Lecture 19

Flow applications

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Previously in CSE 421…

Flow independent of capacity

- So far, for integer capacities:
  - **Vanilla Ford-Fulkerson**: Runtime $O(mC)$
    - Pick any augmenting path
  - **Scaling Ford-Fulkerson**: Runtime $O(m^2 \log C)$
    - Pick the largest augmenting paths
  - **Edmonds-Karp (next)**: Runtime $O(m^2 n)$
    - Pick the shortest augmenting path (in terms of # of edges)

Today

Edmonds-Karp algorithm

- Initialize $f \leftarrow 0$ and $G_f \leftarrow G$
- While BFS starting from s outputs a path $p : s \rightsquigarrow t$ in G_f.
  - Compute bottleneck capacity b and update f and G_f by augmenting f along p at capacity b.
- Output resulting flow f.

Edmonds-Karp

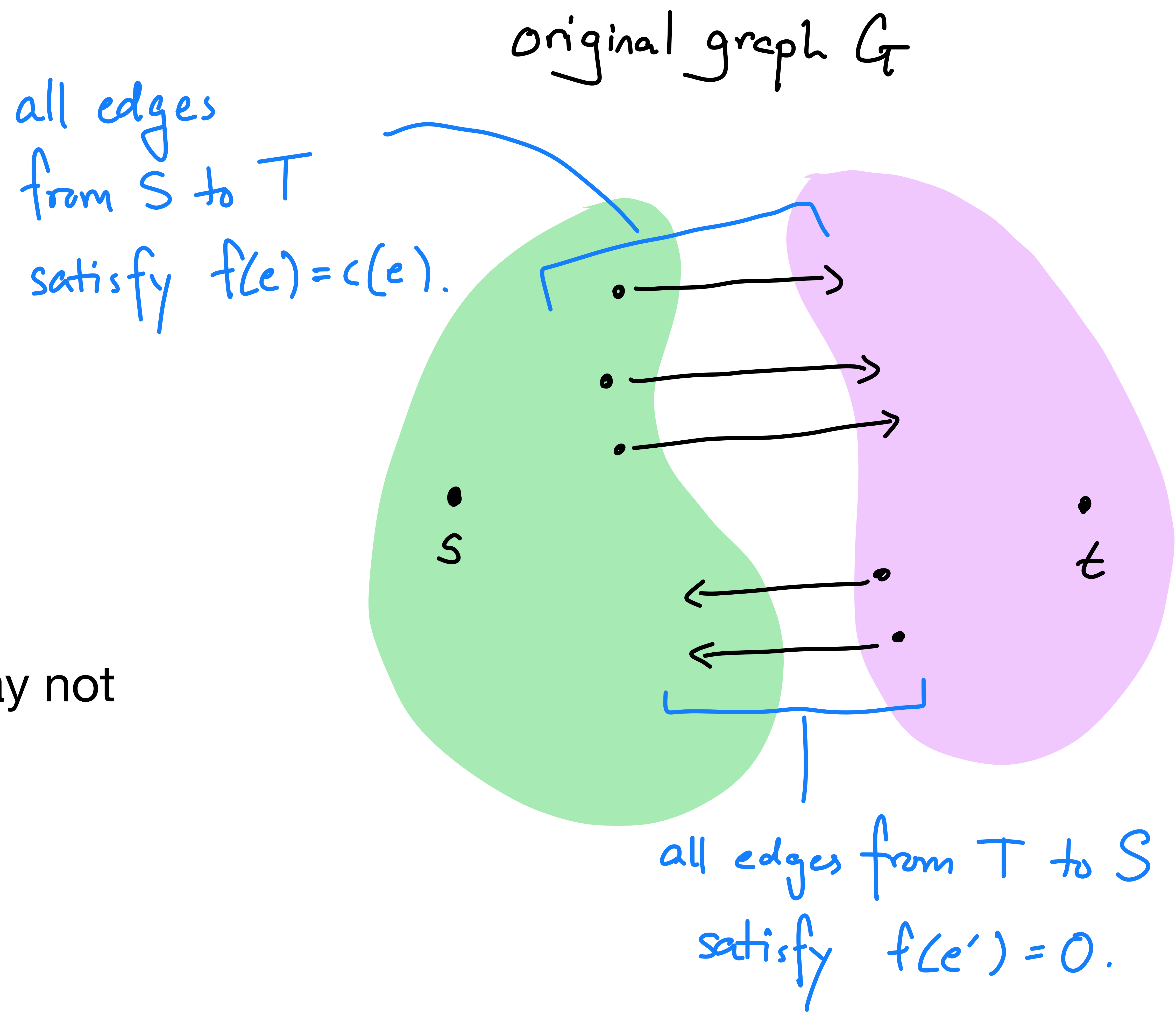
- We know the algorithm: it's BFS based-augumentations.
  - Each run of BFS will compute an augmentation in time $O(m)$.
  - I've claimed the runtime is $O(m^2n)$.
  - Therefore, we need to be able to prove that only $O(mn)$ augmentations are needed.

Edmonds-Karp (sketch)

- Every time an augmenting path is chosen, the bottleneck edge e becomes saturated — i.e. $f(e) = c(e)$
- Suffices to show that each edge e can only be the bottleneck in at most $n/2$ augmenting paths.
- Since there are m edges, this yields a max of $\frac{mn}{2}$ augmenting paths.
- Details are excluded but do use Edmonds-Karp as a subroutine on problem sets and exams.

Maximum flow algs are minimum cut algs

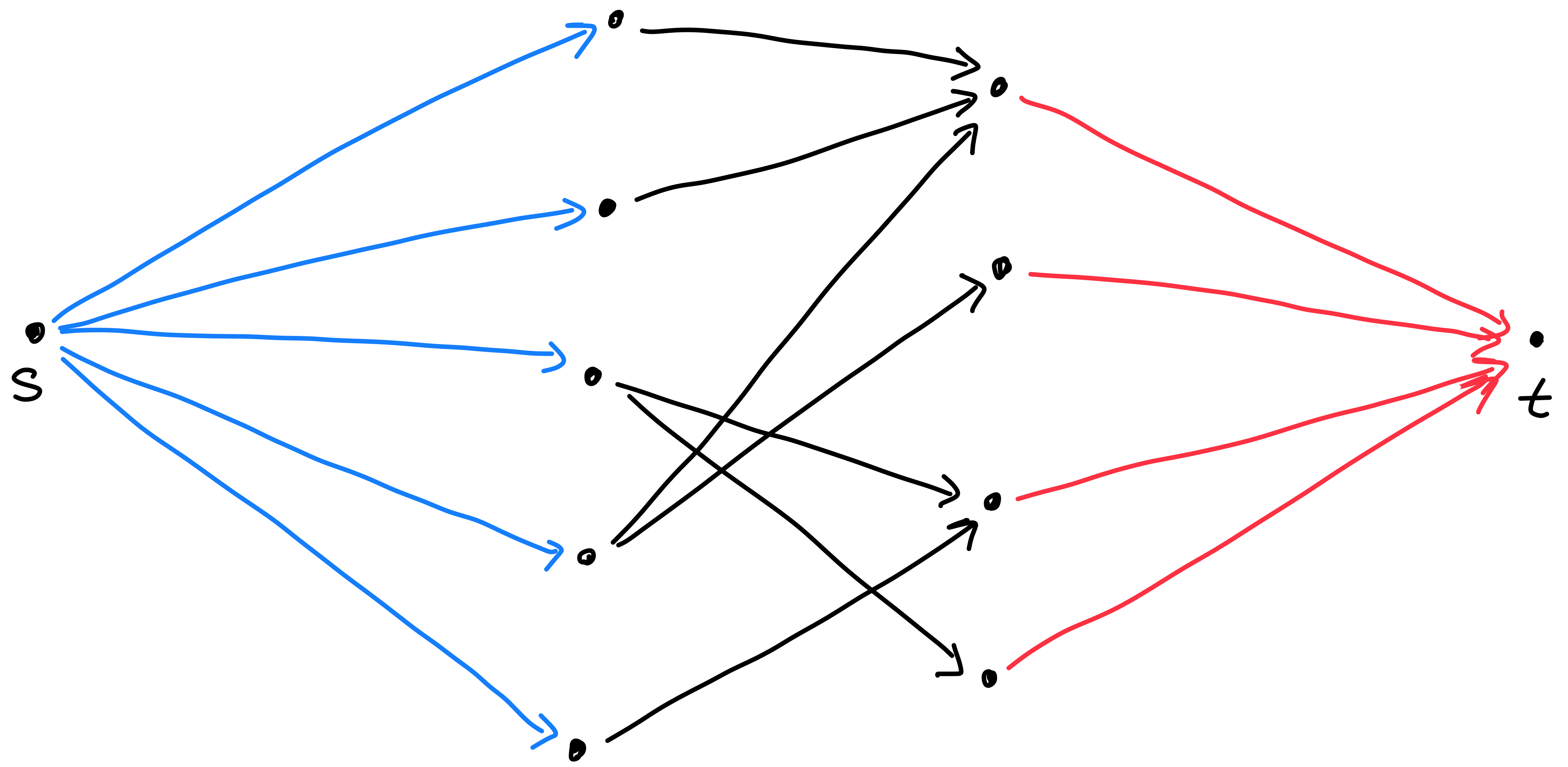
- Given a maximum flow f in a network G, if S is the set of vertices reachable from s in the residual network G_f, then $(S, T := V \backslash S)$ forms a minimum cut
  - Edges from S to T are fully saturated
  - Edges from T to S are completely devoid of flow
  - The min cut may not be unique just as the max flow may not be unique
- Maximum flow and minimum cut are dual problems
  - Two sides of the same coin
  - We will see this come up again in a few lectures!



Applications of max flow/min cut

Recall: bipartite matching

Run Ford-Fulkerson on this graph.



