Lecture 25

Linear programming II

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Final in one week

- DRS requests need to be put in ahead of time.
- This exam is a closed-book exam. You may bring one 8.5 x 11 (handwritten or typeset by you) double-sided cheat sheet to the exam.
- The exam consists of 4 long-form questions each worth ~10 points for a total of 40 points.
- **Topics**:
  - All topics except s*table matchings* and *linear programming*.
  - There will be a NP-reduction problem. The reduction will be given; you will be asked to prove correctness.
- **Suggestions from the TAs who took the practice exam:**
  - Nothing is trying to trick you on the exam.
  - Use extra scratch paper to draw examples and try out your solutions.

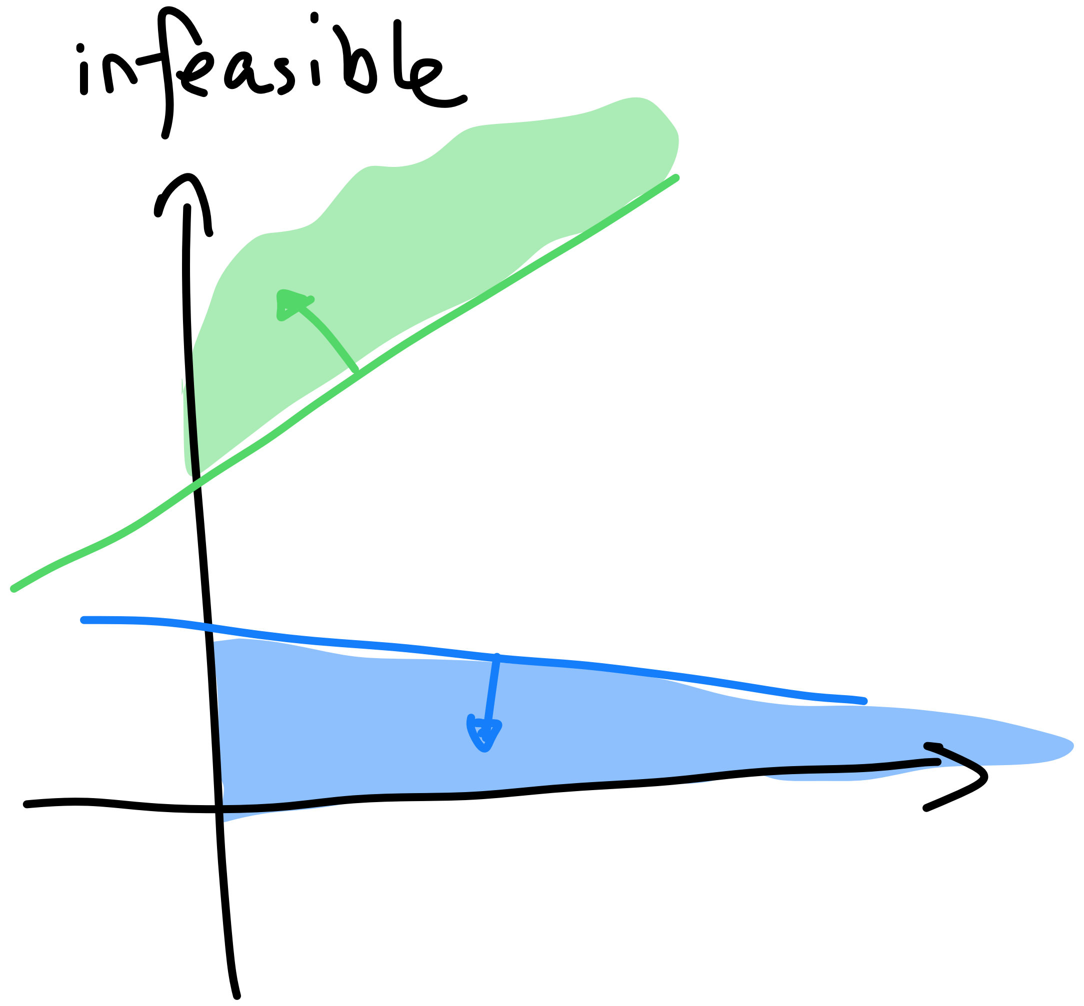
Linear programming feasibility

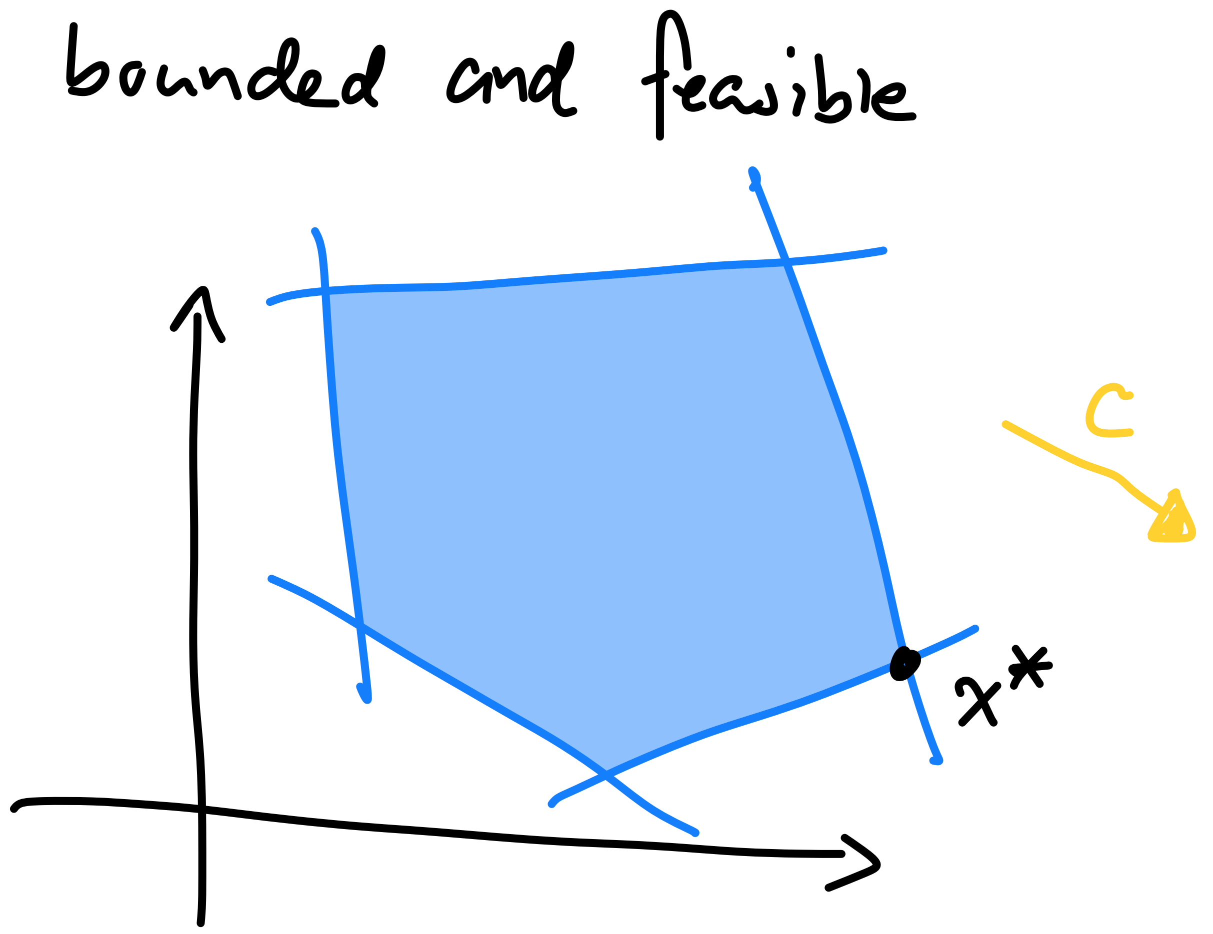
- Recall, the feasible region of a standard LP is $\Gamma = \{x : Ax \leq b, x \geq 0\}$.

