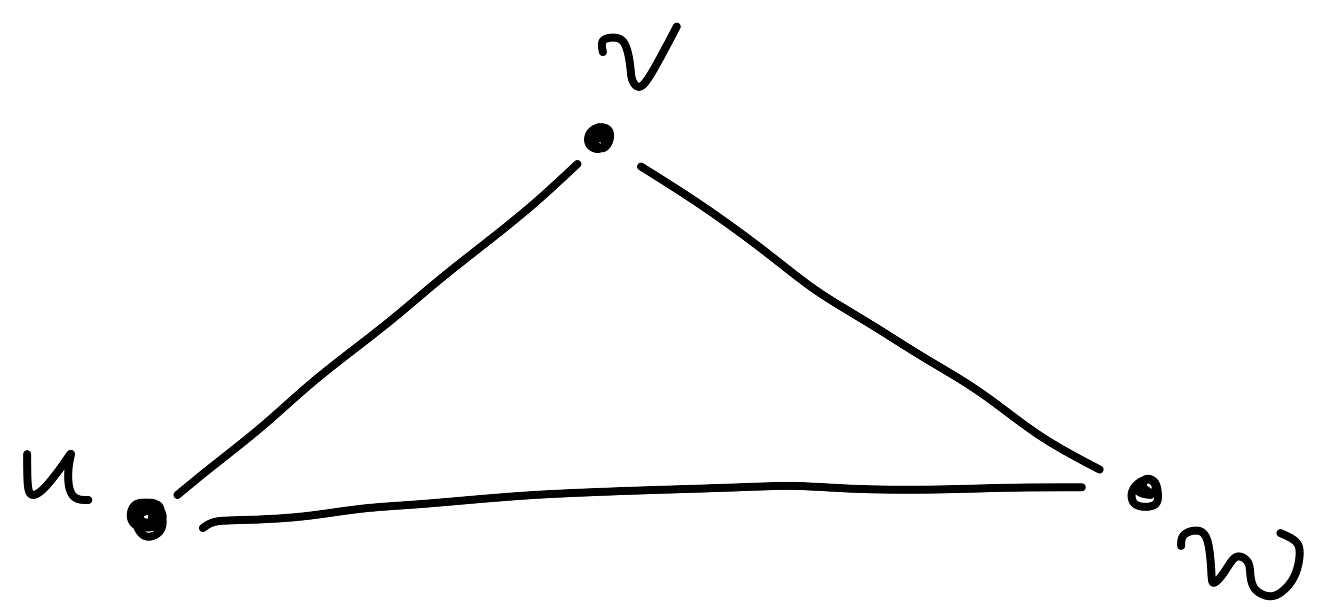
- **Cheritos and Tarjan:**
  - Uses a queue of components
  - Component at head chooses cheapest outgoing edge
  - New merged component goes to tail of the queue
  - $O(m \log \log n)$ time
- **Chazelle:** $O(m \cdot \alpha(m) \cdot \log(m))$ time
- **Karger, Klein, and Tarjan:** $O(m + n)$ time algorithm that works most of the time

Applications of MST

- Network design — minimal connectivity for telephone, electrical, cable, internet networks
- Approximation algorithms for computational problems - traveling problem, Steiner trees
- Indirect applications
  - Max bottleneck paths
  - LDPC error correcting codes
  - Image restoration under Renyí entropy
  - Reducing data storage in sequencing amino acids
  - Modeling local particle interaction in turbulence flows
  - Autoconfig protocol for Ethernet bridging to avoid network cycles

