Lecture 5

Greedy approximation and graph algorithms

Chinmay Nirkhe | CSE 421 Winter 2026

Previously in CSE 421…

Greedy algorithm general strategy

- **Greedy algorithm stays ahead:** Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithms
- **Structural:** Discover a structure-based argument asserting that the greedy solution is at least as good as every possible solution.
- **Exchange argument:** We can gradually transform any solution into the one found by the greedy algorithm with each transform only improving or maintaining the value of the current solution.

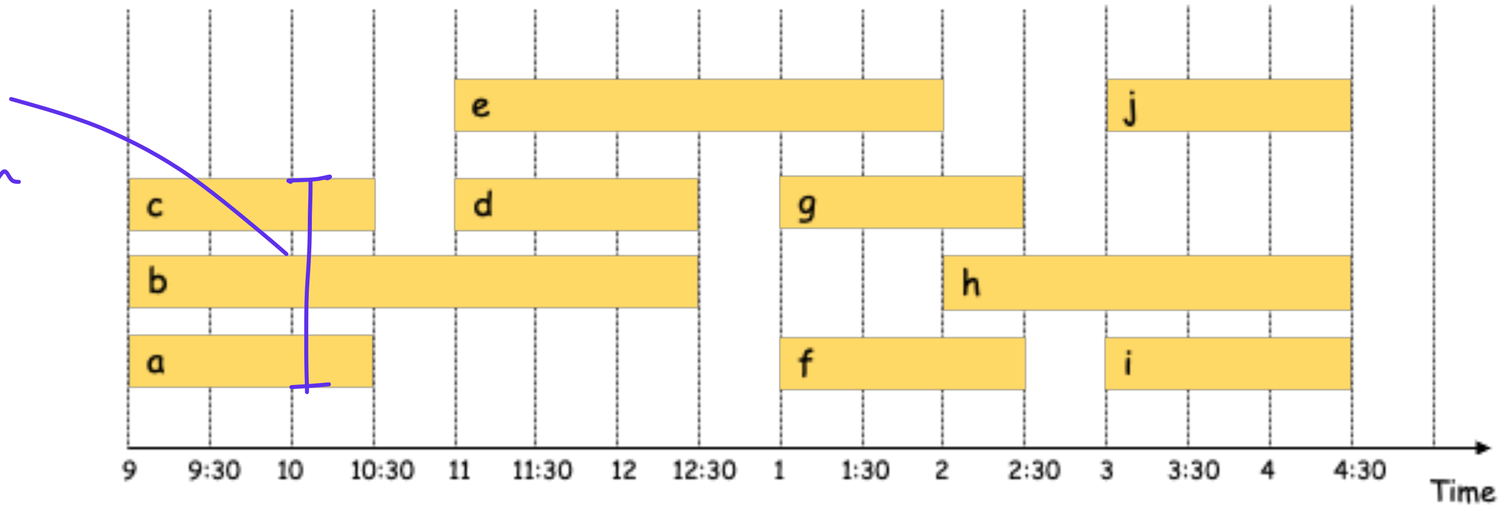
Today

Scheduling all intervals

- **Input:** (s_i, t_i) for $i = 1, \ldots, n$ for n "jobs" each using 1 room.
- **Output:** A scheduling of **all** jobs to rooms using the *minimum number* of rooms so that no two use the same room at the same time.

depth = 3

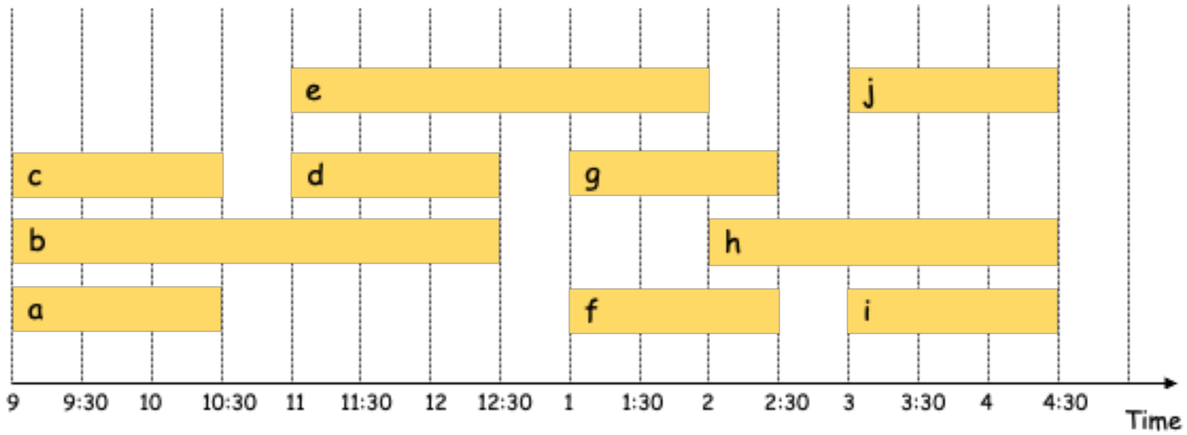
An easy minimum on the number of rooms required.



Scheduling all intervals

A greedy algorithm

- **Greedy strategy:** Increment chronologically and open a new room if all rooms are currently full.
- **Algorithm:**
  - Sort requests by start time $s_1 \leq s_2 \leq \ldots \leq s_n$ in increasing order.
  - Initialize an n sized array $\text{last}(j)$ as zeroes and an n sized array $r(j)$.
  - For $i \leftarrow 1$ to n
    - Find the first j such that $s_i \geq \text{last}(j)$.
    - Then set $\text{last}(j) \leftarrow t_i$ and set $r(i) \leftarrow j$.
  - Return assignment function r.



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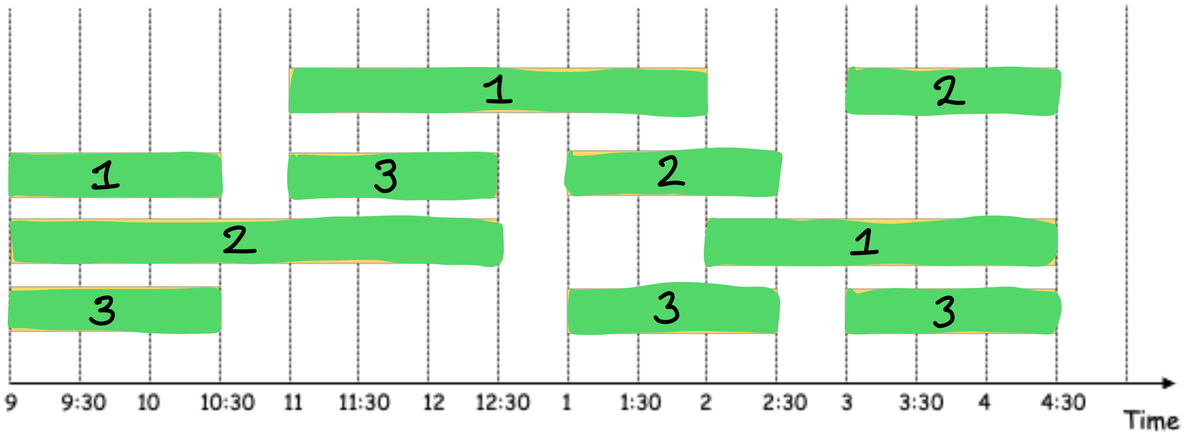

Scheduling all intervals

Proof of correctness

- **Theorem:** The greedy algorithm is minimal: If room j is ever used, then there is a time when $\geq j$ jobs are occurring simultaneously.
- **Proof:**
  - Consider when a new room j is “allocated” for the **first** time. Let job i be the reason.
  - Then, $s_i < \text{last}(j')$ for all $j' < j$.
  - Since $\text{last}(j')$ denotes when the jobs in the other rooms will free up, the i-th job is incompatible with the jobs currently in the other $j - 1$ rooms.
  - Since we sort requests by start time, those jobs all started before s_i and haven’t ended yet.
  - So there are $\geq j$ incompatible requests, requiring at least $\geq j$ rooms.

Scheduling all intervals
Runtime

- **Greedy strategy:** Increment chronologically and open a new room if all rooms are currently full.
- **Algorithm:**
  - Sort requests by start time $s_1 \leq s_2 \leq \ldots \leq s_n$ in increasing order. ← $O(n \log n)$ time
  - Initialize an n sized array $\text{last}(j)$ as zeroes and an n sized array $r(j)$.
  - For $i \leftarrow 1$ to n
    - Find the first j such that $s_i \geq \text{last}(j)$.
    - Then set $\text{last}(j) \leftarrow t_i$ and set $r(i) \leftarrow j$.
  - Return assignment function r. ← $O(1)$

Loop runs $O(n)$ times.

could be slow. $O(n)$ each time.

Total: $O(n^2)$ due to loop.