all edges of capacity 1

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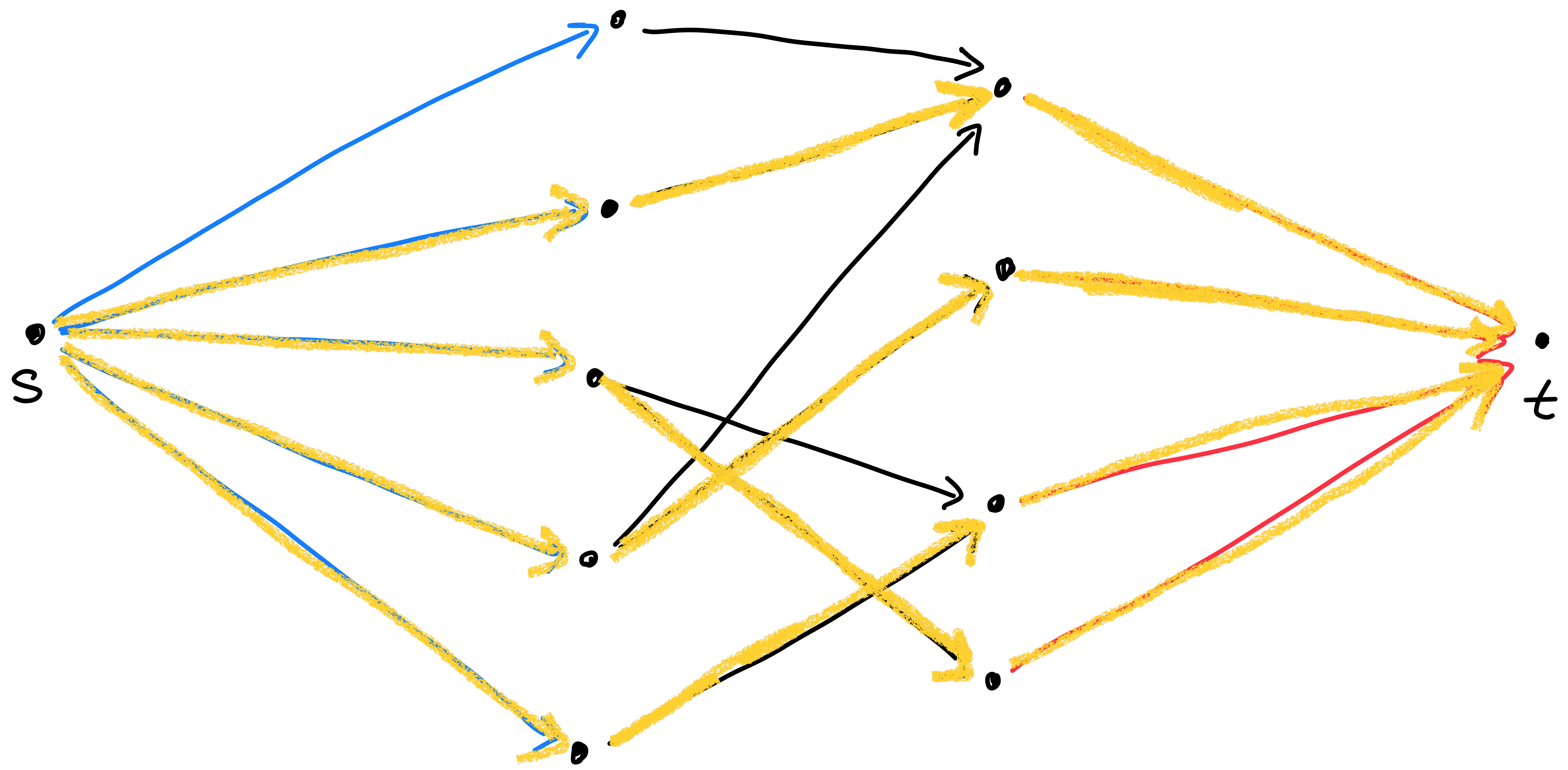


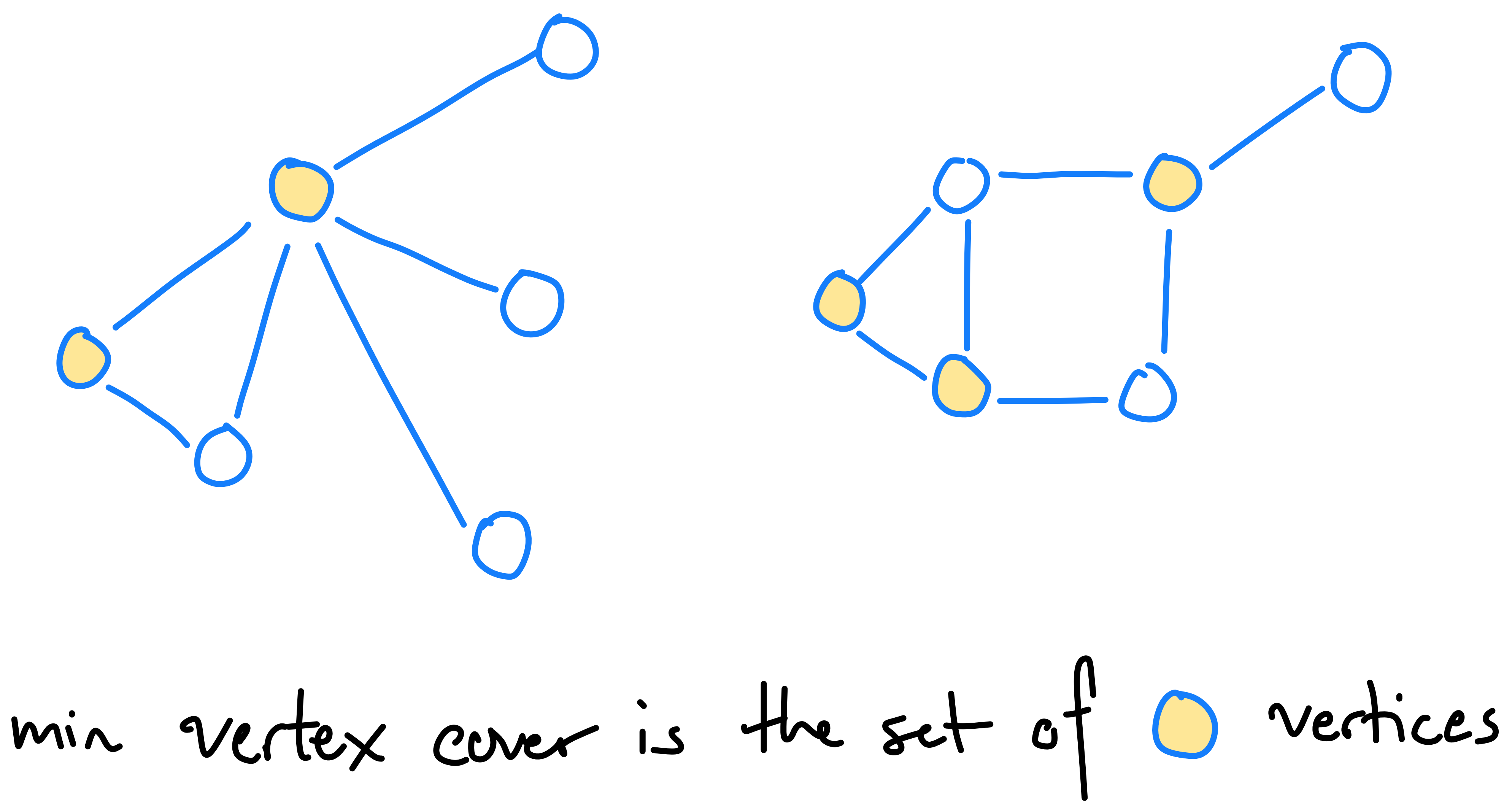
all edges of capacity 1

Recall: bipartite matching

- **Claim:** The edges of flow 1 in the max flow form a maximal bipartite matching.
- **Proof:**
  - Integer flow and bipartite matching equivalence:
    - Since FF only outputs integer flow, and each edge capacity is 1, at most 1 edge leaving a $v \in L$ can be selected. So a integer flow yields a matching of equal size.
    - For every edge $u \to v$ from L to R in the bipartite matching add the flow $s \to u \to v \to t$. All flows will be compatible. So a bipartite matching yields a flow of equal size.
  - By equivalence, max flow will yield a max bipartite matching.

Minimum vertex cover problem

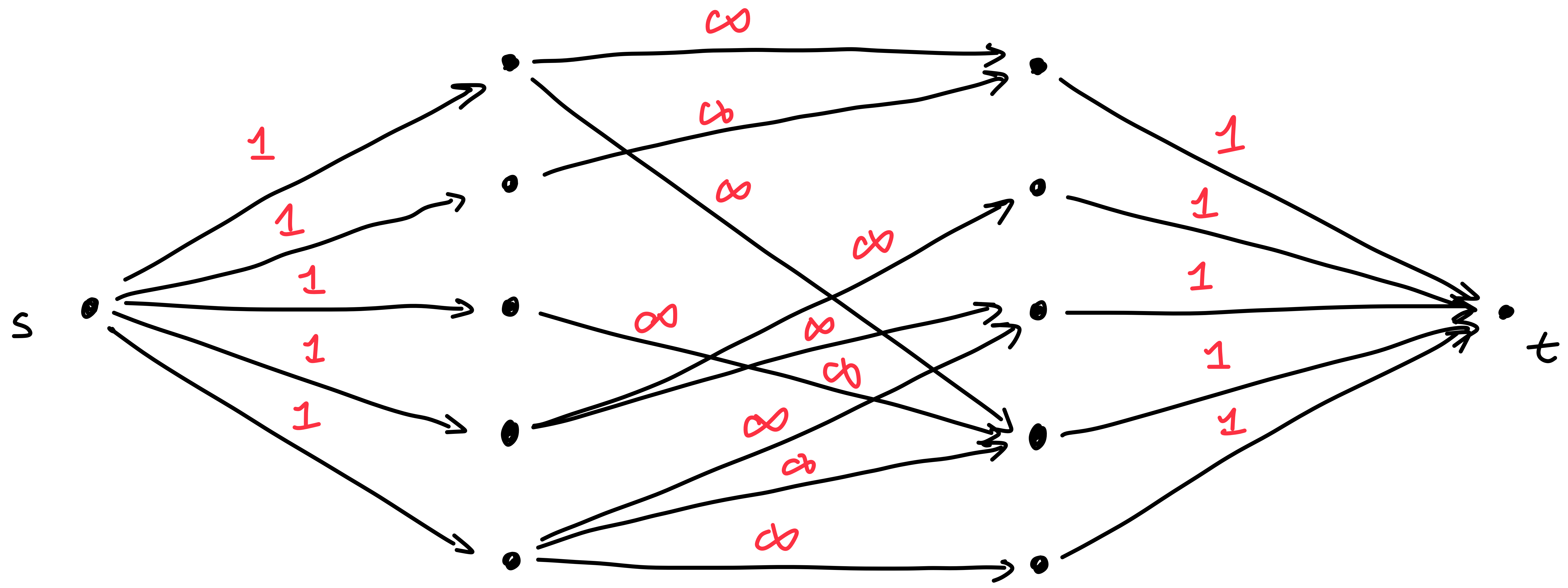
- **Definition:** A subset of vertices $C \subseteq V$ is a *vertex cover* of an undirected graph $G = (V, E)$ iff every edge is touched by some vertex in C.
  - V is a trivial vertex cover for G.
- **Input:** An undirected graph $G = (V, E)$
- **Output:** A minimal vertex cover C for G.



- Min Vertex Cover is a NP-complete problem
- However, min vertex cover on bipartite graphs is efficient!

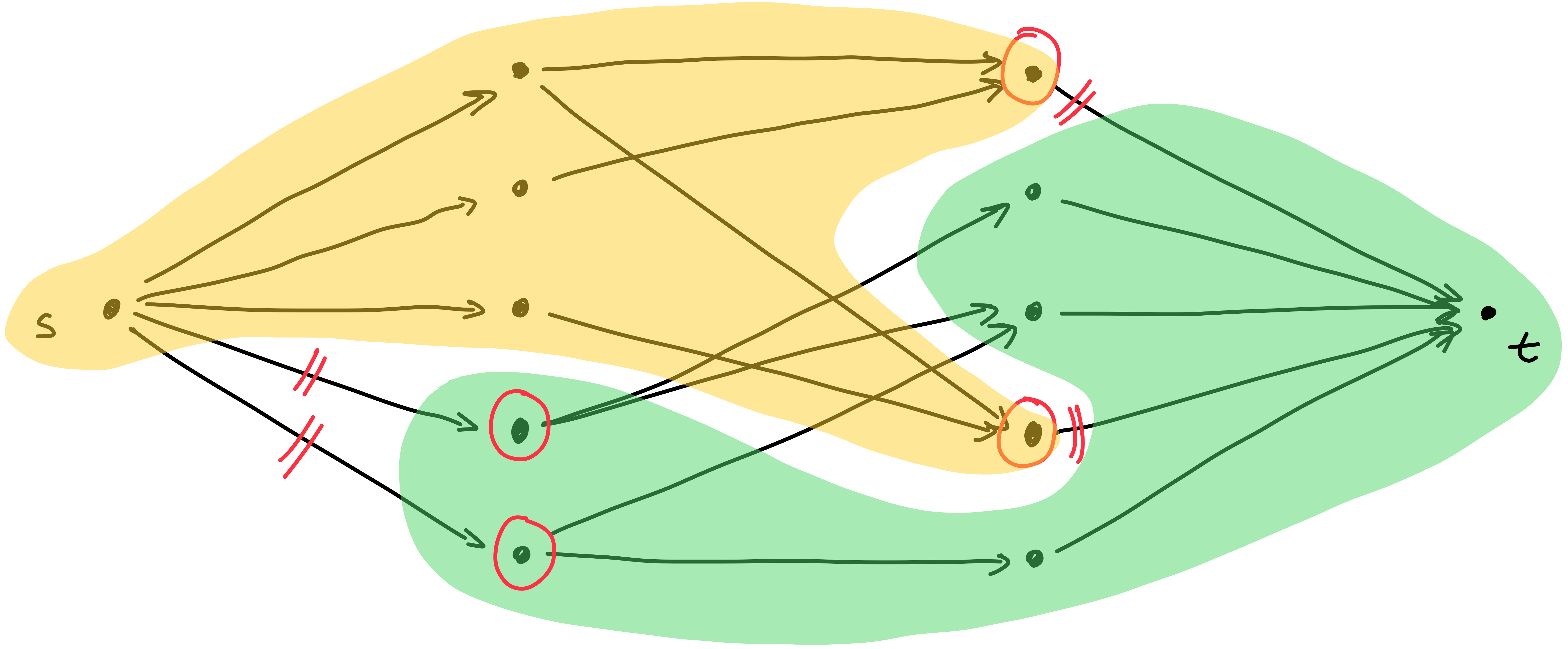
Min cut perspective for bipartite vertex cover

- We could solve the same flow problem if we set the capacity to the edges out of s and into t as 1 and set the middle edges to capacity ∞.