k-clustering of data points

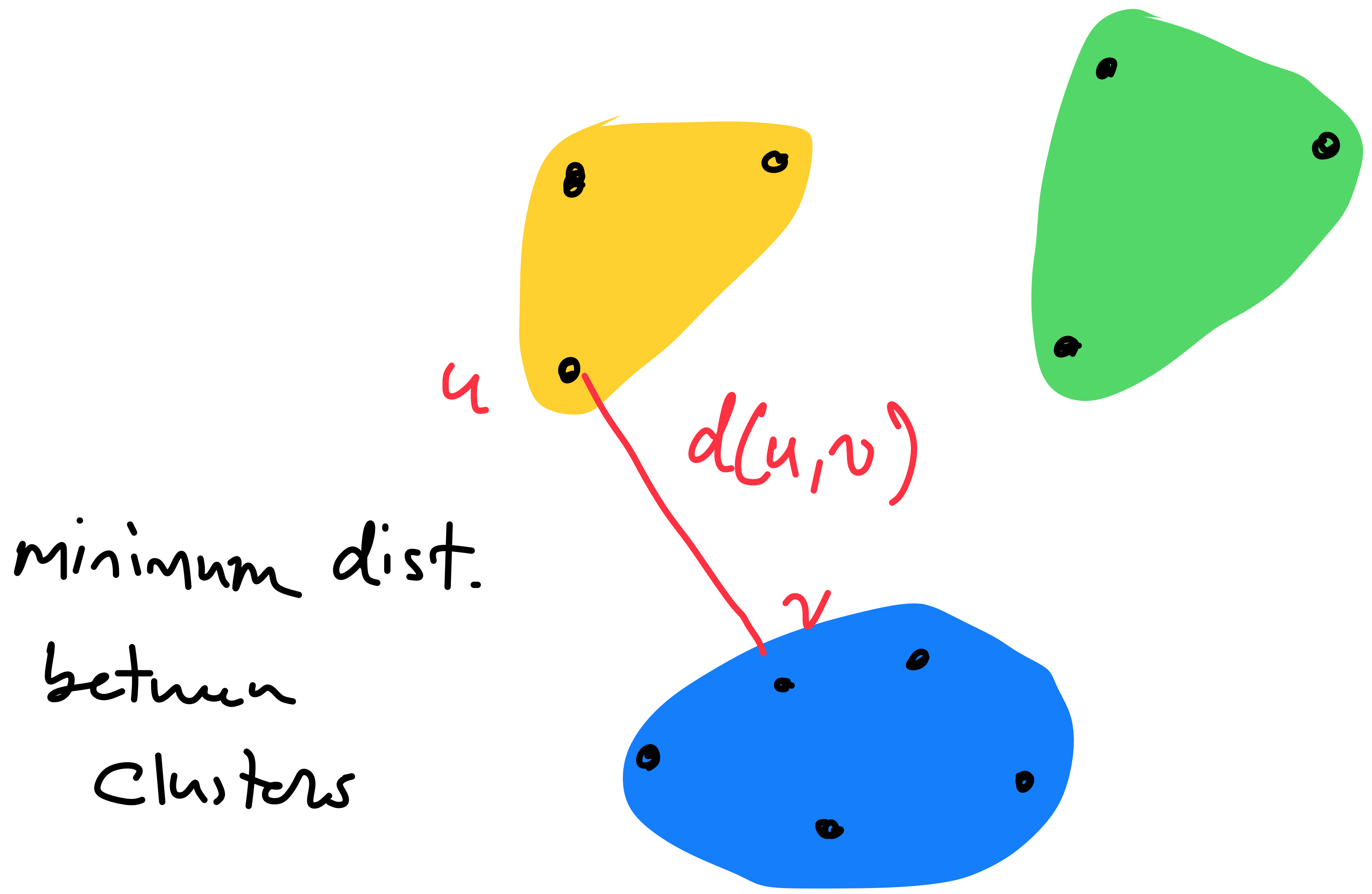
Maximum distance clustering



- **Input:** A set U of n elements, a **metric** $d : U^2 \to \mathbb{R}_{\geq 0}$, and $k \in \mathbb{N}$
  - Metric satisfies $d(u,u) = 0$, $d(u,v) = d(v,u)$
  - and triangle inequality $d(u,v) + d(v,w) \geq d(u,w)$
- **Output:** A clustering function $a : U \to [k]$ maximizing