Scheduling all intervals

Runtime

- **Greedy strategy:** Increment chronologically and open a new room if all rooms are currently full.
- **Algorithm:**
  - Sort requests by start time $s_1 \leq s_2 \leq \ldots \leq s_n$ in increasing order. ← Still $O(n \log n)$.
  - Initialize a **priority queue** Q, $k \leftarrow 0$ and an n sized array $r(j)$. Better data structure
  - For $i \leftarrow 1$ to n
    - Set $j \leftarrow \text{findmin}(Q)$. ← Now only $O(\log k)$ time.
    - If $s_i \geq \text{last}(j)$, schedule job i in room j: $\text{setkey}(j, Q) \leftarrow t_i$ and $r(i) = j$. } Also $O(\log k)$ time.
    - Else, allocate a new room $k \leftarrow k + 1$ and $\text{setkey}(k, Q) \leftarrow t_i$ and $r(i) = k$.
  - Return assignment function r.

Total runtime: $O(n \log k)$ due to better data structure.

Greedy algorithm general strategies

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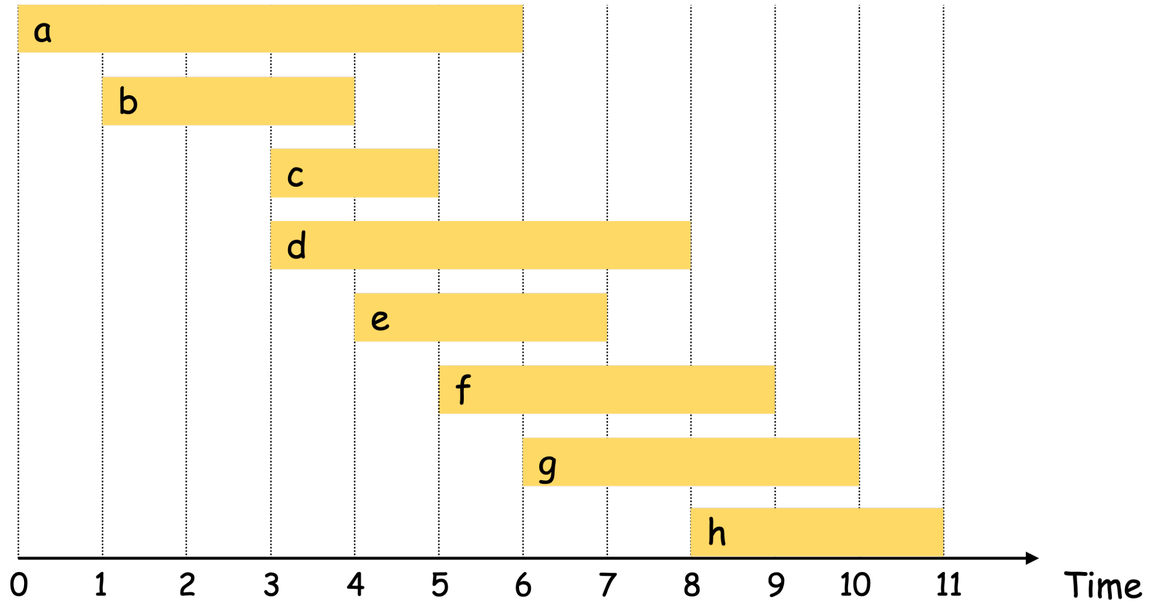
- **Exchange argument:** We can gradually transform any solution into the one found by the greedy algorithm with each transform only improving or maintaining the value of the current solution.

Interval scheduling

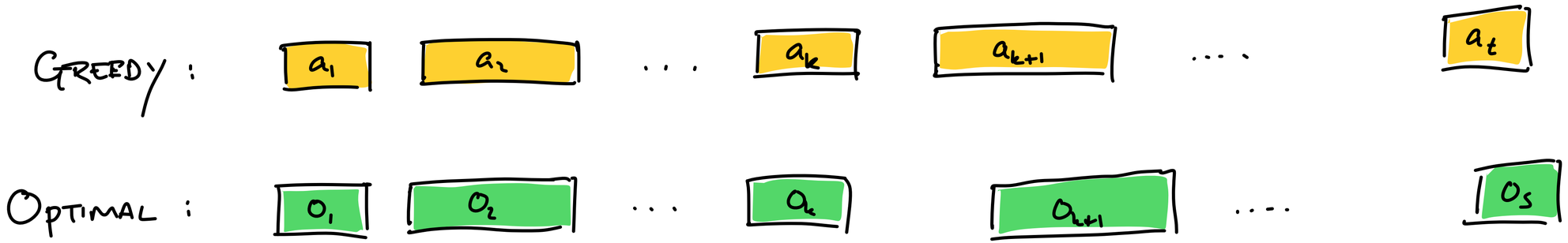
- **Input:** start and end times (s_i, t_i) for $i = 1, \ldots, n$ for n "jobs"
- **Output:** A maximal set of mutually compatible jobs



Greedy algorithm analysis

Contradiction argument edition

- Let $a_1, a_2, \ldots, a_t$ denote the jobs selected by the greedy algorithm.
- Let $o_1, o_2, \ldots, o_s$ denote the jobs selected in an *optimal solution*.
- Assume $a_1 = o_1, a_2 = o_2, \ldots, a_k = o_k$ for the largest possible k.
- Since greedy is **not** optimal (by assumption), $s > k$.



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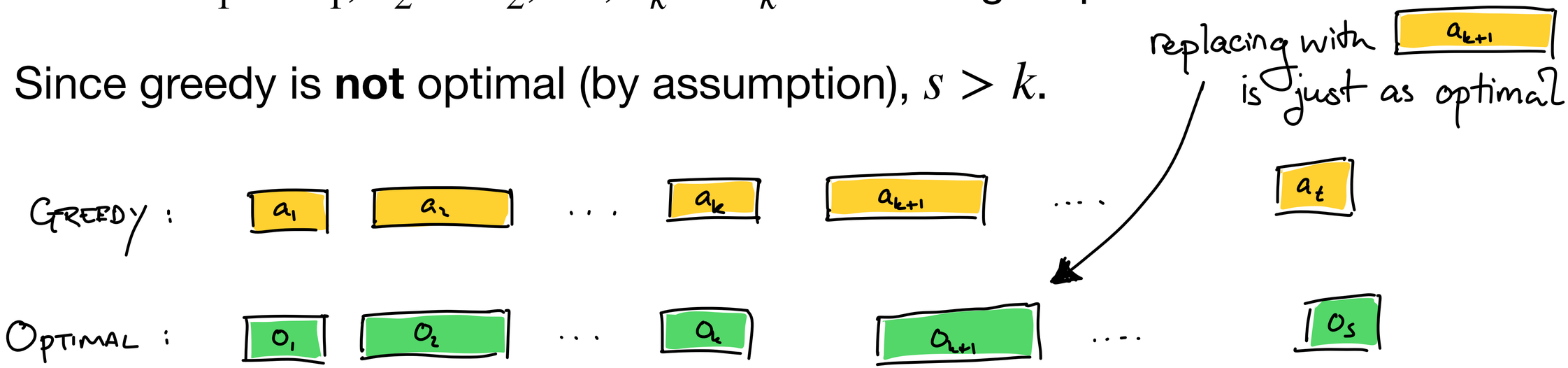

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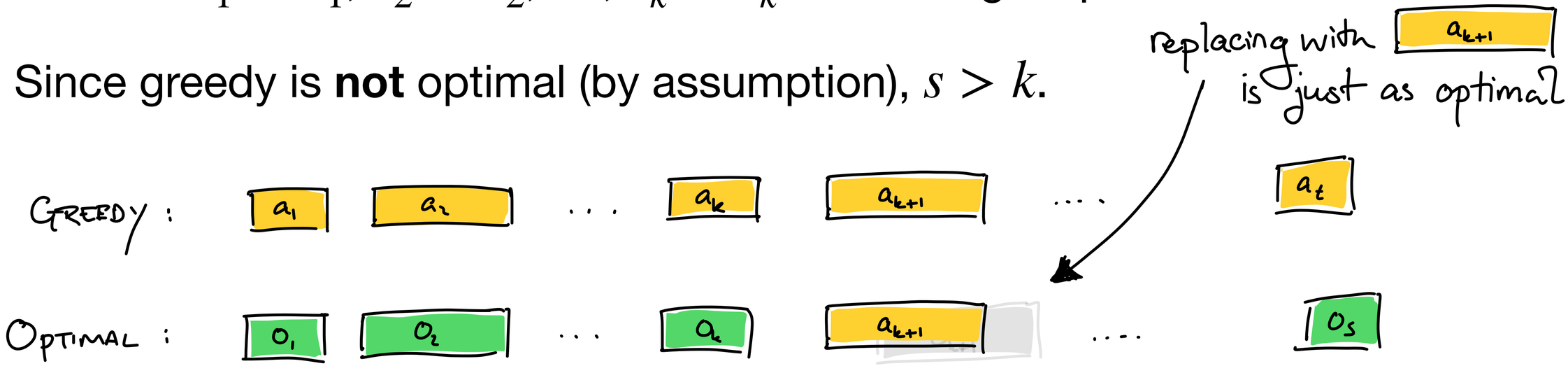

Greedy algorithm analysis

Contradiction argument edition

now we can induct.