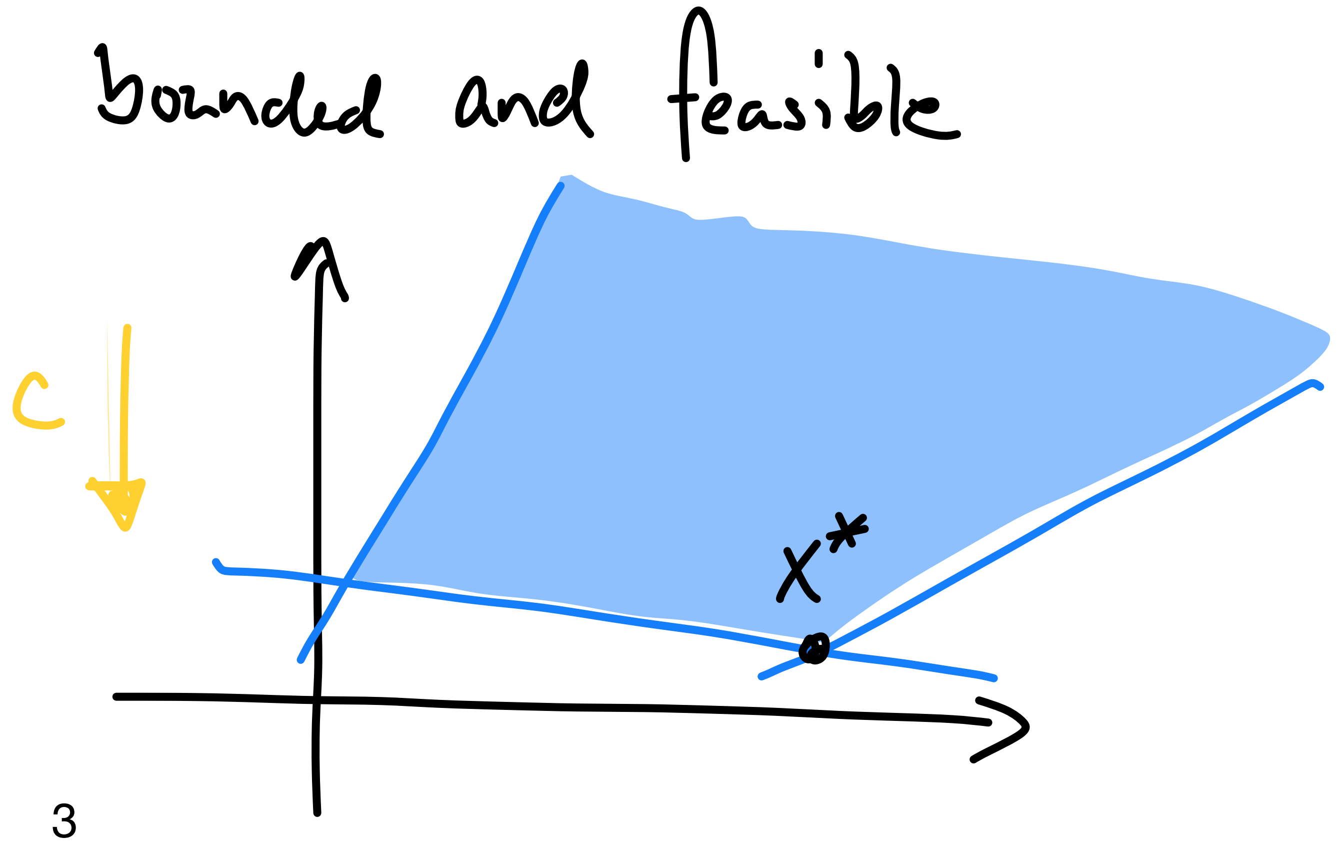
- **Definition**: The LP is *infeasible* if $\Gamma = \emptyset$.