Min cut perspective for bipartite vertex cover

- Vertices of G involved in the min cut (one per edge crossing the cut) forms a minimum size set of vertices of G that block all flow from s to t



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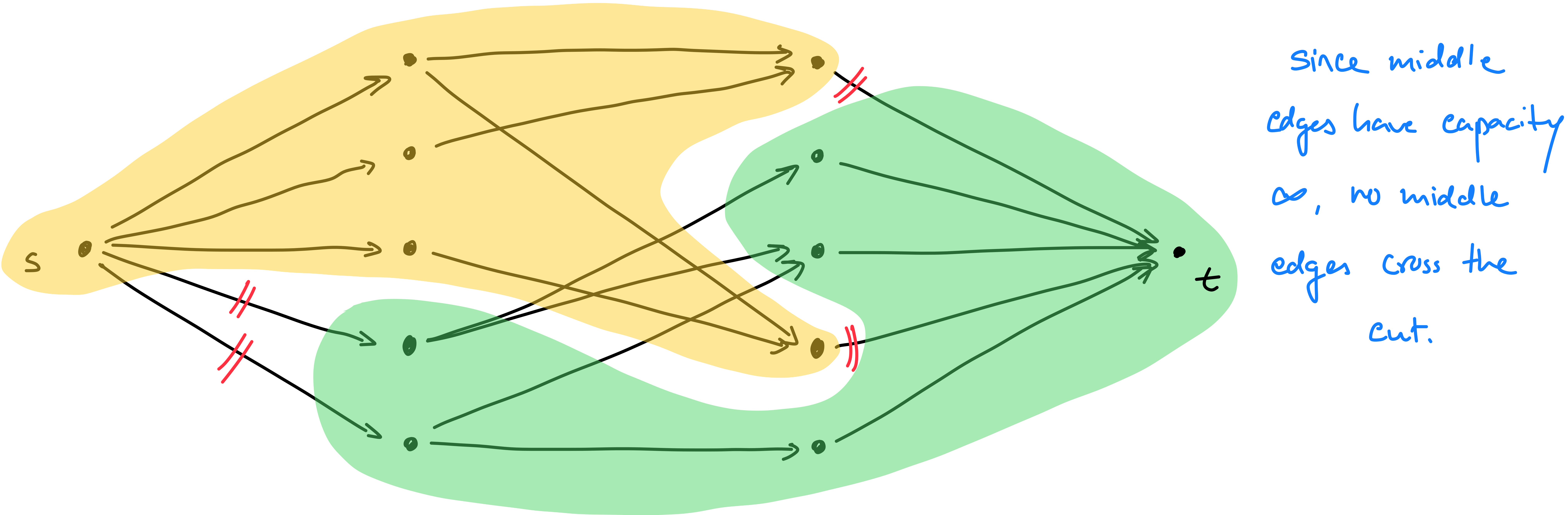
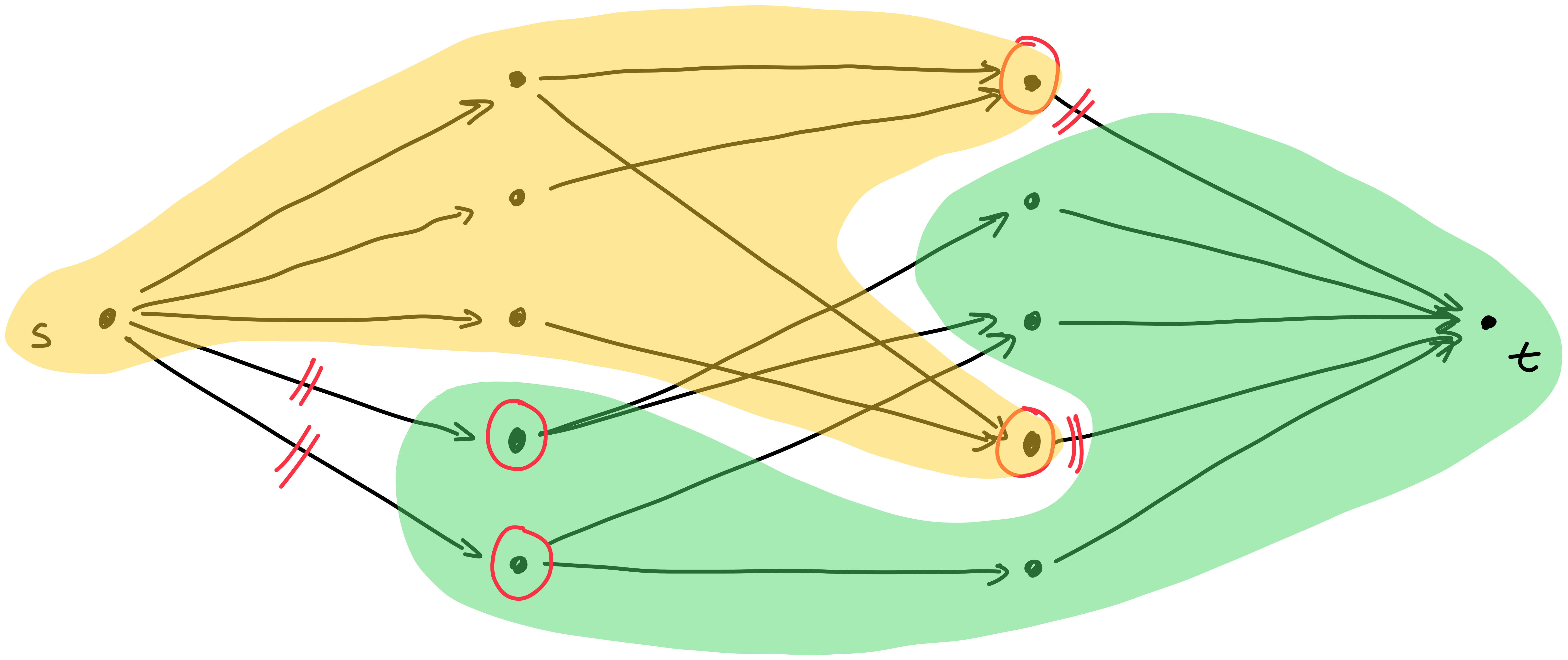


Minimum vertex cover problem

Bipartite graphs

- **Claim:** The min cut we observed just a minute ago generates a vertex cover.



Minimum vertex cover problem

Bipartite graphs

- **Claim:** The min cut we observed just a minute ago generates a min vertex cover.
- **Proof:**
- Suppose it did not generate a vertex cover.
  - Then there is an edge $e = (u, v)$ not covered. We can augment the flow along the path $s \to u \to v \to t$, a contradiction.
- Suppose there is a smaller min vertex cover C'
  - Then the edges connecting s or t to C' form the crossing edges of a smaller min cut. A contradiction.

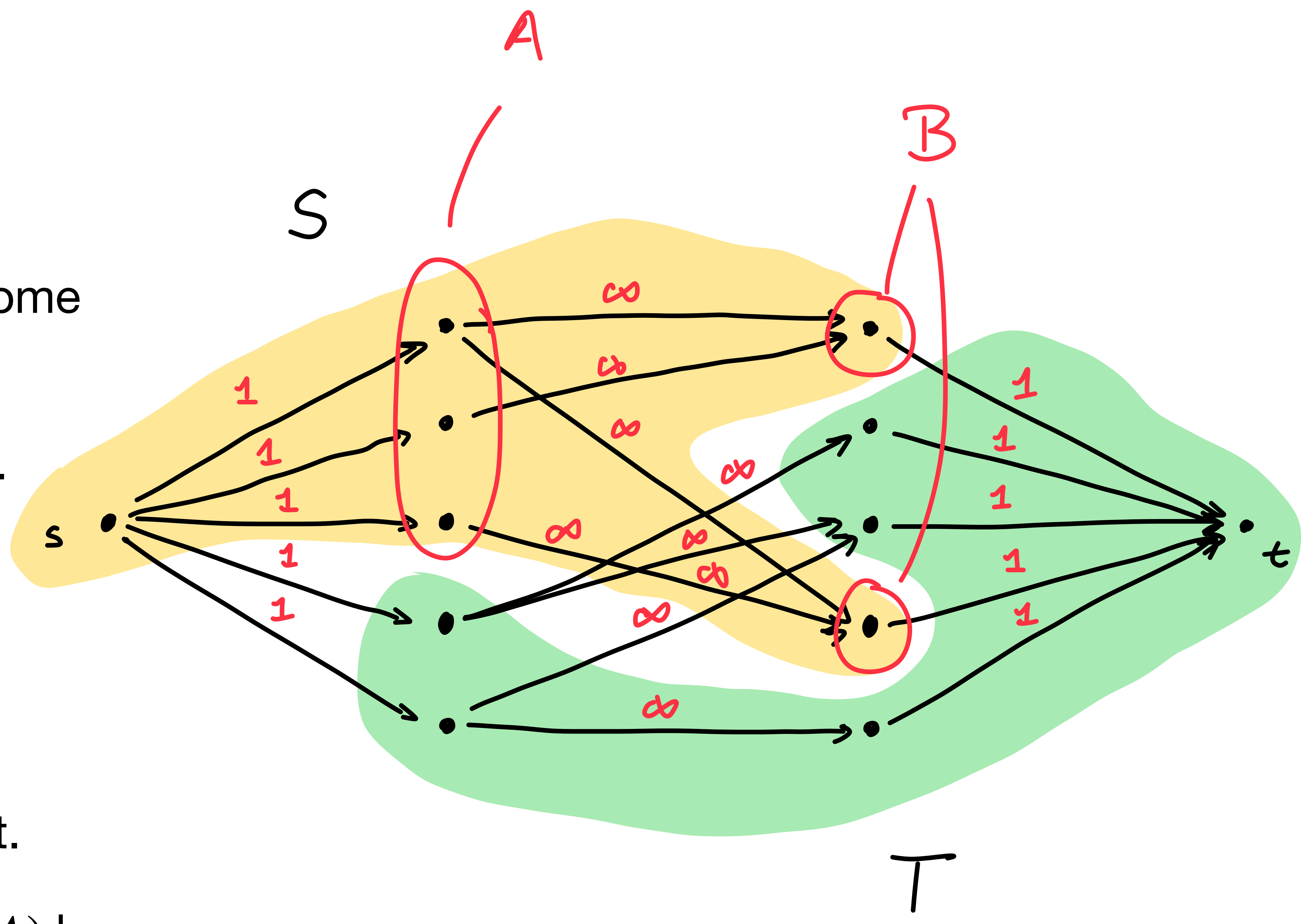
Perfect Matching

- **Definition:** A matching $M \subseteq E$ is perfect iff every vertex participates in some edge of M.
- The previous algorithms give us an algorithm for computing a maximal matching for a bipartite graph.
  - The matching is *perfect* if $|L| = |R|$ and every vertex is matched!
  - The previous algs. also provide a criterion for whether a bipartite graph has a perfect matching: **Hall's theorem**.