- Let $a_1, a_2, \ldots, a_t$ denote the jobs selected by the greedy algorithm.
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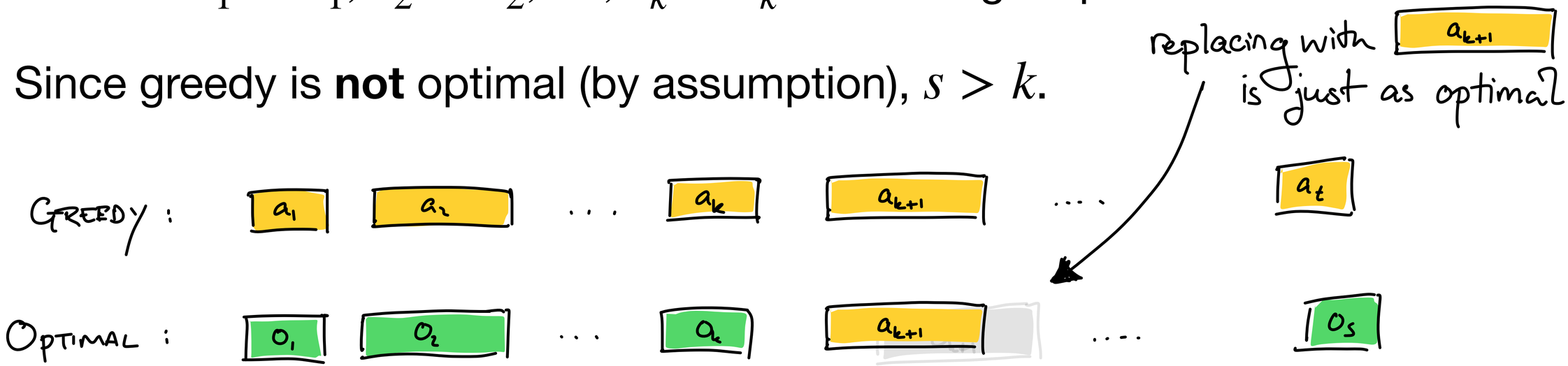
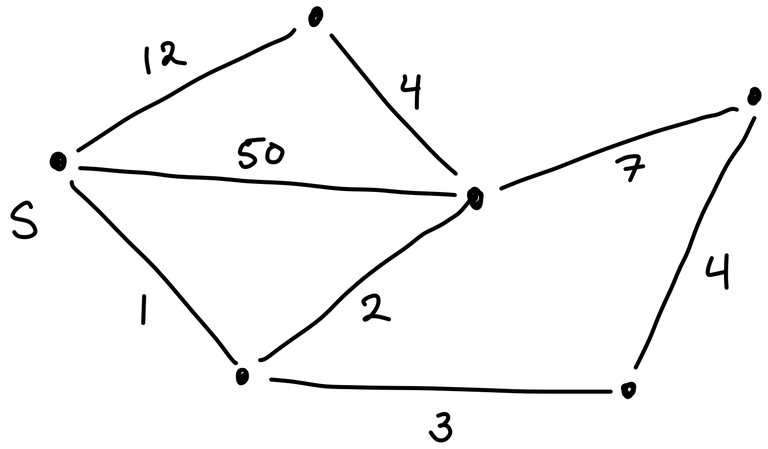


Greedy graph algorithms

Shortest path problem

- **Input:** $G = (V, E)$, edge weights $w : E \to \mathbb{R}_{\geq 0}$, and source $s \in V$.
- **Output:** $d : V \to \mathbb{R}^{\geq 0}$ with $d(u) =$ the min-weight of a path $s \rightsquigarrow u$.



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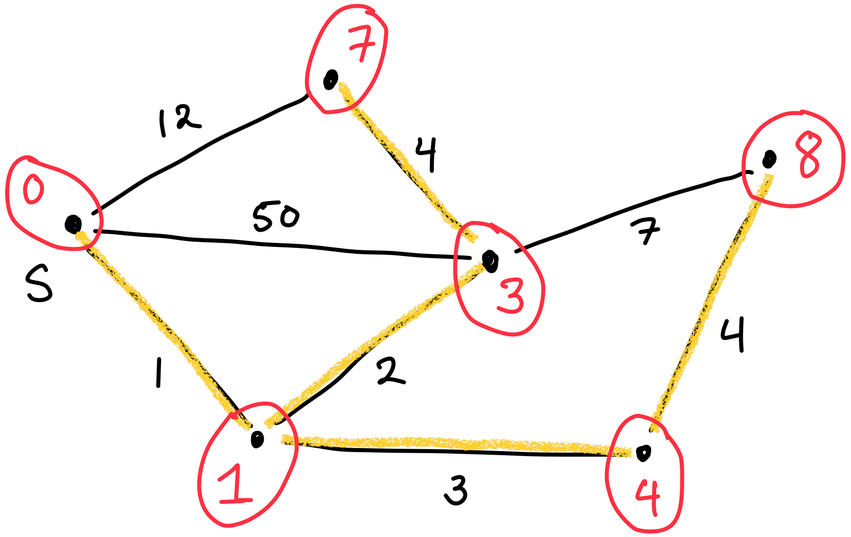


Dijkstra's algorithm

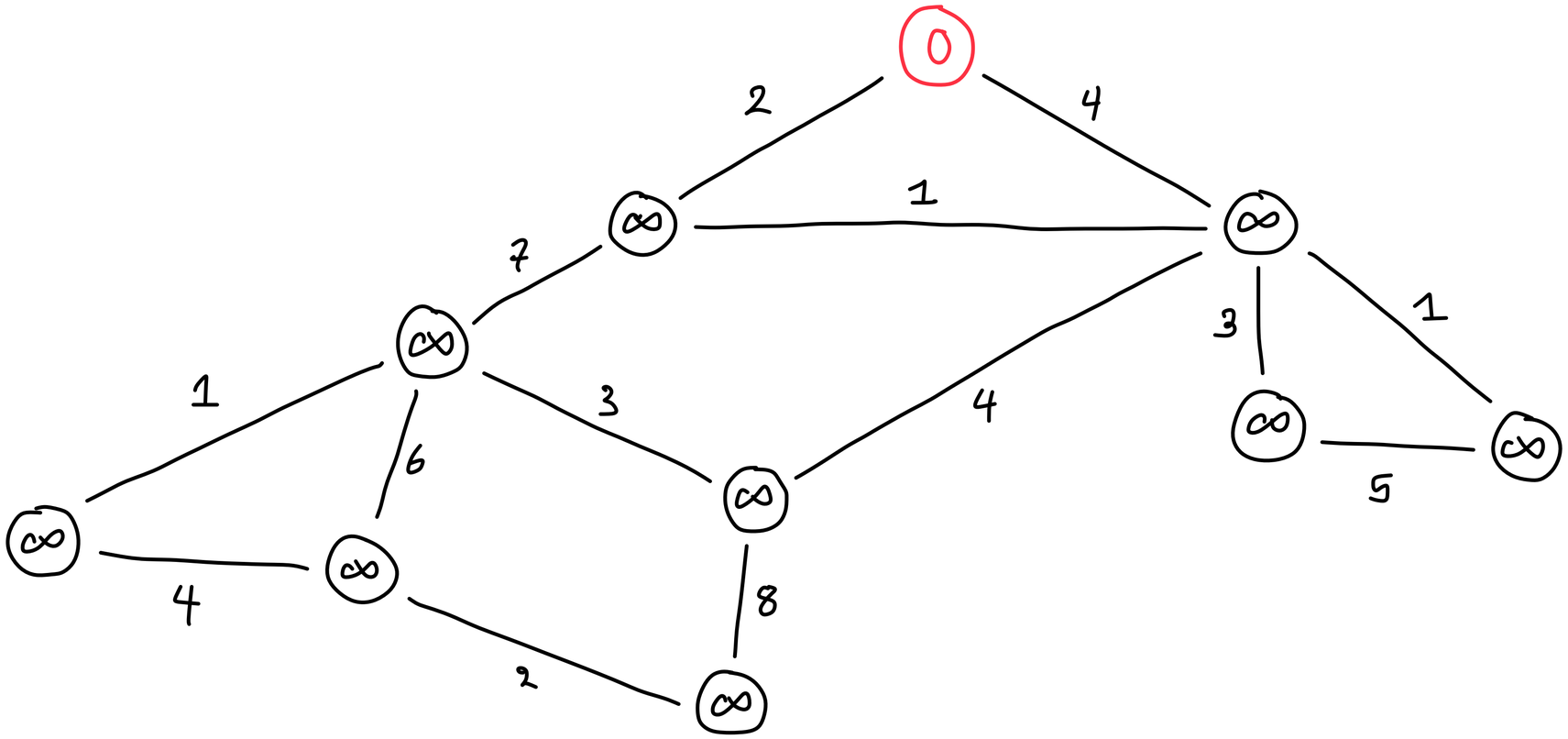
- Initialize $d(v) \leftarrow \infty, p(v) \leftarrow \perp$ ("parent" of v is undefined) for all $v \neq s$.
- Set $d(s) \leftarrow 0,\ p(s) \leftarrow \text{"root"}$
- Create priority queue Q and $\text{insert}(Q, \text{key} = d(v), v)$ for each $v \in V$
- While Q isn't empty, pop minimum key-element u from queue
  - For each neighbor v of u, check if $d(u) + w(u, v) < d(v)$
  - If so, $d(v) \leftarrow d(u) + w(u, v),\ p(v) \leftarrow u$, and $\text{setkey}(Q, \text{key} = d(v), v)$
- Return d, p for distance and parent functions.

once popped off d(u) is fixed forever

update parent of v to be u.