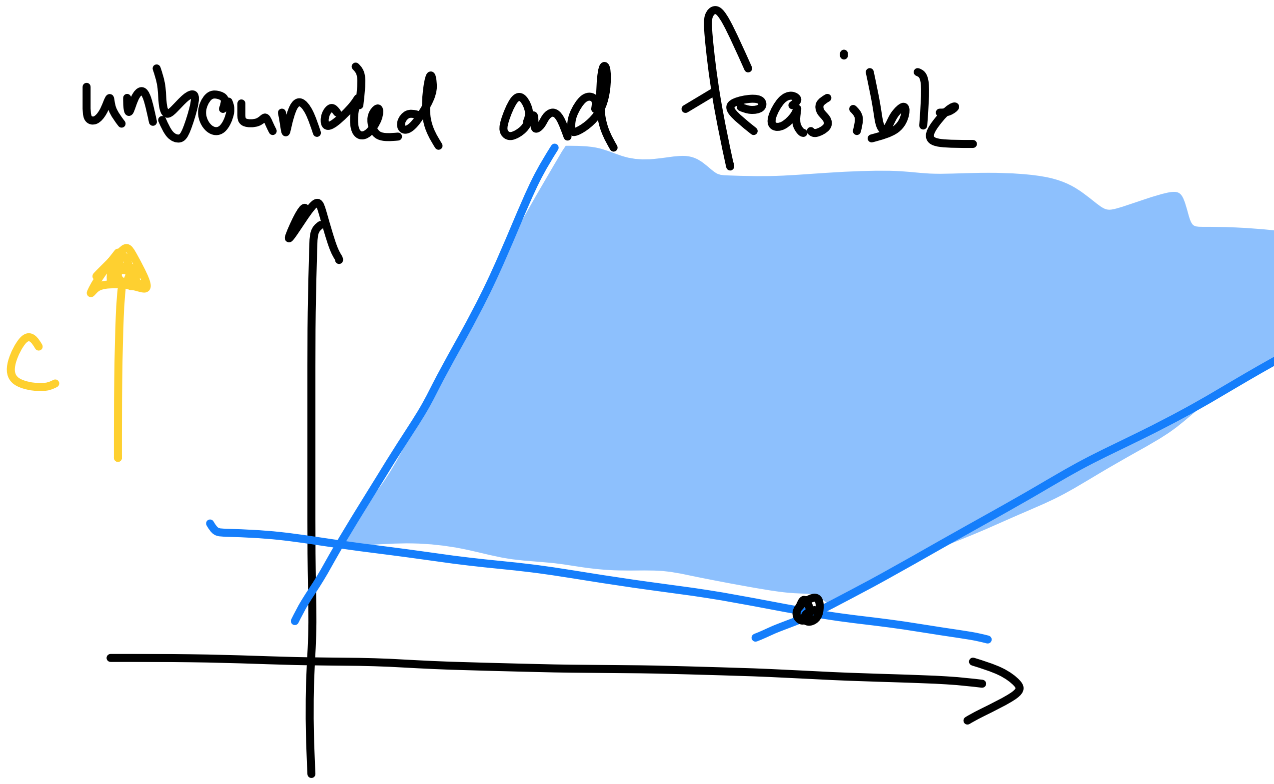
- **Definition**: The LP is *unbounded* if $c^\top x$ can be arbitrarily large for some $x \in \Gamma$.

- Even just deciding if a LP is feasible or not, seems like a challenging problem.



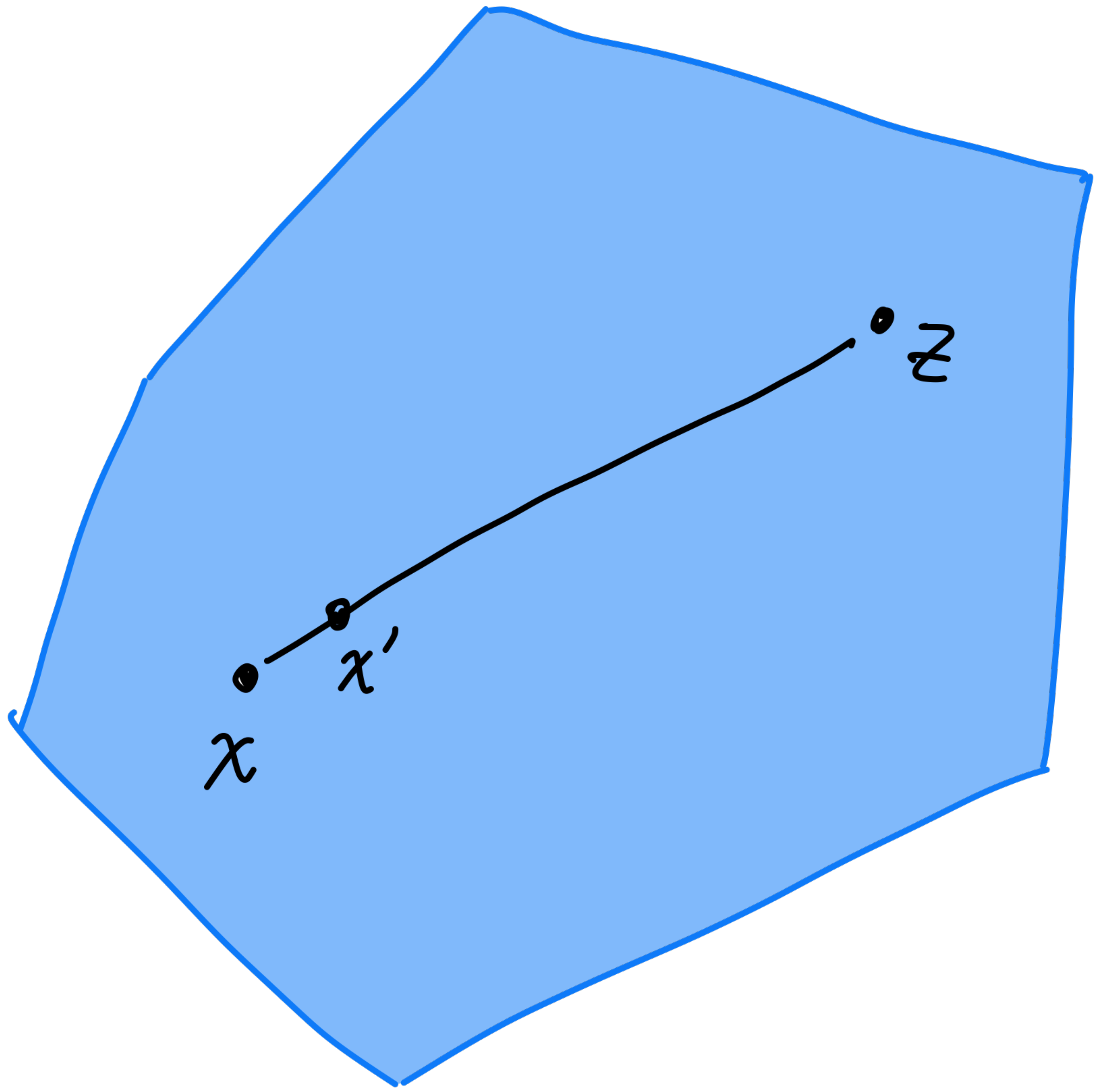






Where are the optimums of LPs

- **Theorem:** If a local optimum exists for an LP, it is a global optimum.
- **Proof:** Recall we are maximizing $c^\top x$ subject to $x \in \Gamma$ and Γ is convex. Assume x is a local optimum but not a global optimum.
  - Then $c^\top x < c^\top z$ for some $z \in \Gamma$ as x is not a global optimum.
  - Consider the line $\overline{xz} \in \Gamma$. Then $x' := x + \epsilon(z - x) \in \Gamma$ for small $\epsilon > 0$ and
  - $c^\top x' = c^\top x + \epsilon c^\top (z - x) > c^\top x$.
  - So x is not a local optimum.
  - This proves the contrapositive.