Hall's theorem

neighbors of the set A in the graph

- **Theorem:** If $|N(A)| \geq |A|$ for all subsets $A \subseteq V$, then there is a perfect matching.
- **Contrapositive:** If there is no perfect matching, then there exists some subset A for which $|N(A)| < |A|$.
- **Proof:** No perfect matching $\implies$ min cut is $< |L|$ in flow network.
  - Let (S, T) be a s-t cut with $c(S, T) < |L|$ (since no perfect matching)
  - Choose $A = S \cap L, B = S \cap R$.
  - Then $N(A) \subseteq B$ since no edges across the middle are in the cut.
  - So $|L| > c(S, T) = (|L| - |A|) + |B| \geq |L| - |A| + |N(A)|$
  - So $|N(A)| < |A|$.

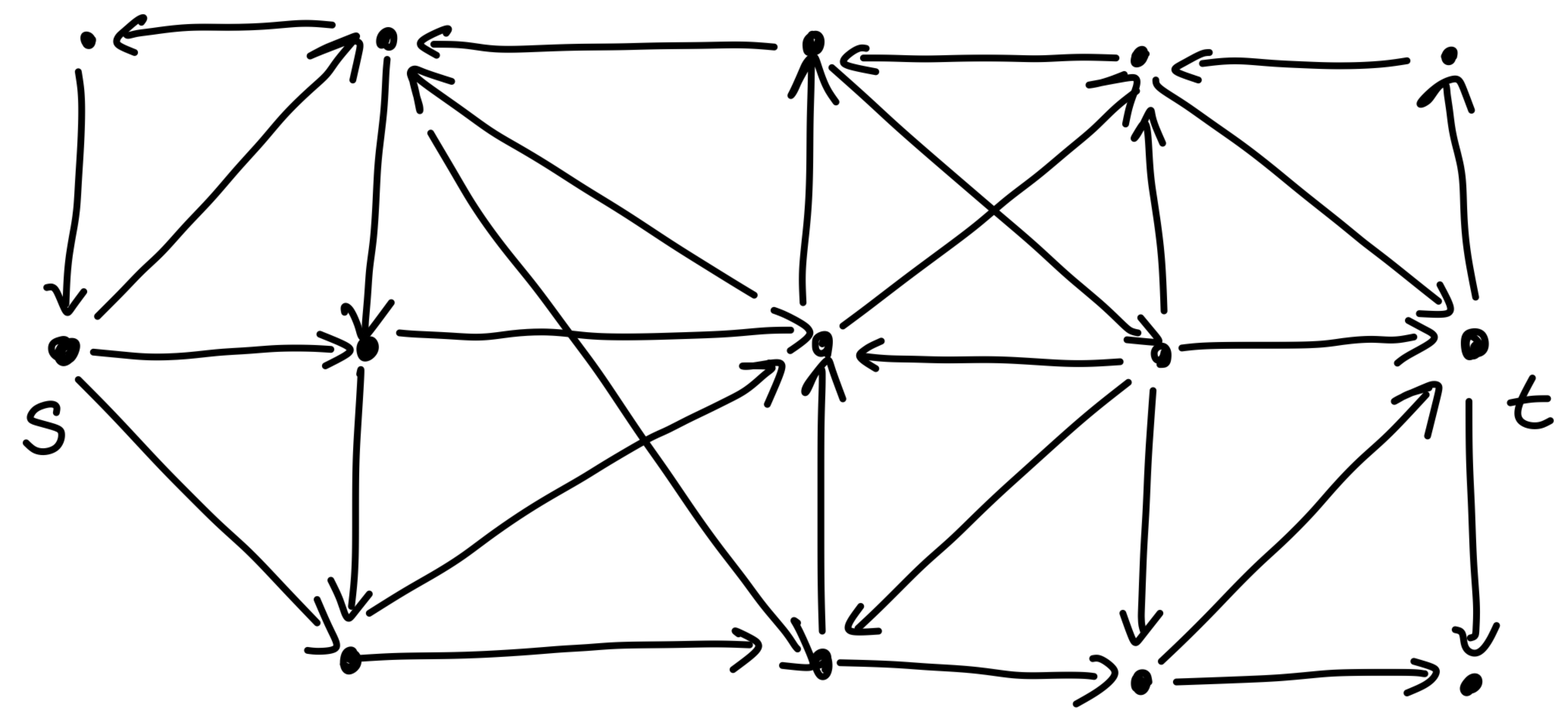


Maximum matching in general graphs

- Bipartite maximum matching runtimes:
  - Generic augmenting path: $O(mn)$
  - State of the art algorithm run in time $O(m^{1+o(1)})$ time with high probability
- General matching algorithm:
  - Solved — $O(mn^{1/2})$ time algorithm exists by Micali-Vazirani
  - Beyond the scope of this course

Edge disjoint paths

- **Input:** A directed graph $G = (V, E)$ with identified vertices s, t
- **Output:** A *maximal* collection of paths $s \rightsquigarrow t$ that share no edges
- **Application**: routing transmissions in communication networks



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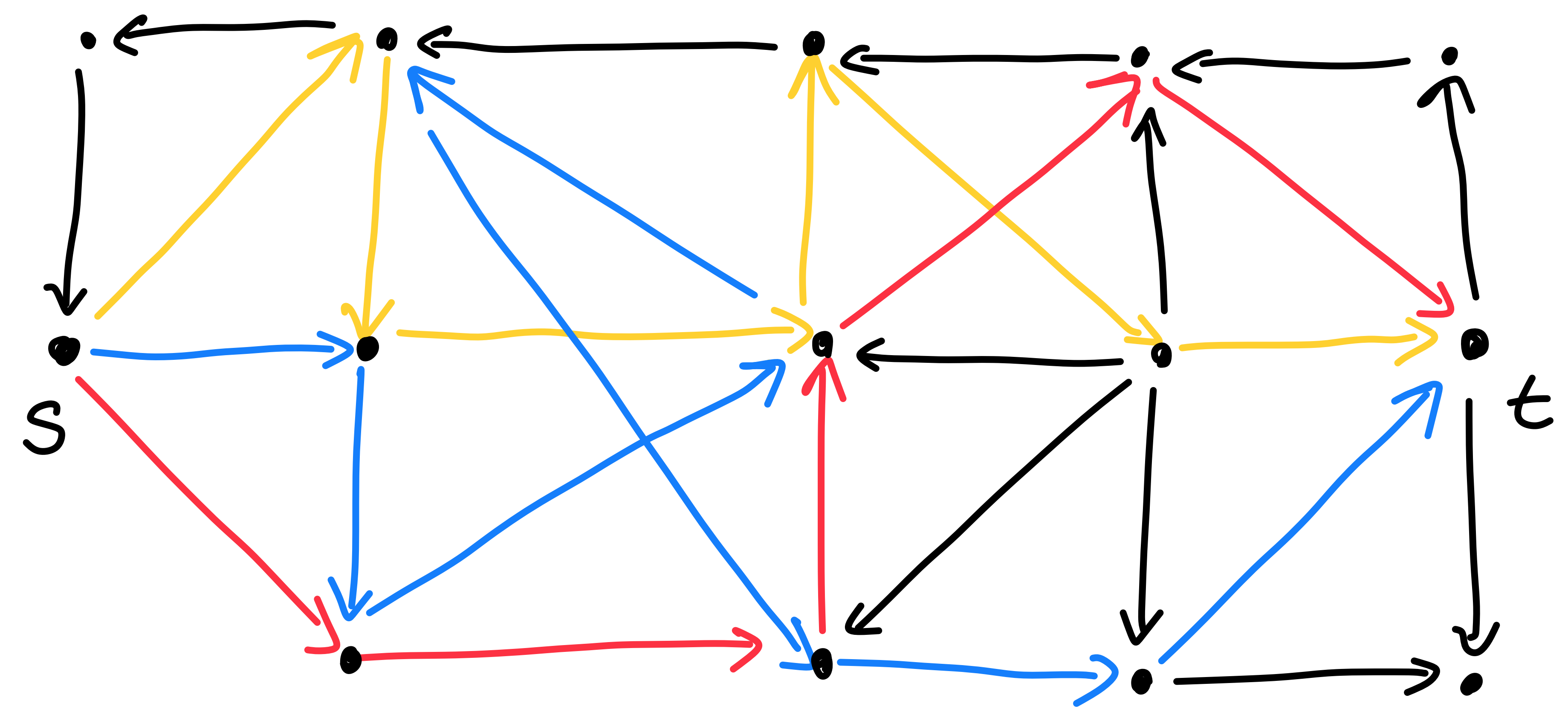
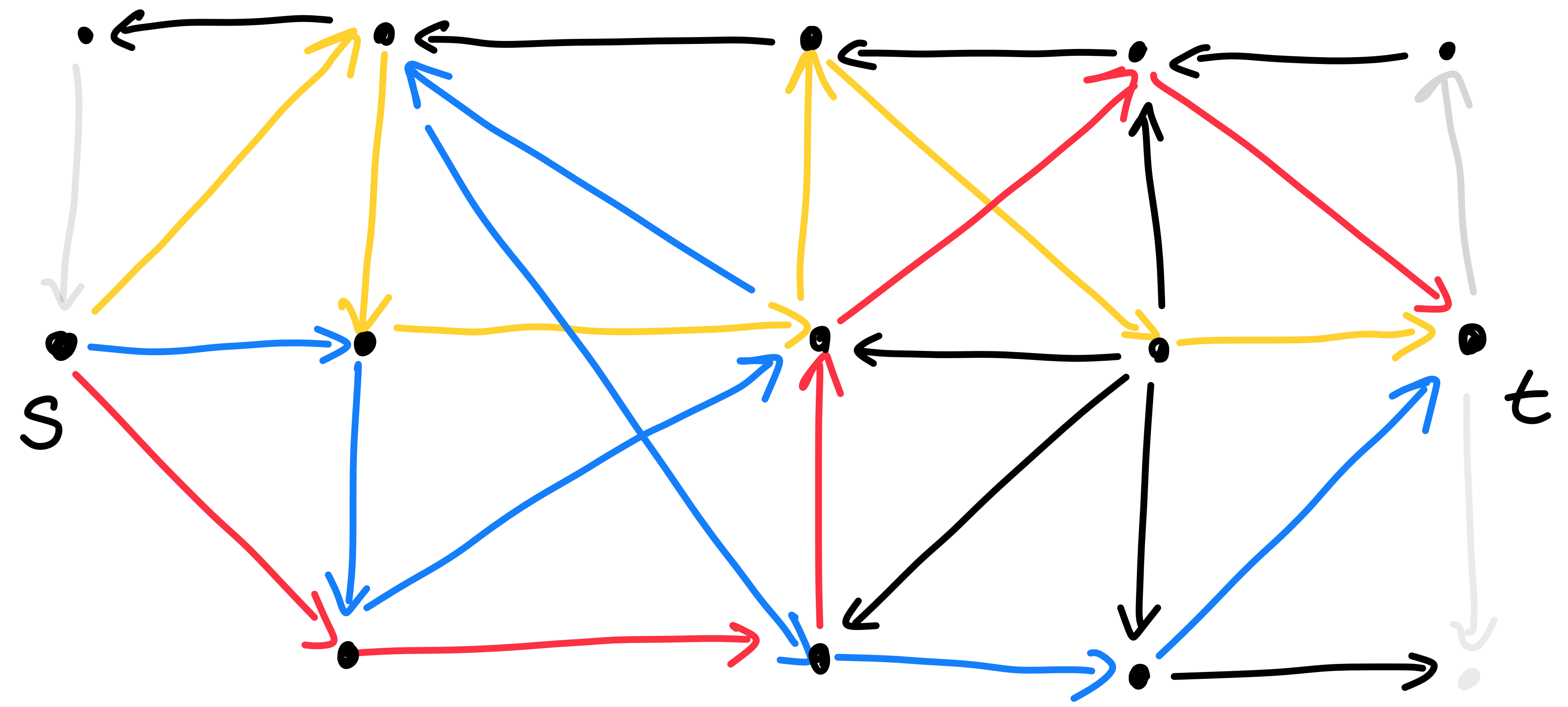


Edge disjoint paths

- **Idea:** Use max flow to calculate edge disjoint paths
- Need to convert our graph to a flow network
  - Remove any edge $\cdot \to s$ and $t \to \cdot$
  - Set capacity of all remaining edges to 1



- **Correctness argument**: Prove a *bijection* between integer flows and edge disjoint paths. Then maximality of flow yields maximal edge disjoint paths.