Dijkstra’s algorithm

Example execution



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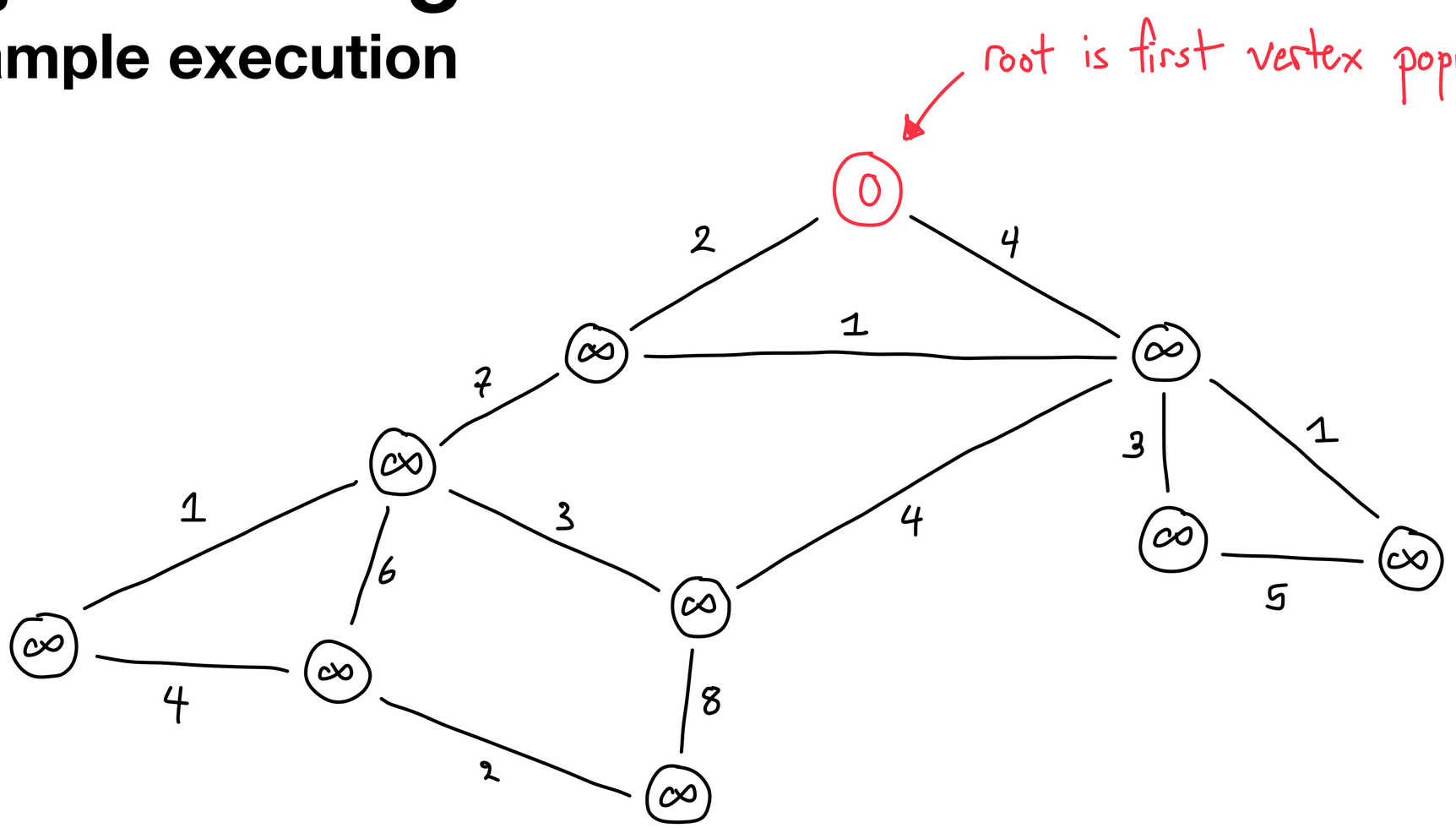

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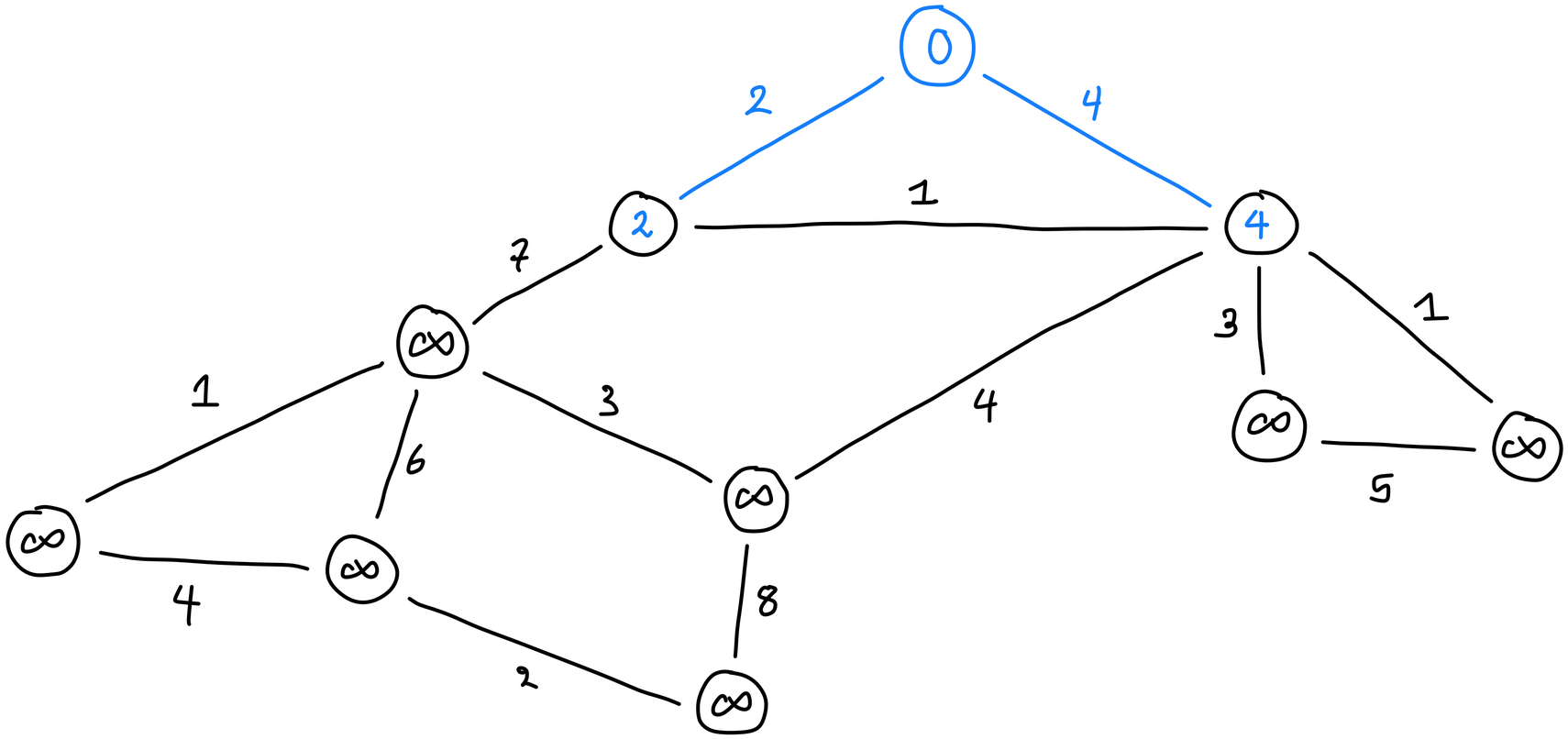