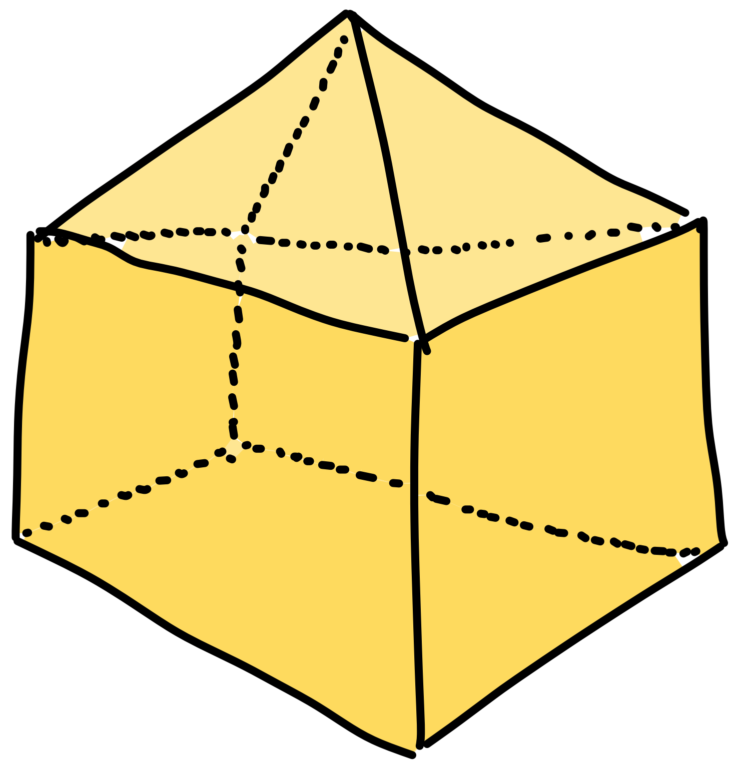


Bounded convex polytopes

- **Definition:** A **vertex** z of a convex polytope Γ is any point such that z is not the midpoint of any line segment $\overline{xy} \in \Gamma$ for $x \neq y$.

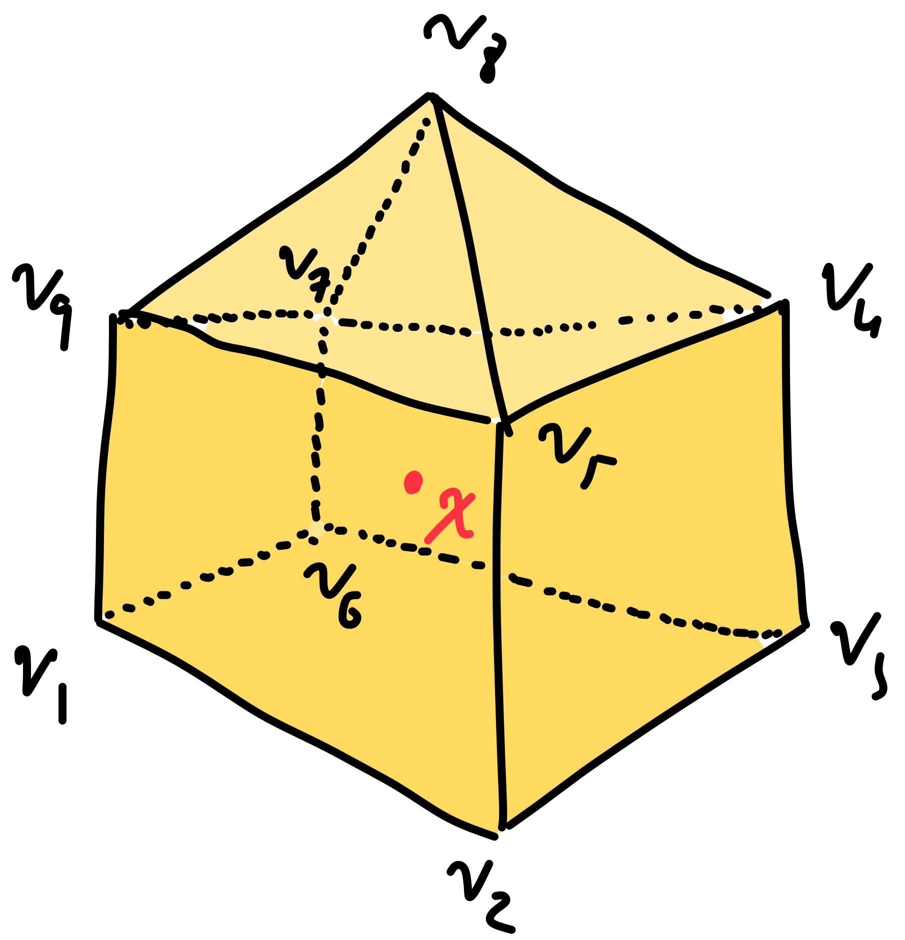
- **Remark:** If $v_1, \ldots, v_k$ are **all** the vertices of a bounded convex polytope Γ, then $\Gamma = \mathbf{conv}(v_1, \ldots, v_k)$, the convex hull of the vertices.

- **Theorem:** If the optimum of a standard linear program whose feasible region is bounded is finite, then the optimum must be achieved at some vertex.



Bounded convex polytopes

- **Theorem:** If the optimum of a standard linear program whose feasible region is bounded is finite, then the optimum must be achieved at some vertex.
- **Proof:** Let $v_1, \ldots, v_k$ be the vertices of the feasible region Γ.
  - Then every point $x \in \Gamma$ equals $\sum_{i=1}^{k} \lambda_i v_i$ for $\lambda \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$.
  - By linearity of objective function,
    - $c^\top x = \sum_{i=1}^{k} \lambda_i c^\top v_i \leq \max_{i=1}^{k} c^\top v_i$
    - So one of the vertices must do better than the vertex x.