Edge disjoint paths

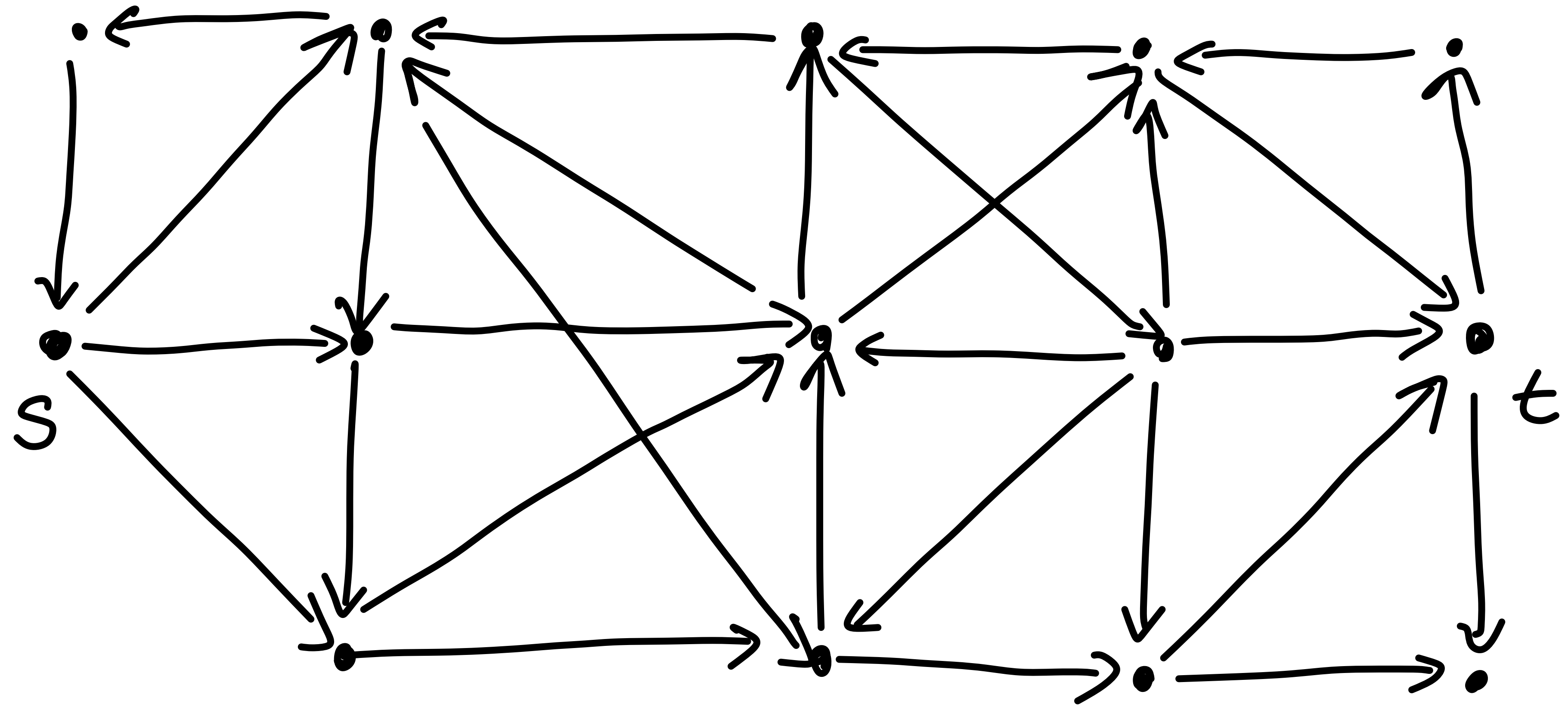
- **Lemma:** Every integer flow is the sum of 1-flow along edge disjoint paths.
- **Proof:**
  - Since capacities are 1, $f(e) \in \{0,1\}$ since it is integer.
  - Then for each edge e, at most one flow along a path can use e.
  - We previously proved that every flow can be decomposed into $\leq m$ paths.
  - Therefore, the paths founds are edge disjoint.

Edge disjoint paths

- **Theorem:** There is a bijection between integer flows and edge disjoint paths.
- **Proof:**
  - Previous lemma converts each integer flow into an edge disjoint path.
  - Sending 1-flow along each edge disjoint path is a valid flow.
    - Conservation of flow follows at every vertex $v \in V \backslash \{s, t\}$ from that of paths.
    - Capacity constraints follow from being a 1-flow and edge disjoint.
  - Together, this proves both directions of the bijection.

Network connectivity

- **Definition:** A set of edges $F \subseteq E$ **disconnects** the source and sink if every path $s \rightsquigarrow t$ must use one edge from F.
- **Input:** directed graph $G = (V, E)$ with source s and sink t
- **Output:** a *minimal* set of edges F that disconnect the source and sink



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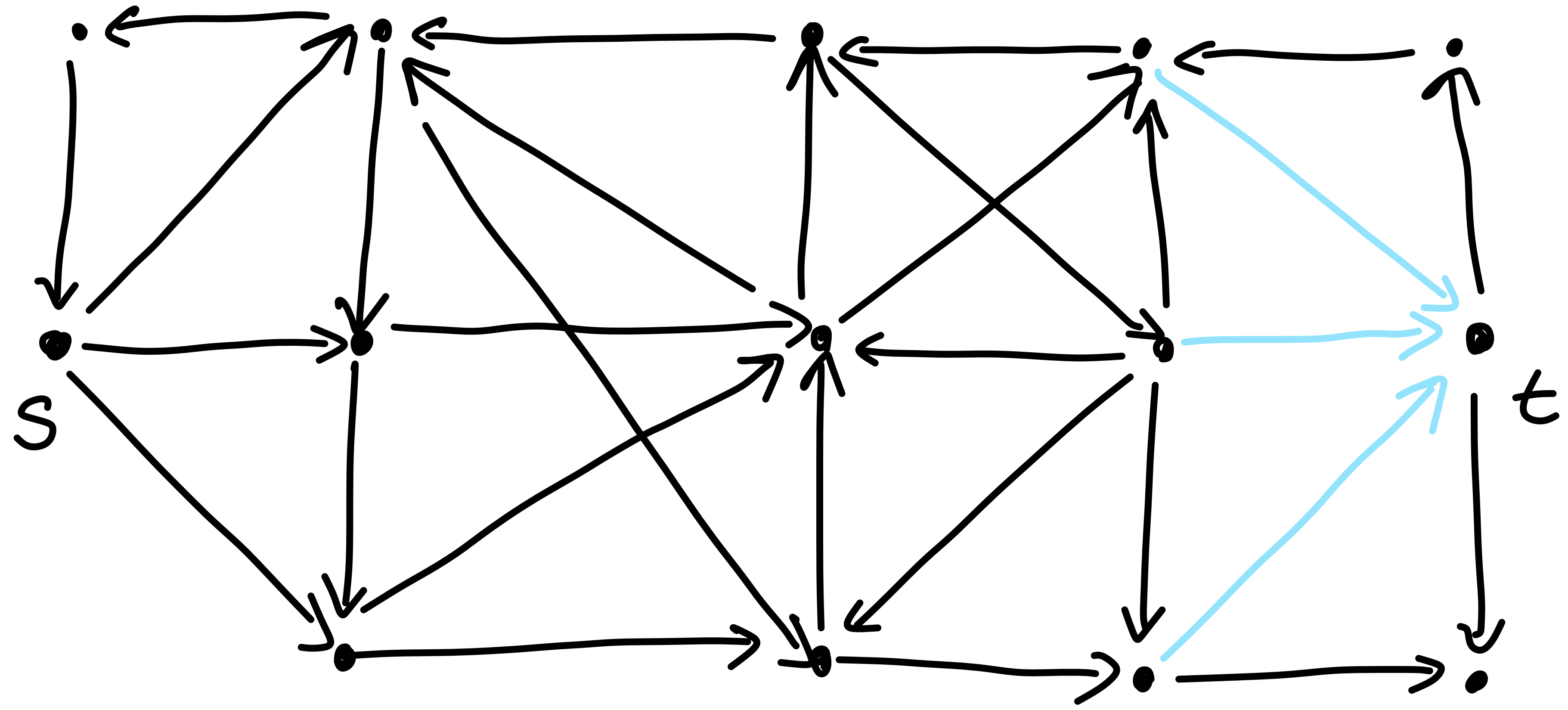