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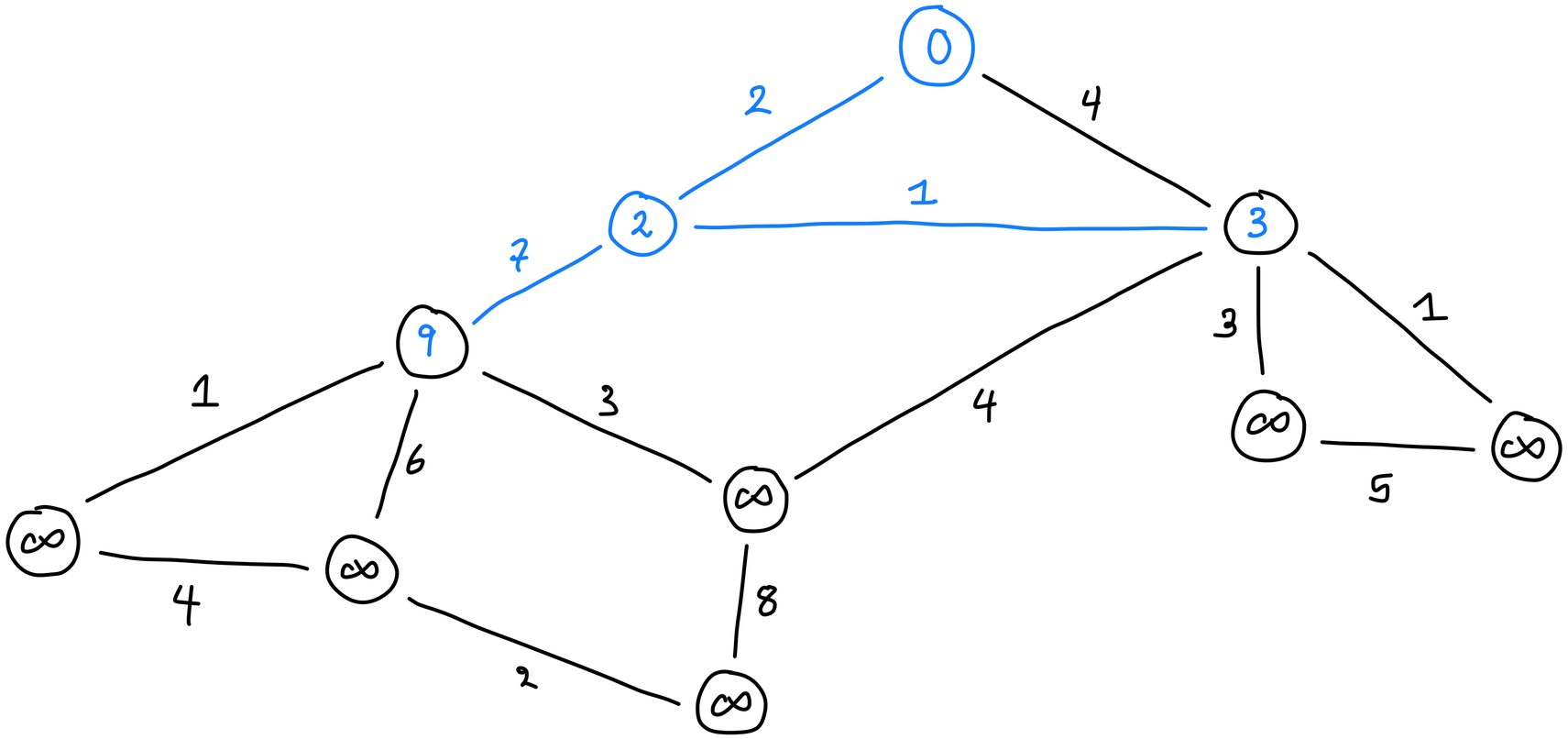

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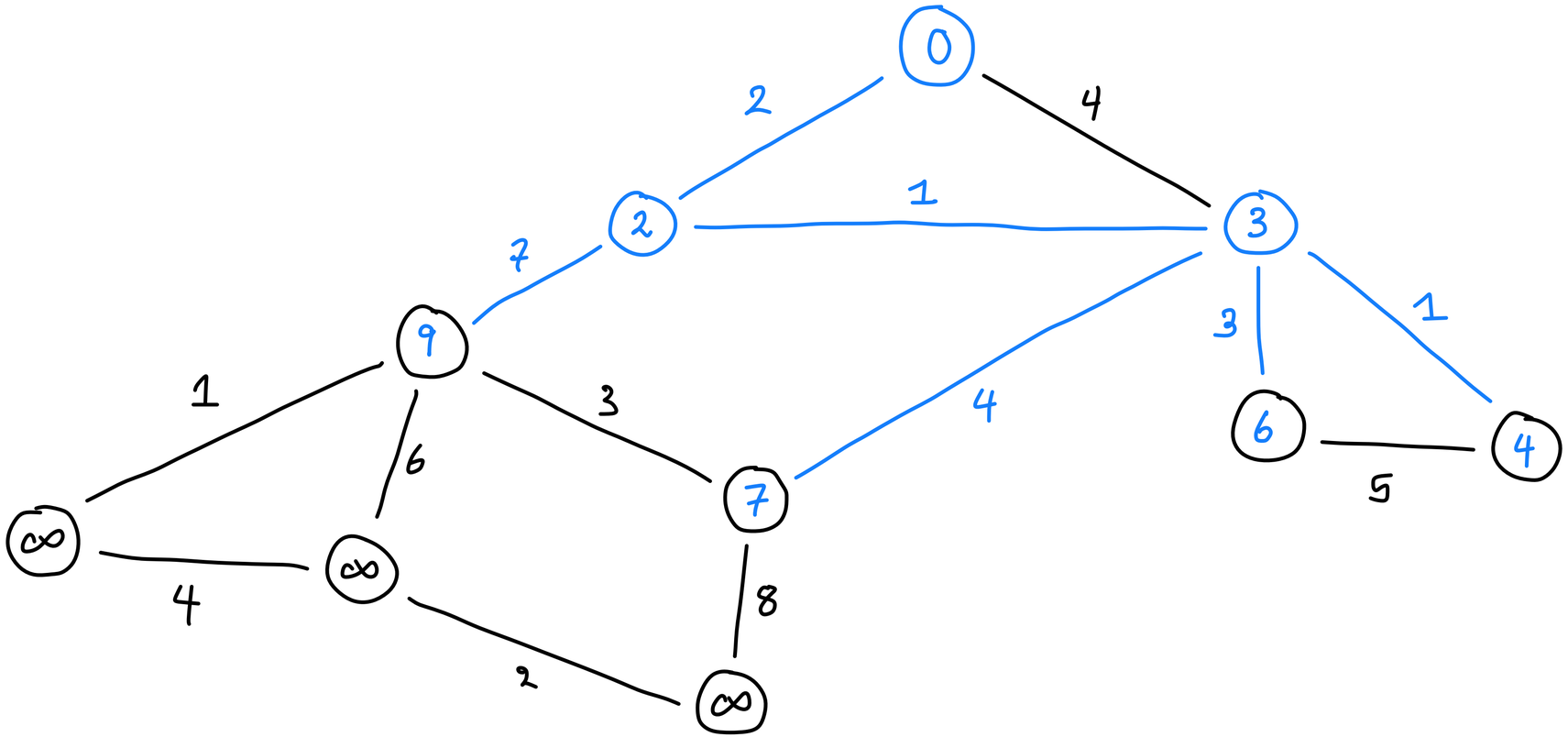