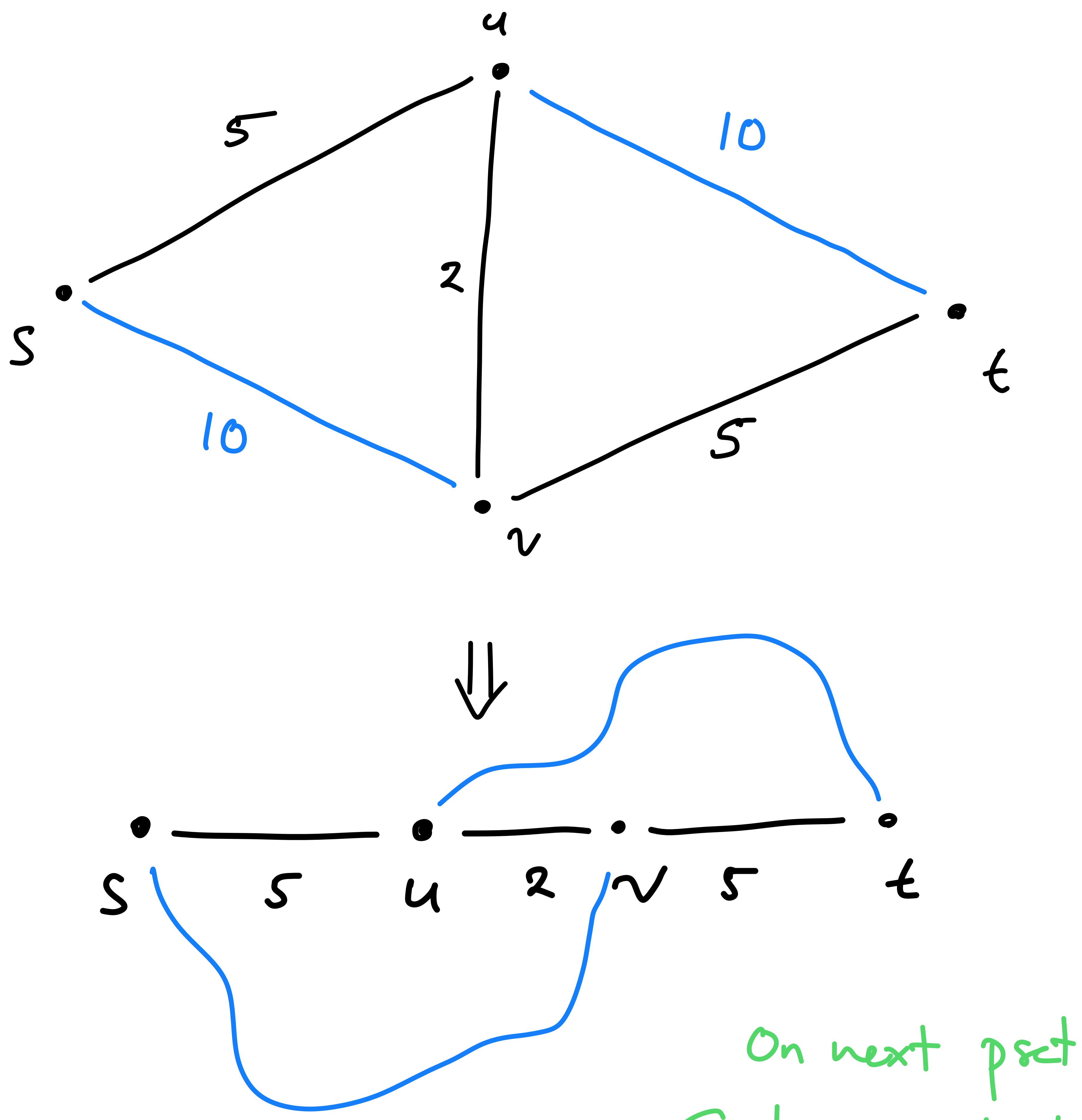


Duality

The string example

Minimization as maximization

- Recall the shortest path problem from s to t
- It is easiest seen as a *minimization* problem
- Now, imagine each edge is a piece of yarn of length $w(e)$ with knots tied at the vertices
  - Pull the yarn apart at s and t till it is taut
  - The strings that are taut form the shortest path from s to t
  - And yet pulling the yarn sounds like a *maximization* problem

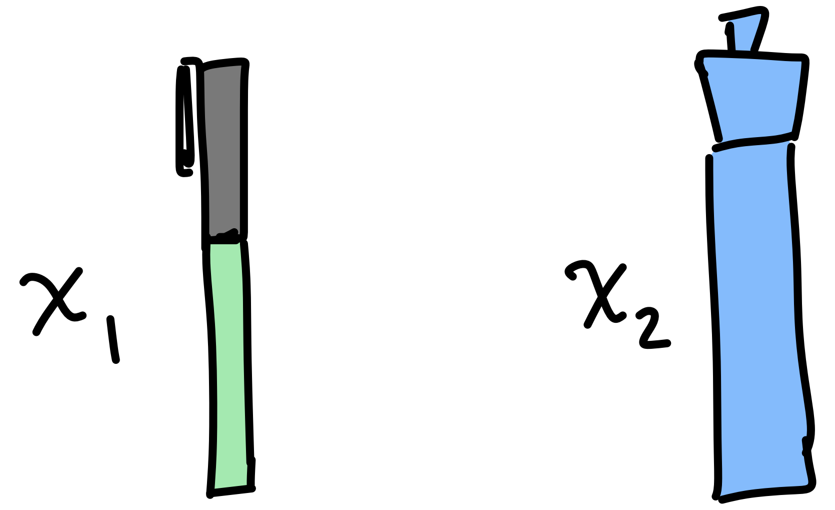


On next pset, how to convert to lin. eqs!

$$\begin{cases} \max & |x_t - x_s| \\ \text{s.t.} & \forall e=(u,v),\ |x_u - x_v| \le d(e) \\ & x \ge 0. \end{cases}$$

Linear program duality

- Consider a salesman who sells either pens or markers.
- He has L labor, I ink, and P plastic.
- He sells pens for S_1 and markers for S_2.



$$
\begin{aligned}
\max \quad & S_1 x_1 + S_2 x_2 \\
\text{s.t.} \quad & L_1 x_1 + L_2 x_2 \le L \\
& I_1 x_1 + I_2 x_2 \le I \\
& P_1 x_1 + P_2 x_2 \le P \\
& x_1, x_2 \ge 0.
\end{aligned}
$$

Linear programming duality

- Now let's imagine there are market prices for the 3 materials: y_L, y_I, y_P.
- It is only economical to **buy** a pen if $y_L L_1 + y_I I_1 + y_P P_1 \geq S_1$
  - The left hand side is the cost to make a pen **at market price**
  - And the right hand side is the cost to **buy a pen**
  - Similarly, buy markers only if $y_L L_2 + y_I I_2 + y_P P_2 \geq S_2$.
- The dual perspective is that the market **minimizes** the total materials price while still being able to sell pens and markers. This is the **dual problem**.