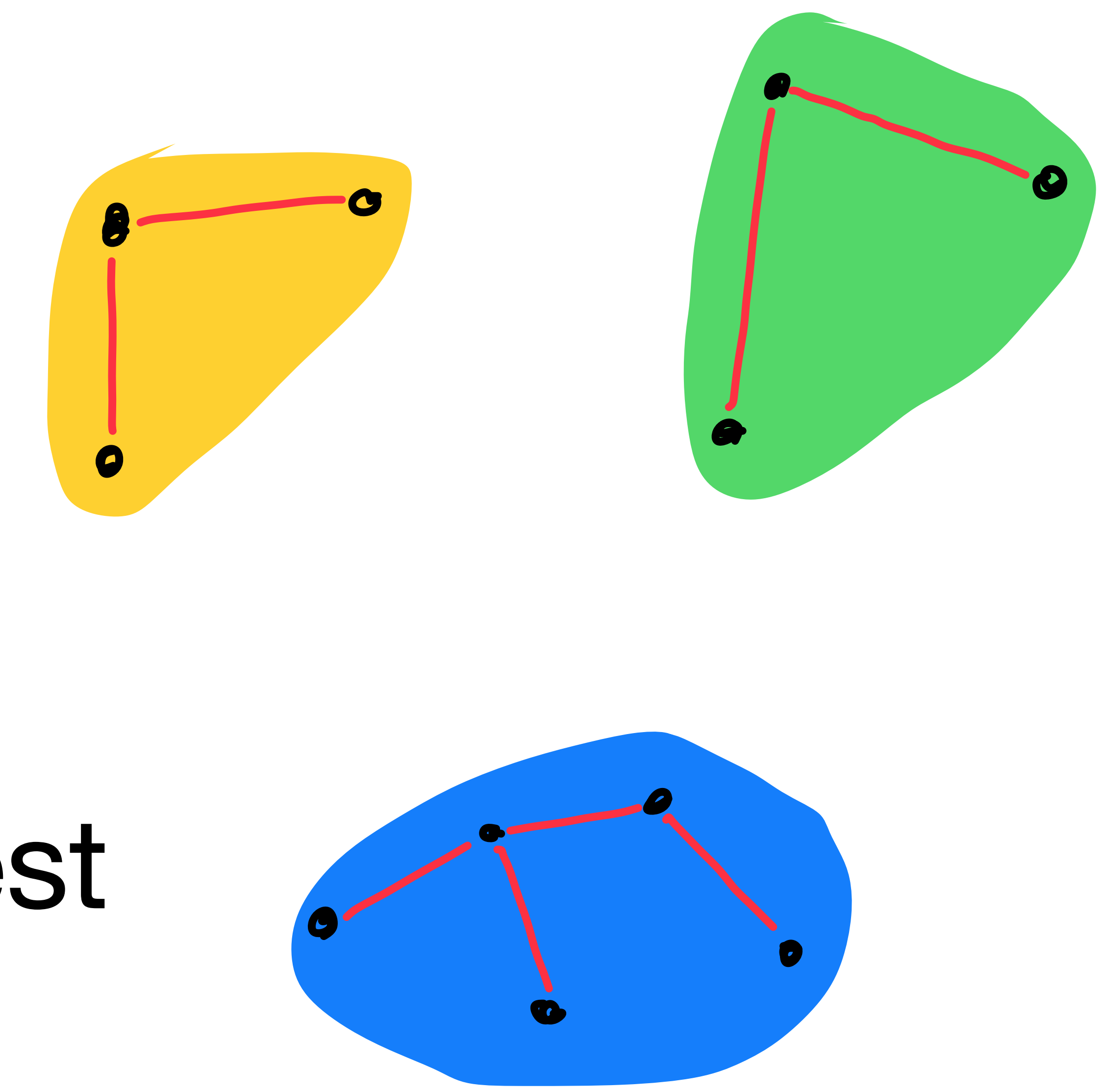
$$\min_{u,v \in U:\ a(u) \neq a(v)} d(u,v),$$

the minimum distance between the clusters



Kruskal's based algorithm

- Let $V = U$ and $E = V^2$ (all-to-all) with weight $w(e) = d(e)$.
- Run Kruskal's until $n - k$ edges are added.
  - Ensures that there are k trees in the forest.
  - Assign a cluster for every tree.
- Alternatively, run any MST algorithm and delete the heaviest $k - 1$ edges from the output tree.



Maximum distance clustering optimality

- Let d^* be the dist. between clustering a generated by Kruskal's
- By our alg. design, $d^* \geq d(u, v)$ for u, v in the same cluster: $a(u) = a(v)$.
- Consider a *different* clustering $b : U \to [k]$
  - There exist two points such that $a(u) = a(v)$ but $b(u) \neq b(v)$.
  - Then spacing between clusters of b is at most $d(u, v) \leq d^*$.
  - So b is no better than a so a is optimal.