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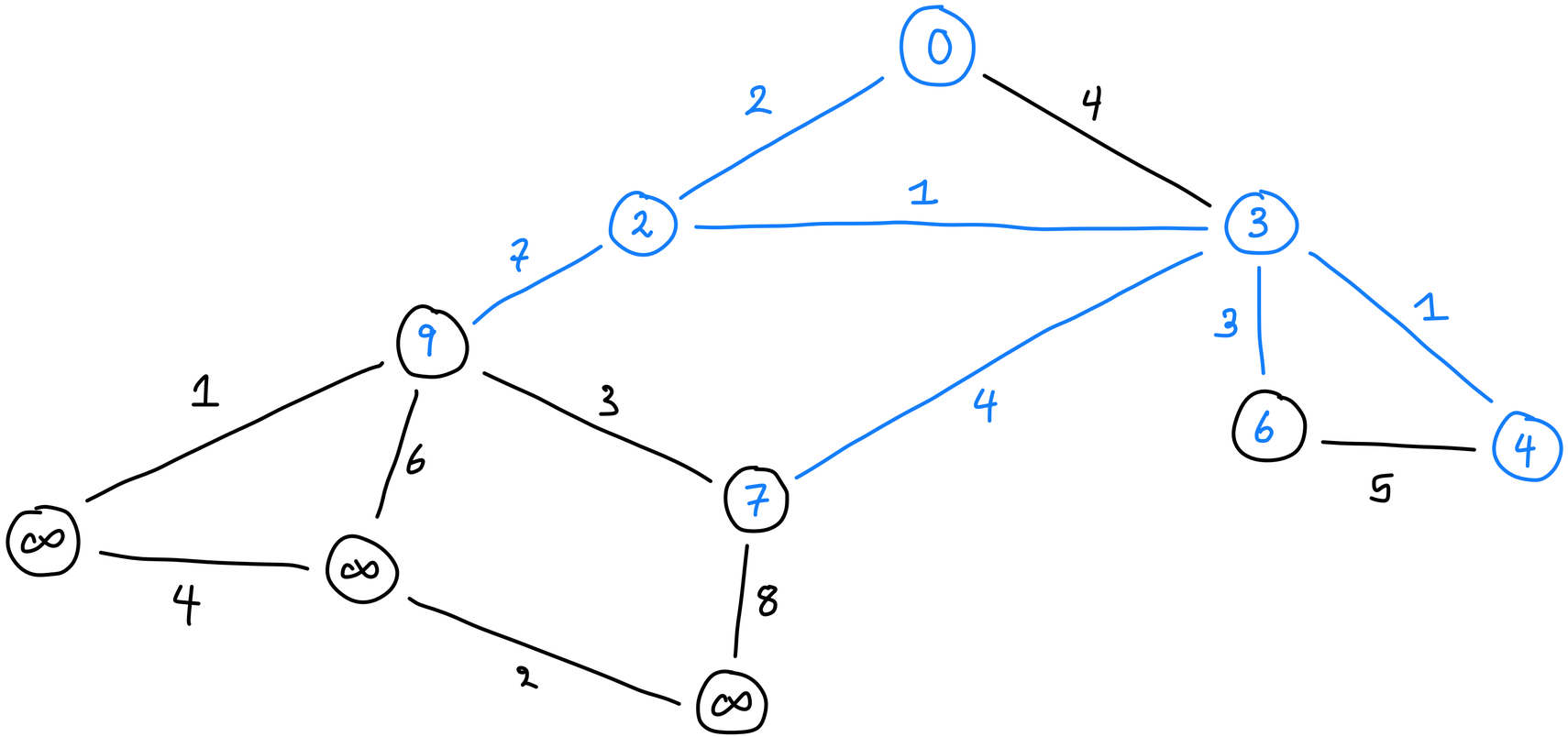

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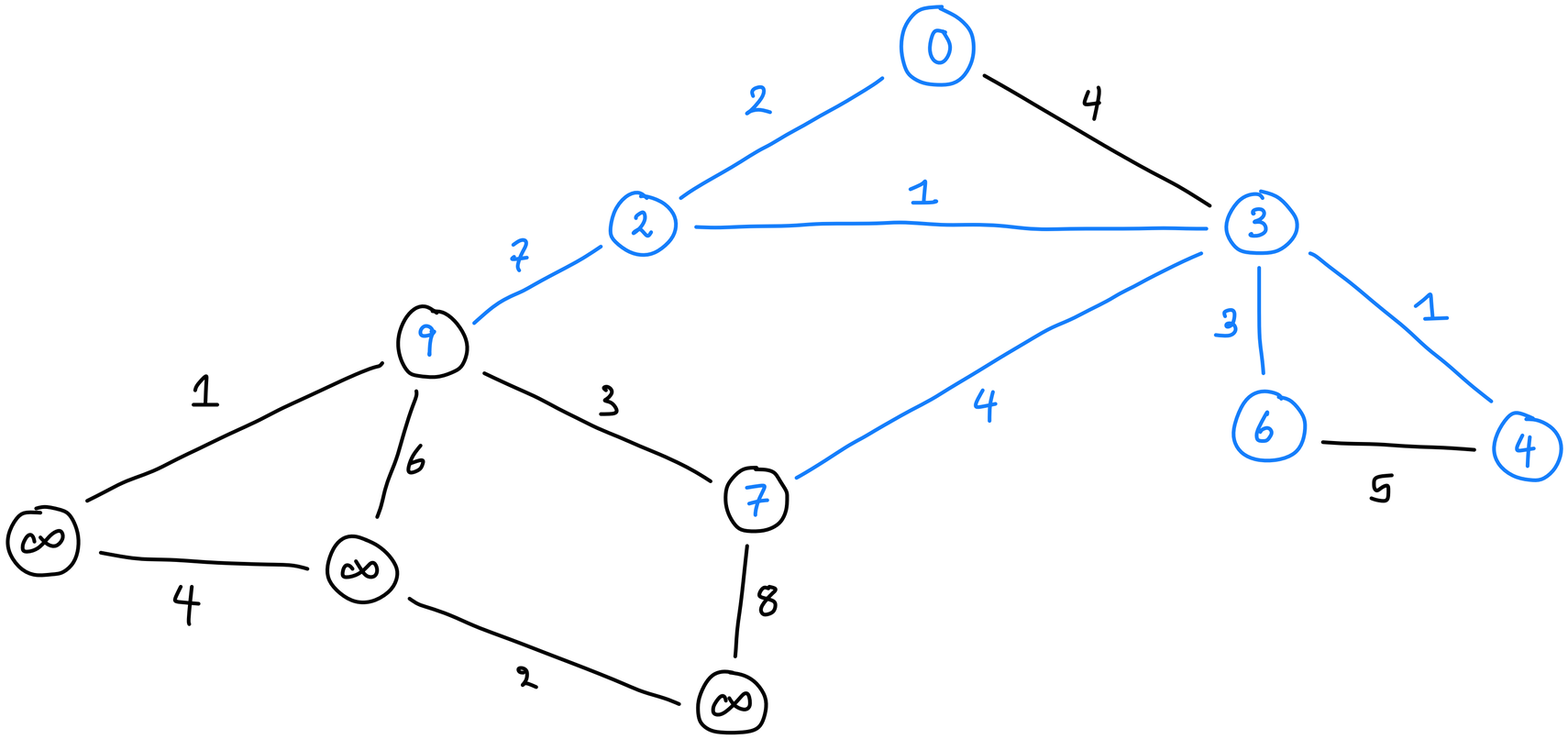
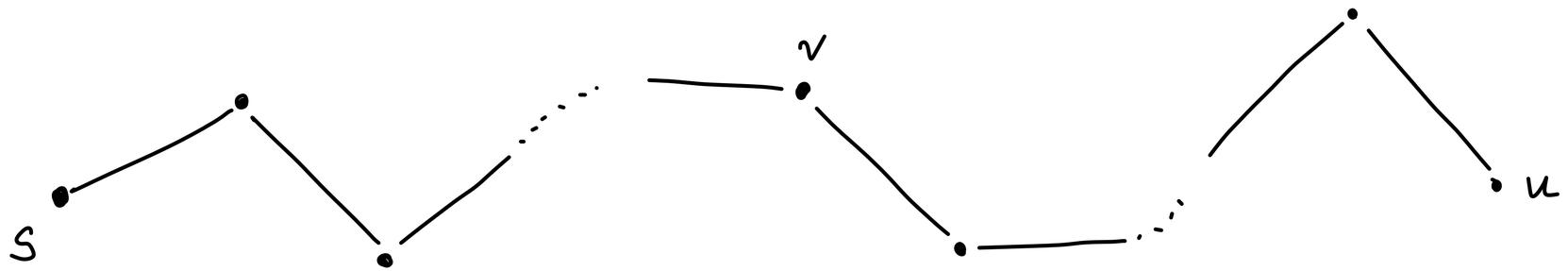


Dijkstra's algorithm

Proof of correctness

- **Lemma:** If q is a path $s \rightsquigarrow u$ of minimal weight to u, then for any vertex v on q, the subpath from s to v is of minimal weight.
- **Proof:**