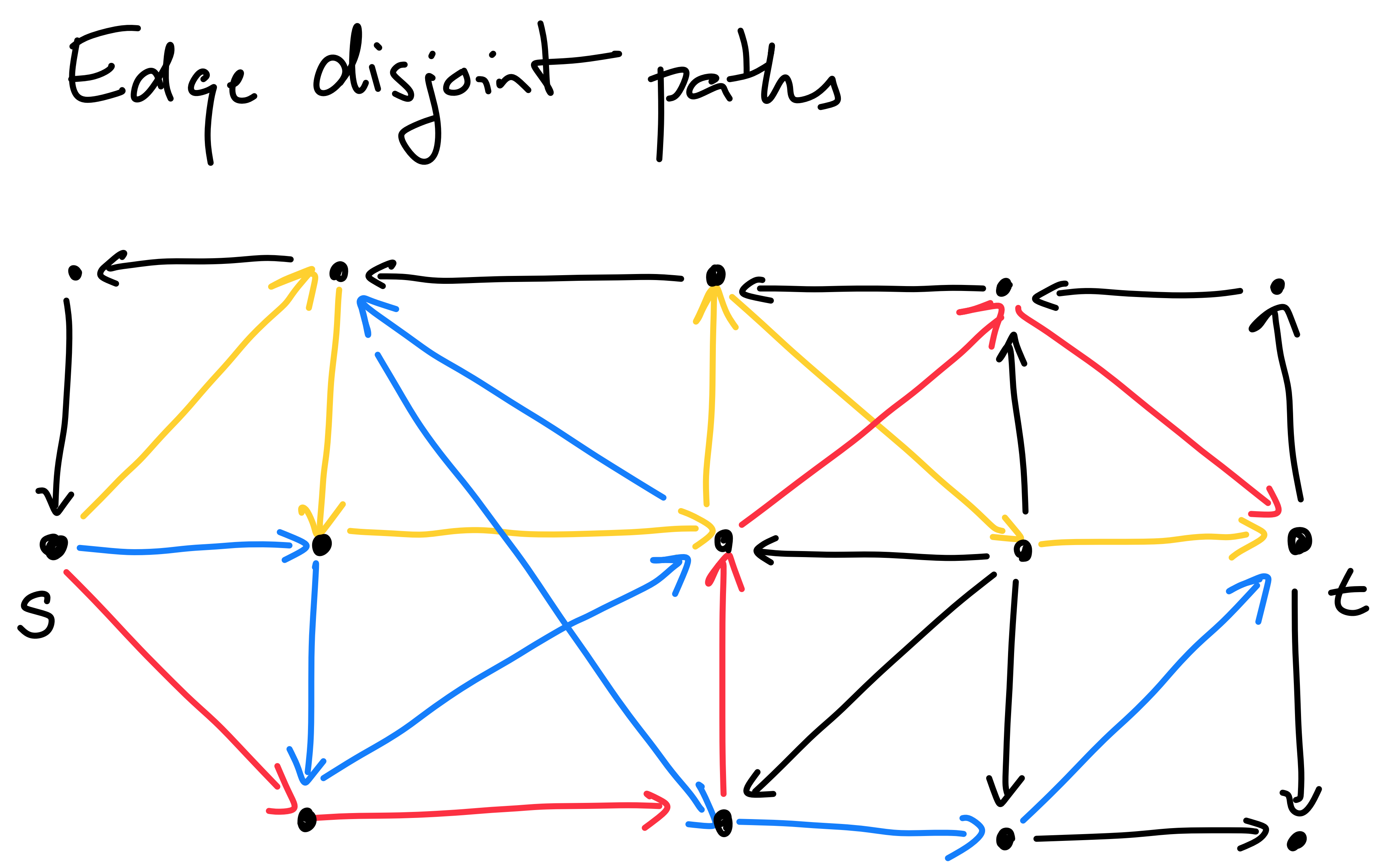
Network connectivity

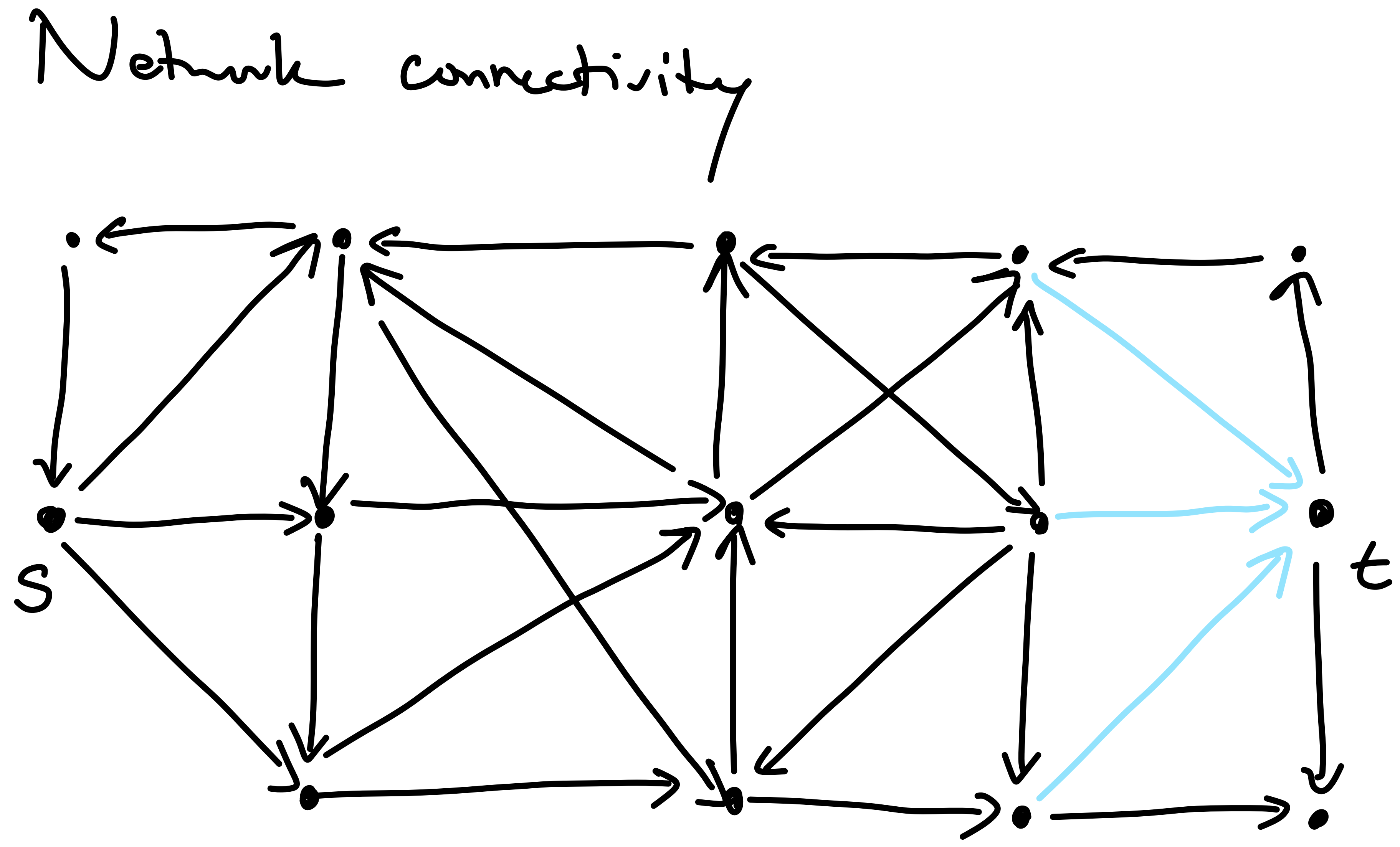
- **Idea:** Use min cut to calculate minimal network disconnecting set
- Again, need to convert our graph to a flow network
  - Remove any edge $\cdot \to s$ and $t \to \cdot$
  - Set capacity of all remaining edges to 1

- **Correctness argument**: Prove a *bijection* between cuts and network disconnecting sets. Then minimality of cut yields minimal disconnecting set.

Network connectivity

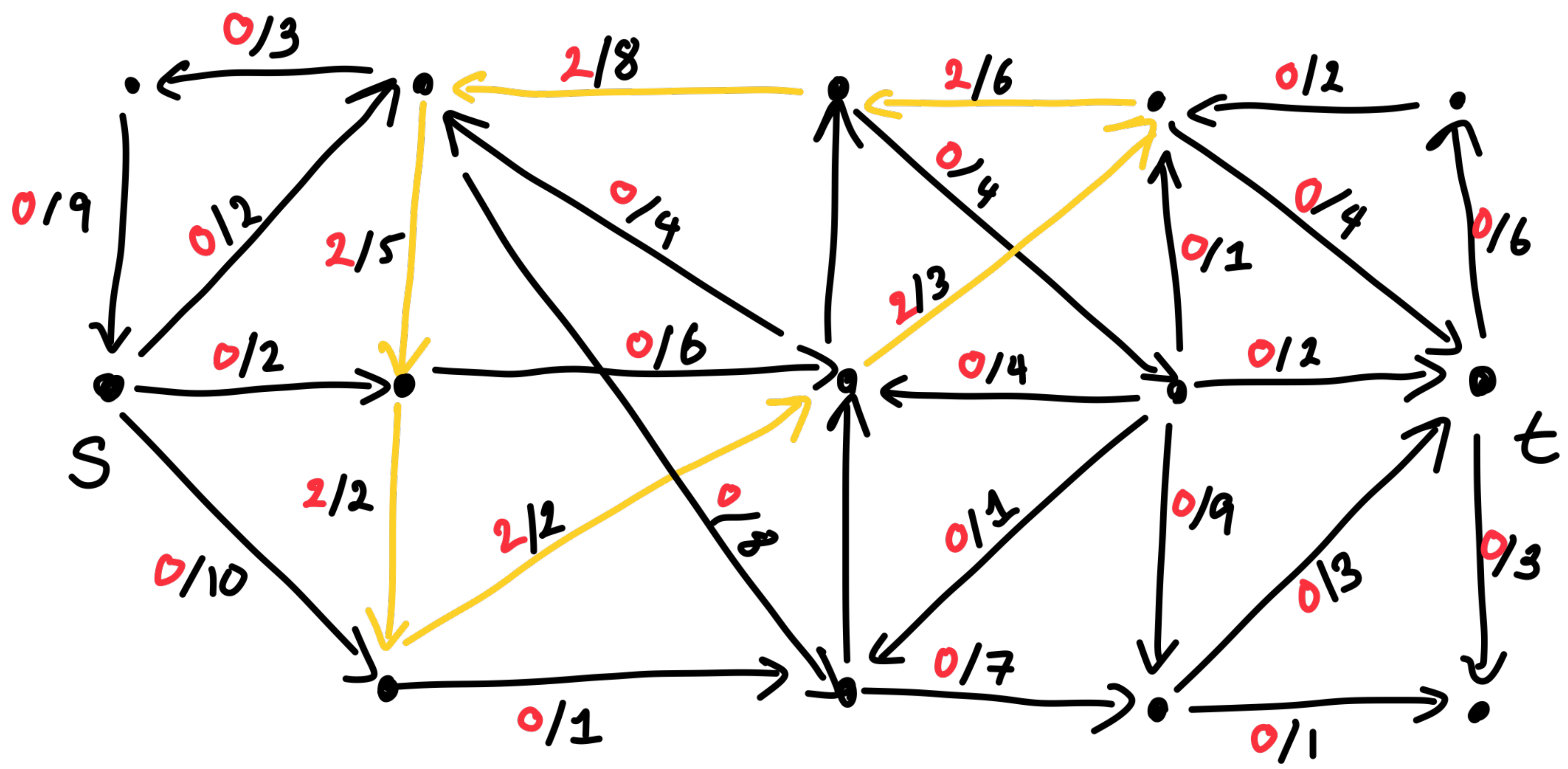
- Network connectivity and edge disjoint paths use the same reduction
    - Network connectivity is equivalent to min cut
    - Edge disjoint paths is equivalent to max flow
- **Menger's theorem:** the maximum number of edge disjoint s-t paths is equal to the minimum size of a disconnecting set





Directed flow cycle

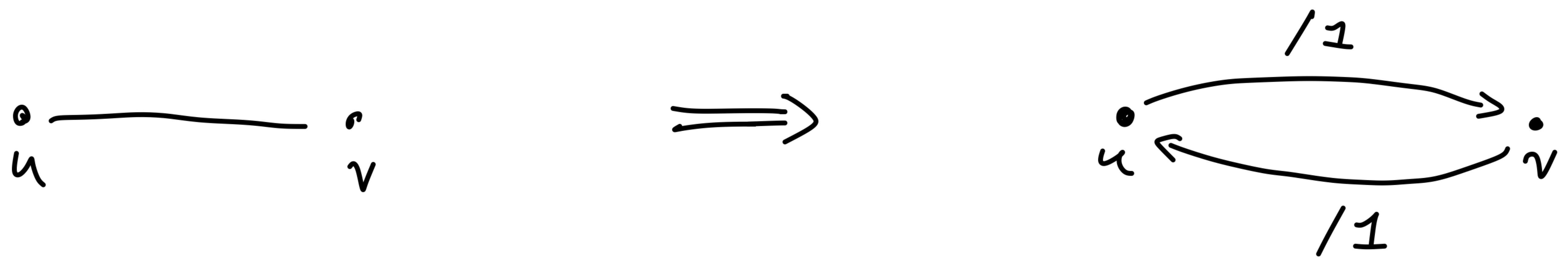
- **Definition:** A directed flow cycle is a flow of value 0 but $f \not\equiv 0$ on every edge
- **Examples:**