Linear programming duality

- Now let's imagine there are market prices for the 3 materials: y_L, y_I, y_P.
- Recall, L_1 is the amount of labor required for a pen, I_1 is the amount of ink required for a pen, etc.
- It is only economical to **buy** a pen if $y_L L_1 + y_I I_1 + y_P P_1 \geq S_1$
    - The left hand side is the cost to make a pen **at market price**
    - And the right hand side is the cost to **buy a pen**
    - Similarly, buy markers only if $y_L L_2 + y_I I_2 + y_P P_2 \geq S_2$.
- The dual perspective is calculating the **minimal** total materials price ($y_L L + y_I I + y_P P$) while still able to sell pens and markers. This is the **dual problem**.
- The primal perspective is calculating the **maximal** total profit subject to the material restrictions.

Linear programming duality

- Now let's imagine there are market prices for the 3 materials: y_L, y_I, y_P.
- Recall, L_1 is the amount of labor required for a pen, I_1 is the amount of ink required for a pen, etc.
- If $y_L L_1 + y_I I_1 + y_P P_1 < S_1$, then it would not be economical to **buy** a pen
  - The left hand side is the cost to make a pen **at market price**
  - And the right hand side is the cost to **buy a pen**
  - Buy pens only if $y_L L_1 + y_I I_1 + y_P P_1 \geq S_1$ and buy markers only if $y_L L_2 + y_I I_2 + y_P P_2 \geq S_2$.
- The dual perspective is calculating the **minimal** total materials price ($y_L L + y_I I + y_P P$) while still able to sell pens and markers. This is the **dual problem**.
- The primal perspective is calculating the **maximal** total profit subject to the material restrictions.

Linear programming duality

$$\max \quad S_1 x_1 + S_2 x_2$$

$$\text{s.t.} \quad L_1 x_1 + L_2 x_2 \le L$$

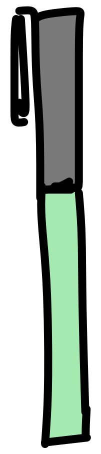
$$I_1 x_1 + I_2 x_2 \le I$$

$$P_1 x_1 + P_2 x_2 \le P$$

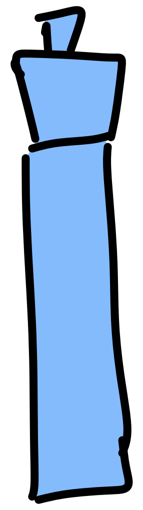
$$x_1, x_2 \ge 0.$$

number of pens & markers

x_1



x_2



$$\min \quad y_L L + y_I I + y_P P$$

$$\text{s.t.} \quad y_L L_1 + y_I I_1 + y_P P_1 \ge S_1$$

$$y_L L_2 + y_I I_2 + y_P P_2 \ge S_2$$

$$y_L, y_I, y_P \ge 0.$$

price of materials

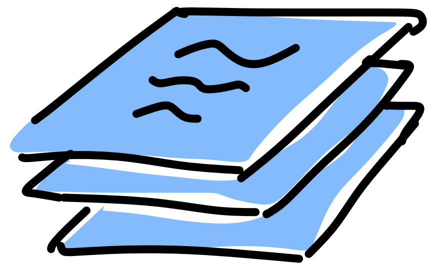
y_L

labor

y_I

ink

y_P



plastic

Linear programming duality

$$\max \quad S_1 x_1 + S_2 x_2$$

$$\text{s.t.} \quad L_1 x_1 + L_2 x_2 \le L$$

$$I_1 x_1 + I_2 x_2 \le I$$

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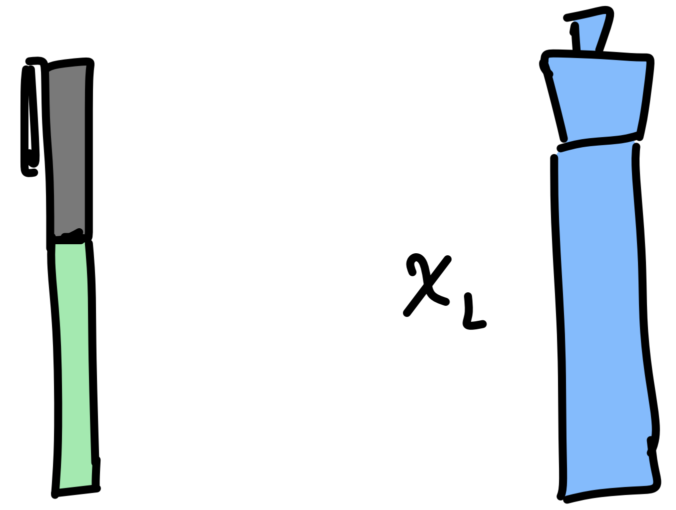
$$x_1, x_2 \ge 0.$$

number of pens & markers

x_1



x_2

$$\min \quad y_L L + y_I I + y_P P$$

$$\text{s.t.} \quad y_L L_1 + y_I I_1 + y_P P_1 \ge S_1$$

$$y_L L_2 + y_I I_2 + y_P P_2 \ge S_2$$

$$y_L, y_I, y_P \ge 0.$$

price of materials

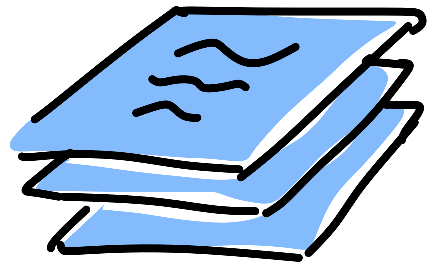
y_L

labor

y_I

ink

y_P



plastic

Linear programming duality

$$\max \quad S_1 x_1 + S_2 x_2$$

$$\text{s.t.} \quad L_1 x_1 + L_2 x_2 \le L$$

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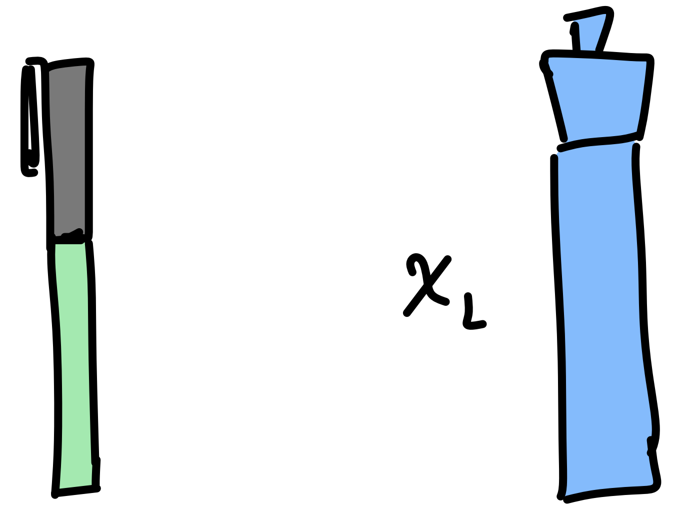
$$P_1 x_1 + P_2 x_2 \le P$$