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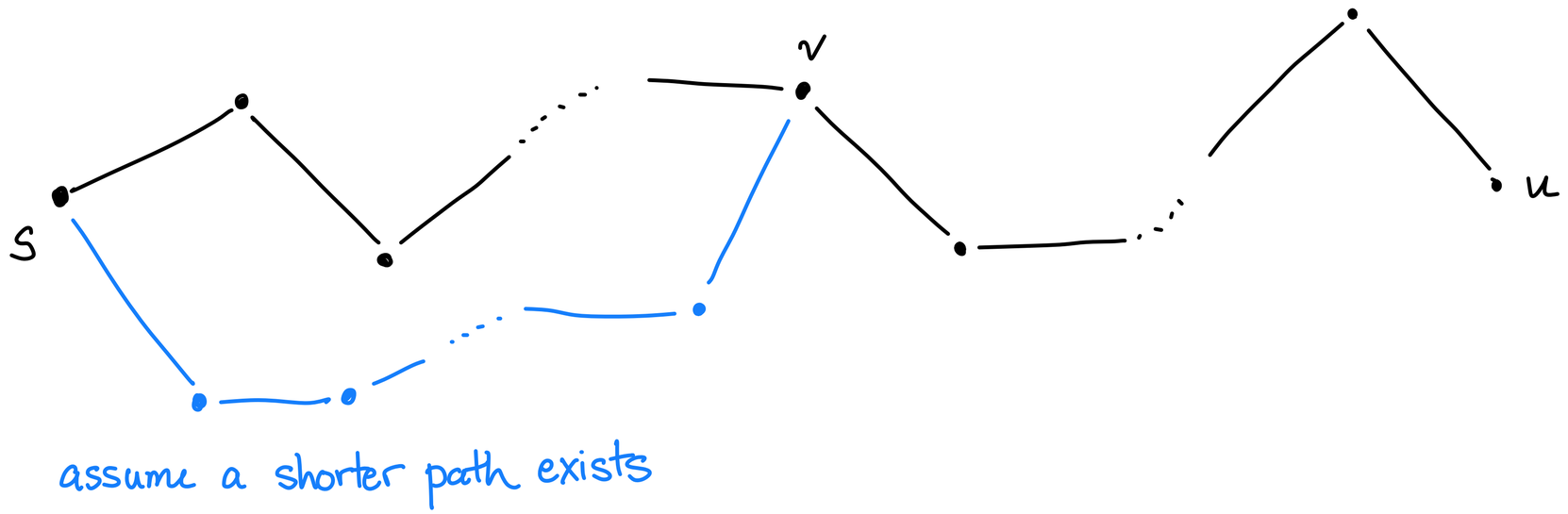

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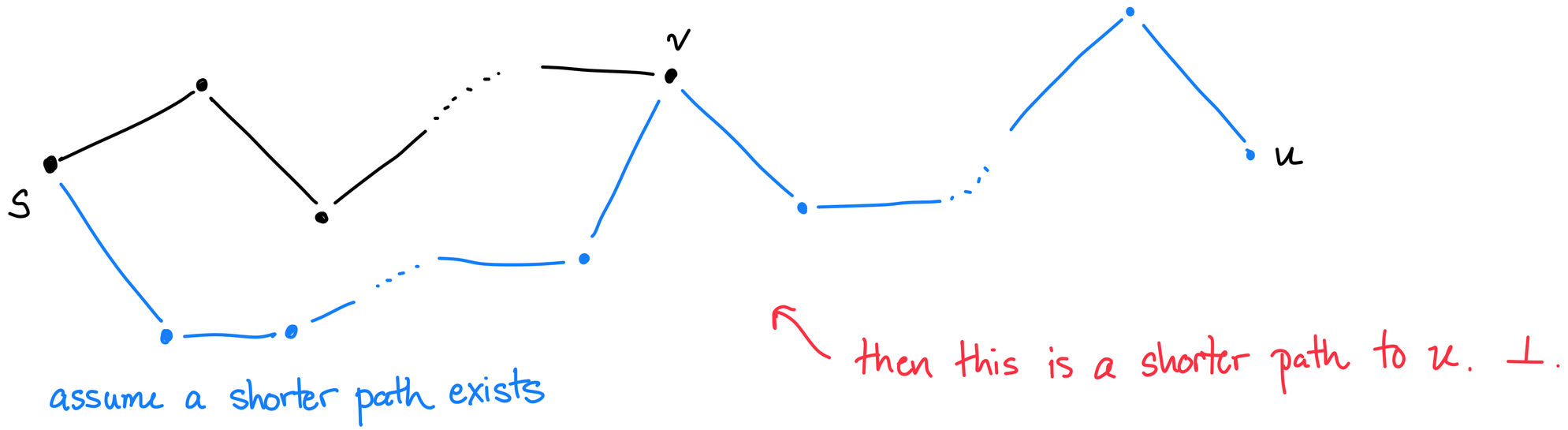
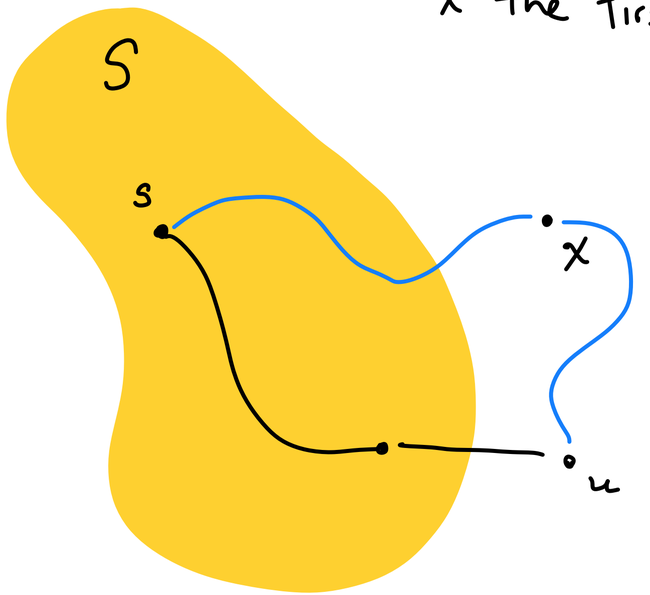


Dijkstra's algorithm

Proof of correctness

- **Claim:** During run, let S be the set of vertices popped off Q. At that moment,
  - for $y \in S$, $d(y)$ = length of shortest path $s \rightsquigarrow y$ and
  - for $x \notin S$, $d(x)$ = length of shortest path $s \rightsquigarrow x$ with only the last edge leaving S.
- **Proof:** By induction. Let u be the next vertex popped off.



Since u is popped off, $d(u) \leq d(x)$.

Then, length of blue path ≥ length of black path

since x ? u has non-negative weight.

So, length of path to u is correct.

Dijkstra's algorithm

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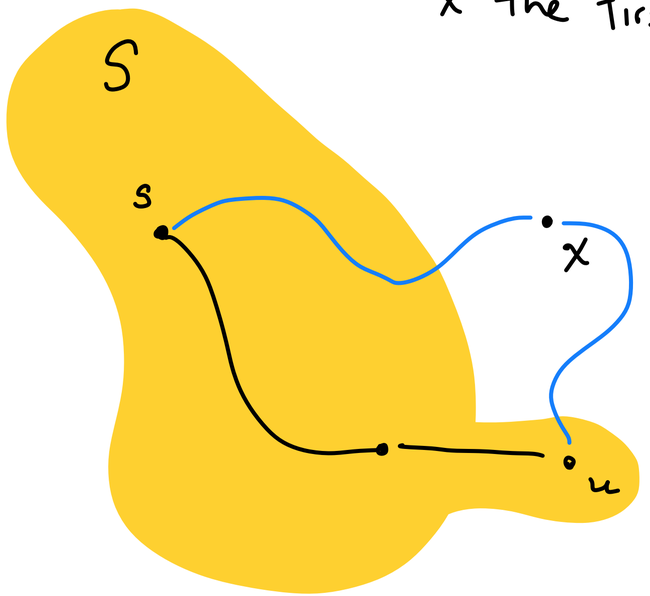

Consider any other path with x the first vertex outside S.

Since u is popped off, $d(u) \leq d(x)$.

Then, length of blue path ≥ length of black path since x ⋯ u has non-negative weight.

So, length of path to u is correct.

Dijkstra's algorithm other perks

- The assignment of parent $p(u)$ generates a tree of shortest paths with root s
- If you only want to calculate the shortest path to vertex u, can abort the algorithm as soon as u is popped from the queue.
  - This follows from the correctness claim in the previous slide
  - For the vertices in S, the distance is minimal over *all* paths and not just the ones contained in S.
- Dijkstra's algorithm also works for *directed* graphs. Similar proof — feel free to use both directed and undirected versions in your psets/exams (without proof).

Dijkstra's algorithm

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- Return d, p for distance and parent functions.

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update parent of v to be u.