- Directed flow cycles can be removed by running graph traversal on f, finding cycles and removing bottleneck flow around the cycle

Undirected graphs

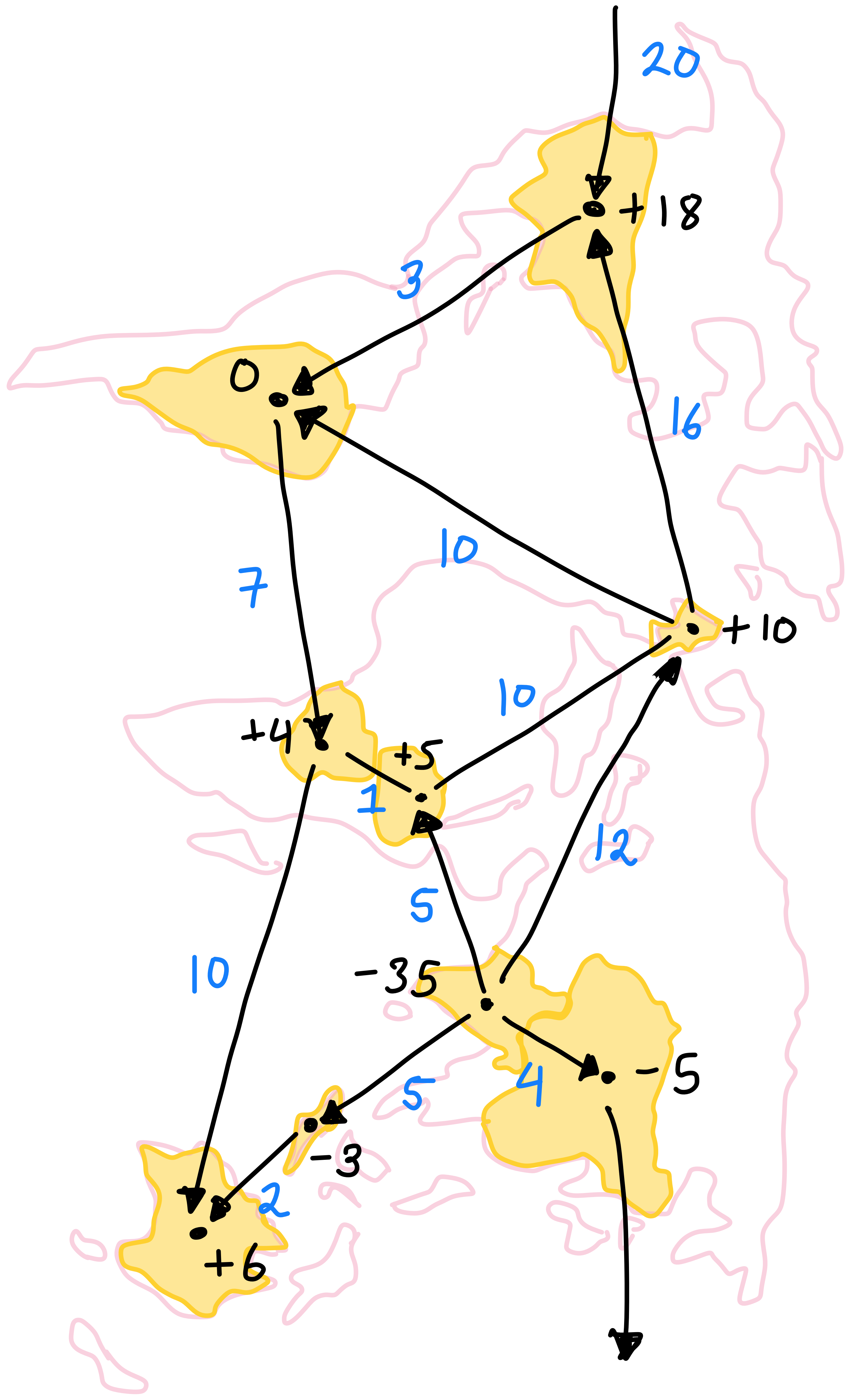
- Edge disjoint path and disconnecting set problems can be solved with flow algorithms for *directed* graphs
- What about undirected graphs?
- **Solution:** Replace each edge (u, v) with directed edges $(u \to v), (v \to u)$



  - Run directed algorithm on new graph
  - Remove any directed flow cycles
  - Include edge $\{u, v\}$ if either edge is used after removing flow cycles

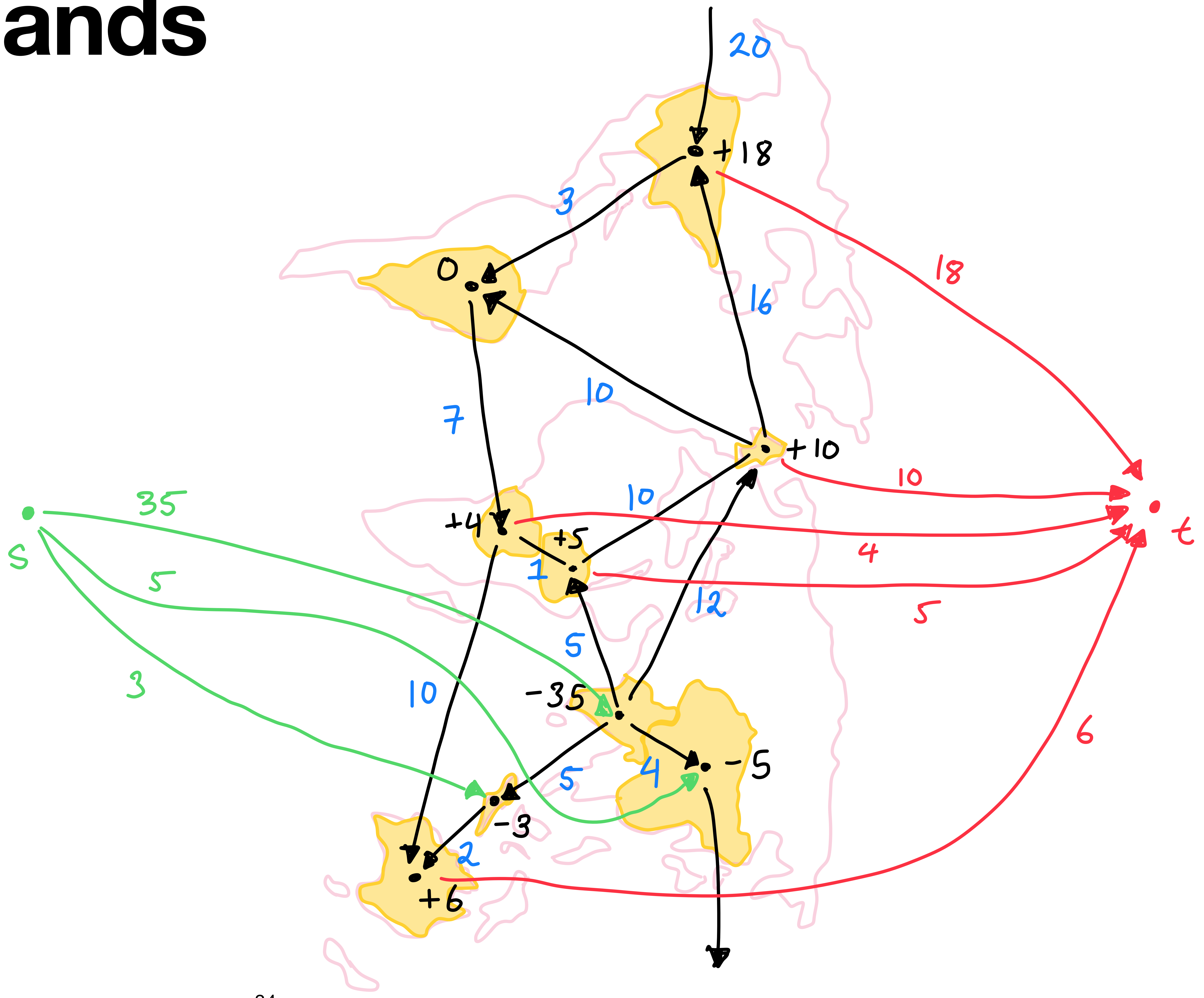
Circulation Demands

- Some countries produce more rice than the consume and some countries consume more rice than the consume
- There are trade routes that describe which countries can trade with which others and at what capacity
- How do we calculate rice routing?
- **Input:** directed graph $G = (V, E)$ with capacities $c : E \to \mathbb{R}_{\geq 0}$ and demand $d : V \to \mathbb{R}$ such that $\sum_{v \in V} d(v) = 0$.
- **Output:** A flow $f : E \to \mathbb{R}$ such that $f^{\text{in}}(v) - f^{\text{out}}(v) = d(v)$



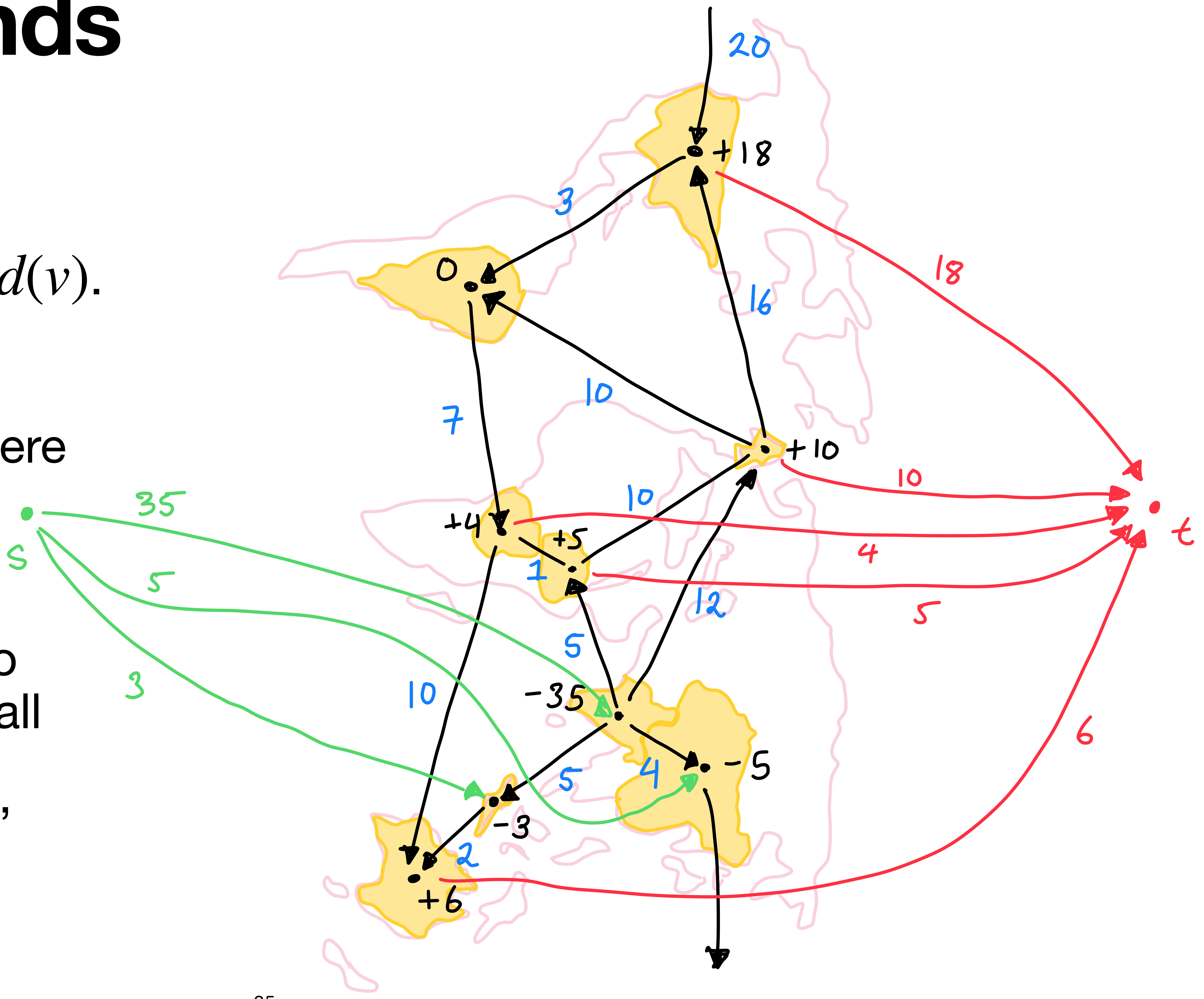
Circulation demands

- Add source s and t to graph
- Add edge $s \to v$ of $-d(v)$ if $d(v) < 0$.
- Add edge $v \to t$ of $d(v)$ if $d(v) \geq 0$.
- Compute max flow on the graph.