$$x_1, x_2 \ge 0.$$

number of pens & markers

x_1 x_2



$$\min \quad y_L L + y_I I + y_P P$$

$$\text{s.t.} \quad y_L L_1 + y_I I_1 + y_P P_1 \ge S_1$$

$$y_L L_2 + y_I I_2 + y_P P_2 \ge S_2$$

$$y_L, y_I, y_P \ge 0.$$

price of materials

y_L y_I y_P

labor ink plastic

Linear programming duality

$$\max \quad S_1 x_1 + S_2 x_2$$

$$\text{s.t.} \quad L_1 x_1 + L_2 x_2 \le L$$

$$I_1 x_1 + I_2 x_2 \le I$$

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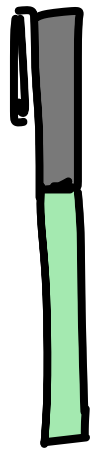
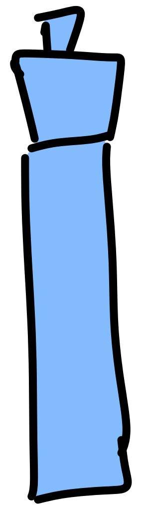
number of pens & markers

x_1



x_2



$$\min \quad y_L L + y_I I + y_P P$$

$$\text{s.t.} \quad y_L L_1 + y_I I_1 + y_P P_1 \ge S_1$$

$$y_L L_2 + y_I I_2 + y_P P_2 \ge S_2$$

$$y_L, y_I, y_P \ge 0.$$

price of materials

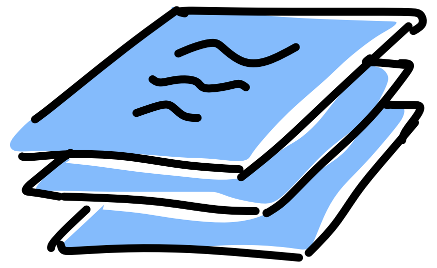
y_L

labor

y_I

ink

y_P



plastic

Linear programming duality

Primal linear program (P)

$$\begin{aligned} \max \quad & c^T x \quad \leftarrow \in \mathbb{R}^n \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

Dual linear program (D)

$$\begin{aligned} \min \quad & b^T y \quad \leftarrow \in \mathbb{R}^m \\ \text{s.t.} \quad & A^T y \ge c \\ & y \ge 0 \end{aligned}$$

Linear programming duality

(Weak duality)

- **Theorem**:
  - If $x \in \mathbb{R}^n$ is feasible for $(\mathscr{P})$ and $y \in \mathbb{R}^m$ is feasible for $(\mathscr{D})$, then $c^\top x \leq y^\top Ax \leq b^\top y$.
  - If $(\mathscr{P})$ is unbounded, then $(\mathscr{D})$ is infeasible.
  - If $(\mathscr{D})$ is unbounded, then $(\mathscr{P})$ is infeasible.
  - If $c^\top x = b^\top y$ for $x \in \mathbb{R}^n$ is feasible for $(\mathscr{P})$ and $y \in \mathbb{R}^m$ is feasible for $(\mathscr{D})$, then x is an optimal solution for $(\mathscr{P})$ and y is an optimal solution for $(\mathscr{D})$.

Proving weak duality

- Let's prove when both LPs are feasible, that $c^\top x \le y^\top Ax \le b^\top y$.

Since x is feasible for (P),

$$Ax \le b, \quad x \ge 0. \quad (1)$$

Then, $y^T(Ax) \le y^T b$ by (1).

$$= b^T y$$

Since y is feasible for (D),

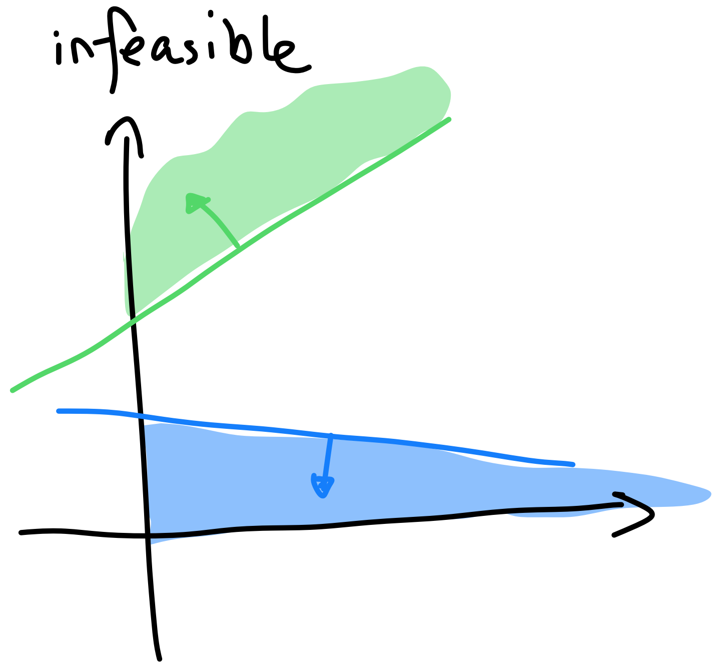
$$A^T y \ge c, \quad y \ge 0. \quad (2)$$