Priority queue data structure review

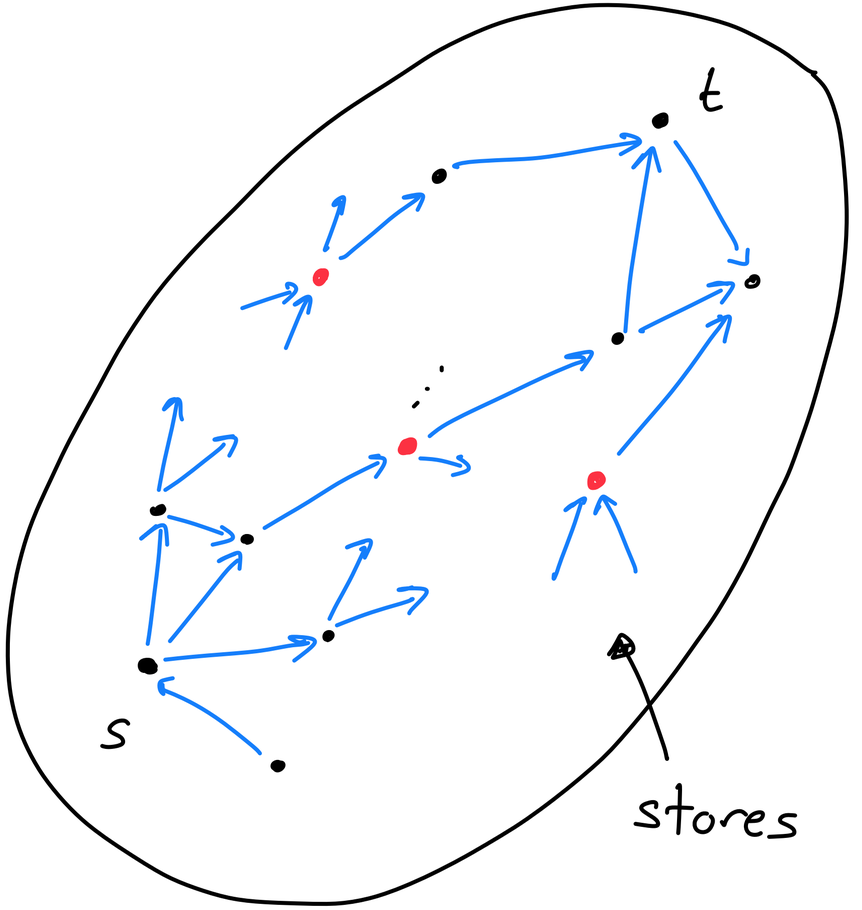
- Each element v in the queue is associated with a key k
- Operations allowed by the data structure
  - $\text{insert}(v, k)$
  - $(v, k) \leftarrow \text{findmin}(Q)$ or $(v, k) \leftarrow \text{deletemin}(Q)$
  - $\text{decreasekey}(v, k)$ if v is already in the queue.
- Implementations
  - With arrays: $O(n)$ time for find-min or delete, and $O(1)$ time for set and decrease
  - With heaps: $O(\log n)$ time for insert, delete, decrease and $O(1)$ for find-min

Dijkstra's algorithm

- The algorithm has $O(n)$ insertions, $O(n)$ delete-mins since each vertex is added and deleted once
- And $O(m)$ decrease-keys with each decrease-key corresponding to an edge
- Implementation based runtimes
  - Array has insert $O(1)$, delete-min $O(n)$, and decrease-key $O(1)$
  - Array has total $O(n + n^2 + m) = O(n^2)$ time
  - Heap has insert, delete-min, and decrease-key $O(\log n)$
  - Heap has total $O(m \log n)$ time

Example problem: Johnny's birthday present

- Consider a city expressed as a *directed weighted* graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}_{\geq 0}$.
- Johnny's mother starts at $s \in V$ and needs to get to the birthday party at location $t \in V$
- She forgot to buy a birthday present though and can find one at vertices $V' \subseteq V$.
- **Goal:** Calculate *the* shortest path from $s \rightsquigarrow t$ that includes a vertex of V'.

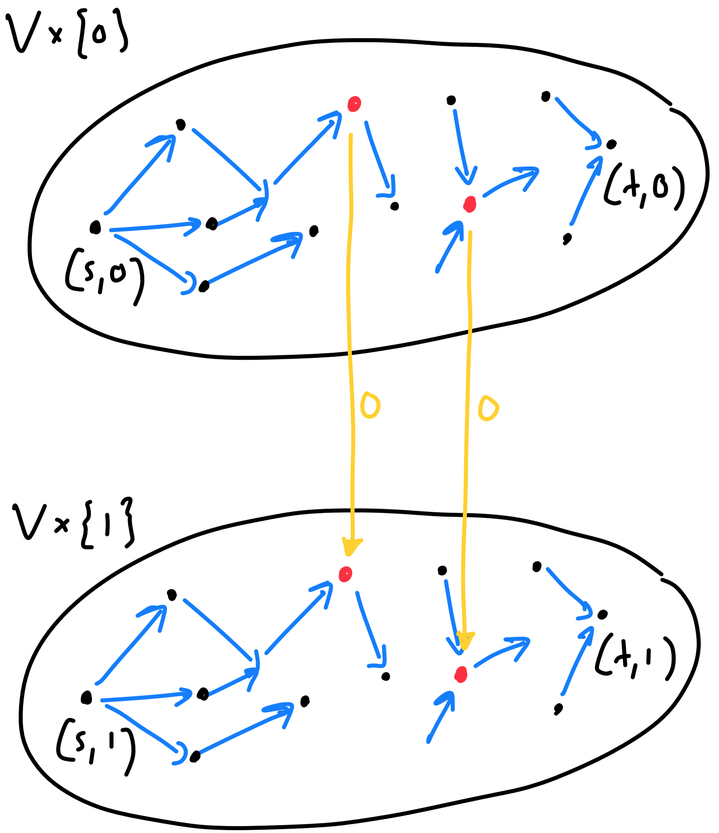


Johnny's birthday present

- **Naive algorithm**: Brute-force search + Dijkstra's algorithm
- We want to compute $\min_{u \in V'} d(s,u) + d(u,t)$
- Iterate over each "midpoint" $u \in V'$,
  - And use Dijkstra's twice to compute $d(s,u) + d(u,t)$
  - Keep track of the minimum as we go along
- Runtime is $O(|V'|) \cdot \text{Runtime}(\text{Dijkstra's}) = O(|V'| m \log n)$
- Can we do better?

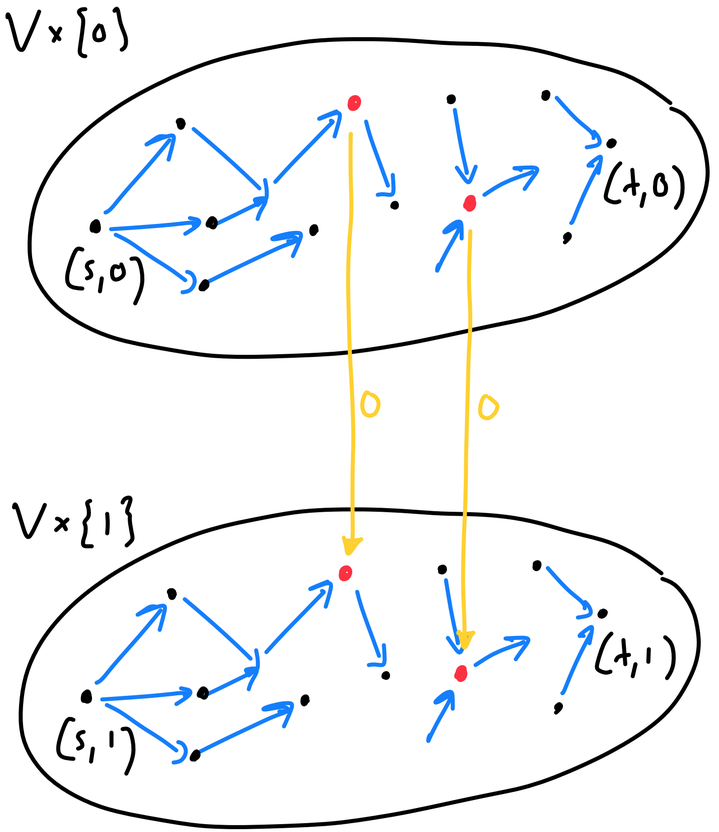
Johnny's birthday present

- The key will be finding a better graph.
- Consider a graph with vertex set $V_2 = V \times \{0,1\}$ with a vertex $(v, b) \in V_2$ indicating location x [whether store has been visited]
- **Starting vertex:** $(s,0)$ since item isn't acquired
- **End vertex:** $(t,1)$ since item is acquired
- Now we are looking for shortest path from $(s,0)$ to $(t,1)$ in graph $G_2 = (V_2, E_2)$. But what is E_2?



Johnny's birthday present

- **Algorithm:**
  - Let $V_2 = V \times \{0,1\}$.
  - Construct edge set E_2 by including $(u,0) \to (v,0)$ and $(u,1) \to (v,1)$ whenever $u \to v$ in E. Set the weight of the new edges to be that of $u \to v$
  - For every $v \in V'$, include edge $(v,0) \to (v,1)$ of weight 0.
  - Run Dijkstra's on G_2 from $(s,0)$ to $(t,1)$.
- **Runtime:** $O((2m+n)\log(2n)) = O(m \log n)$



Johnny's birthday present

- **Correctness:**
  - There is a 1-to-1 correspondence between
    - paths $s \rightsquigarrow v \rightsquigarrow t$ for $v \in V'$ in G.
    - And paths $(s,0) \rightsquigarrow (t,1)$ in G_2.
  - This is because any path in G_2 must descend in the second coordinate once and this can only happen at a vertex $v \in V'$. No edges allow ascending in the second coordinate.
  - In particular, both paths in the correspondence have the same length.
  - Therefore, optimizing over the second set of paths is equivalent to the optimizing over the first.
  - Dijkstra's algorithm optimizes over the second set of paths.