And, $c^T x \le (A^T y)^T x$

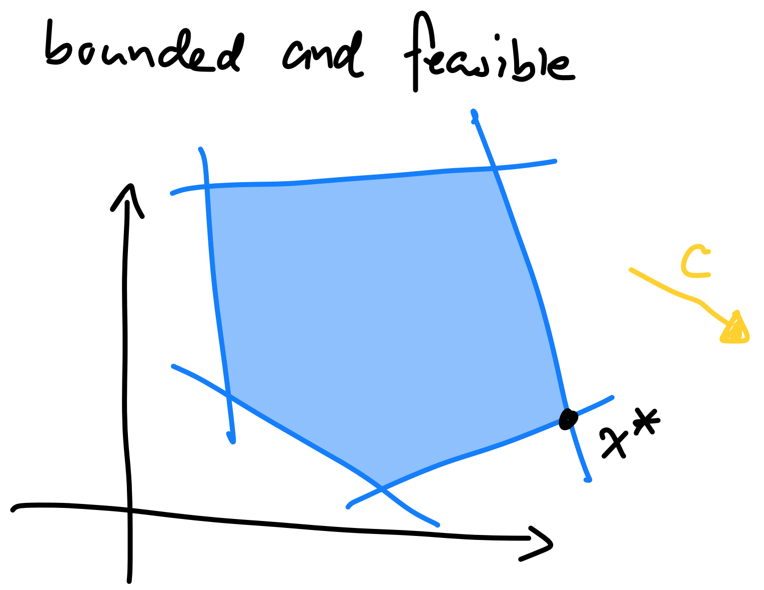
$$= (y^T A) x$$

$$= y^T A x.$$

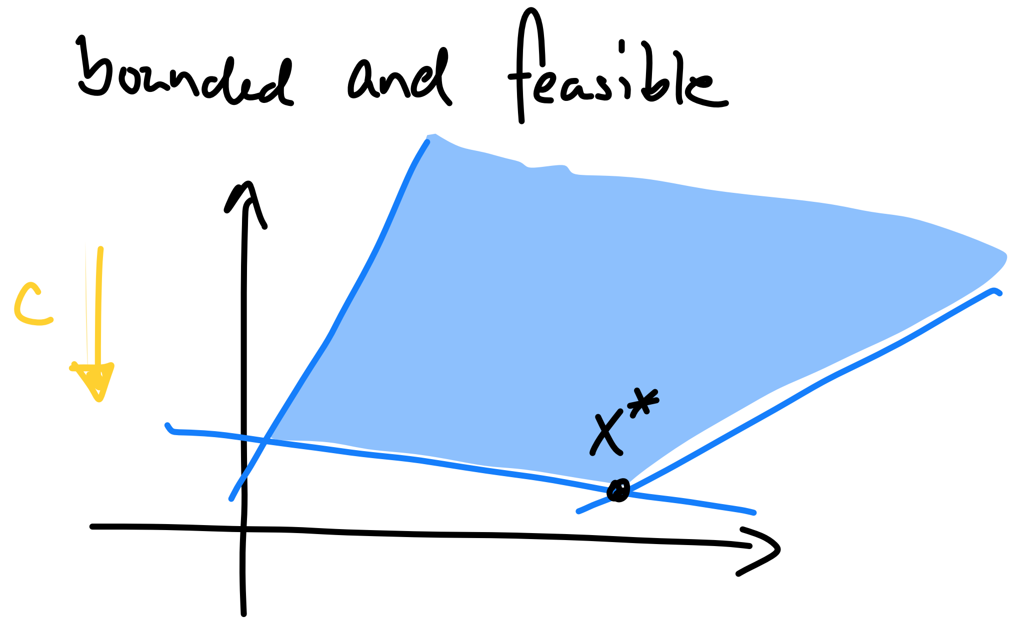
Proving weak duality



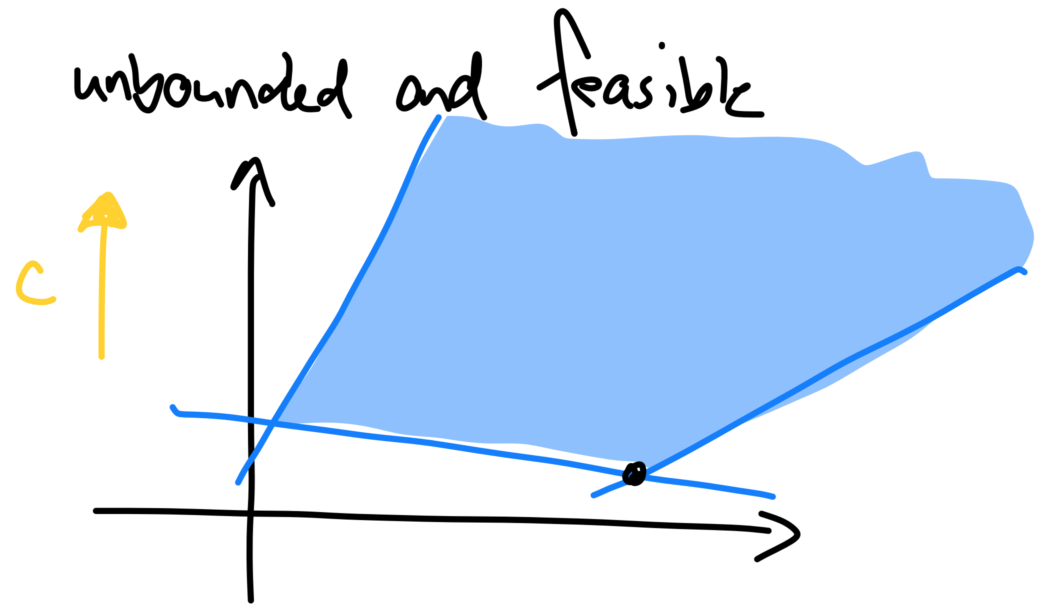
- If $(\mathscr{P})$ is unbounded
    - Then for all $N \in \mathbb{N}$, there exists $x \in \Gamma$ such that $N < c^\top x$.



- If $(\mathscr{D})$ is feasible,
    - then for any feasible y, $c^\top x \leq y^\top A x \leq b^\top y$.



- Together, this proves that $b^\top y$ is not finite, a contradiction.
- Therefore, if $(\mathscr{P})$ is unbounded, then $(\mathscr{D})$ is infeasible.



- Similarly, if $(\mathscr{D})$ is unbounded, then $(\mathscr{P})$ is infeasible.

Proving weak duality

- Lastly, since $c^\top x = b^\top y$ for some feasible x and feasible y,
- Assume for contradiction, there exists x' s.t. $c^\top x' > c^\top x = b^\top y$.
  - Then, $c^\top x' \leq y^\top A x' \leq y^\top b$ by first argument in weak quality.
  - This is a contradiction, proving no x' exists. So x is optimal.
- Similar argument proves that y is also optimal.

Lessons from duality

- We reproved the max flow/min cut duality from the flow unit of this course
- **Observation:** Min cut does not have an intuitive poly-sized LP
  - However, it does have a m variable and $|P|$ equations sized LP
  - Therefore, its has a dual (max flow) with $|P|$ variables and m equations
  - Max flow also has a simple poly-sized LP and an efficient algorithm
- Intuitively, this is why we solve min cut by solving max flow and looking at saturated edges. It's sometimes algorithmically easier to solve a problem over its dual.