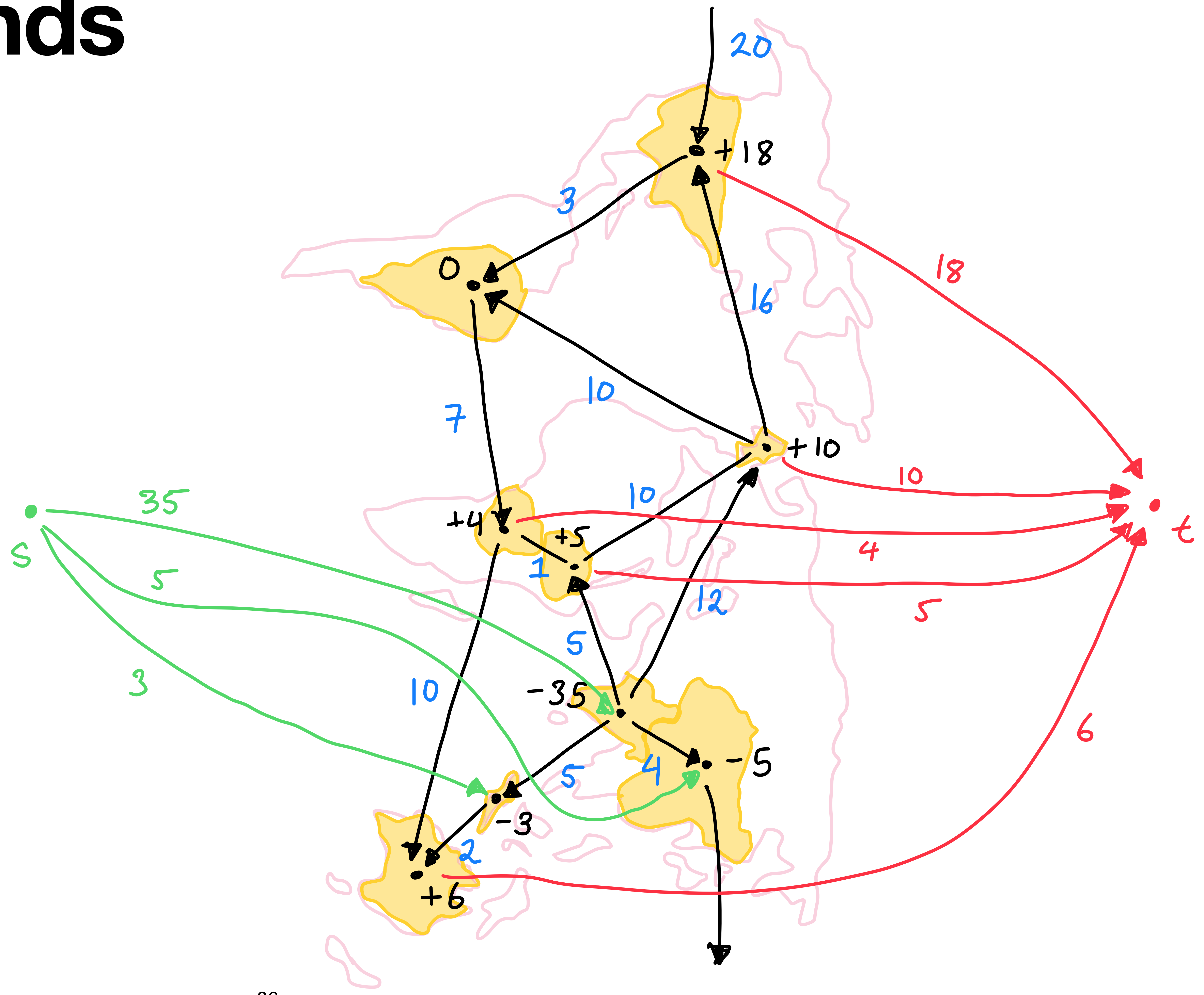
Capacity demands

- **Theorem:** Let $D^+ = \sum_{v:d(v)\geq 0} d(v)$.
  - Then if, max flow = D^+, there is a *circulation* meeting all capacities and demands.
  - If max flow $< D^+$, then no circulation exists meeting all capacities and demands. $D^+ - v(f)$ is the "wasted" production.



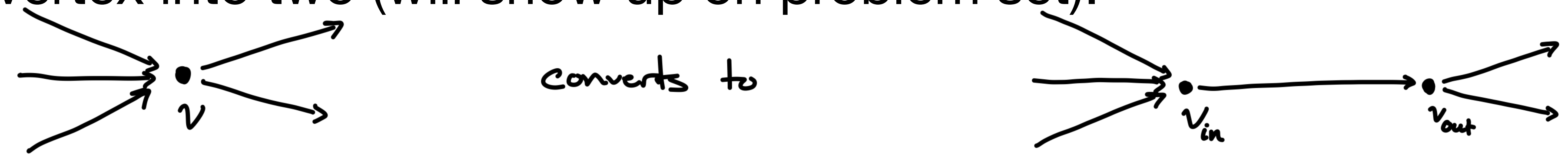
Capacity demands

- When does a circulation not exist? When max flow = min cut $< D^+$.
- Min-cut between "source" and "sink" vertices is smaller than demand.
- Look at India: The trade network is too small to satisfy the output.



General max flow/min cut algorithmic paradigm

- If source and sink are not obvious, they may need to be added to the graph
- We need to choose capacity limits for edges: 0, 1, ∞ or an input from the problem are logical choices
  - Edges of capacity ∞ **cannot** cross the cut. Equivalently, edges of free flow.
- Undirected graphs will need to be converted to directed equivalents
  - Unnecessary flow cycles can be removed after flow is calculated
- Split a vertex into two (will show up on problem set):



- Choose correct version of flow algorithm based on capacities

Cut like problems

- Until now, most of the problems looked mostly “flow”-like
- Max flow = min cut tells us that there are probably many “cut”-like problems we can also solve
- Next: an examples of a cut-like problem
    - Goal here is to get you to see flow networks appear in unexpected situations
    - This is at the heart of learning how to design algorithms

Baseball winner

- Imagine a simplified scenario — the team(s) that wins the most games overall is crowned the winner(s).
- Midway through the season, we have the following win totals for the teams

Team	Wins	Games remaining vs Angels	Games remaining vs Rangers	Games remaining vs Athletics	Games remaining vs Mariners
Angels	81	—	5	4	3
Rangers	80	5	—	3	4
Athletics	69	4	3	—	5
Mariners	70	3	4	5	—

Could the Mariners possibly win or tie for first?

Baseball winner

- Best case is Mariners win out — 82 wins
- Still depends on how the other teams play each other. How do we algorithmically calculate this?
- In order to win or tie, Mariners must have a run total at least as high as every other team.

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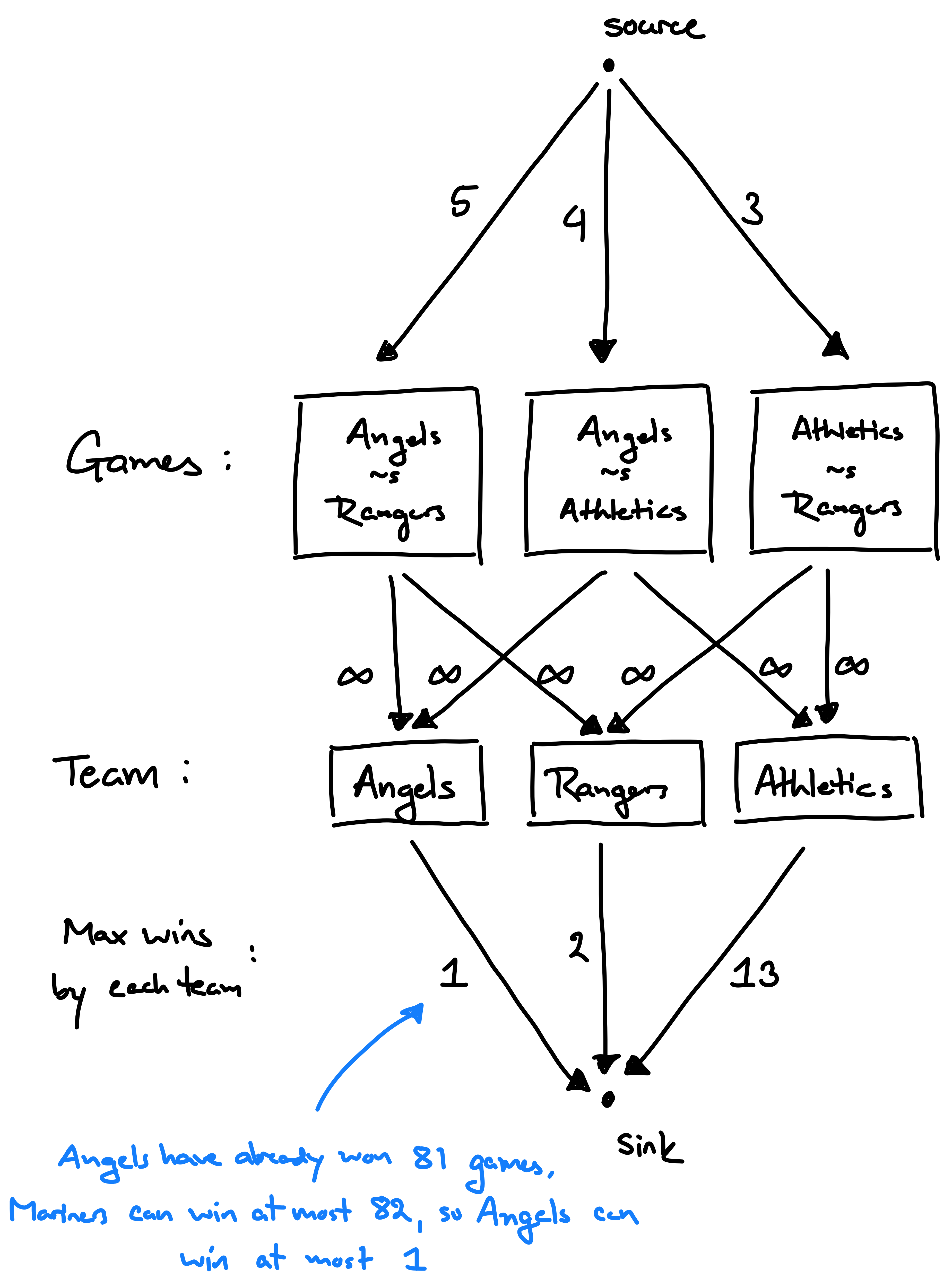

Baseball winner

- If there was a way that the games could play out such that no team amassed > 82 wins then there would be a flow of value $5 + 4 + 3 = 12$ in this network.

- However, the min cut equals = 10

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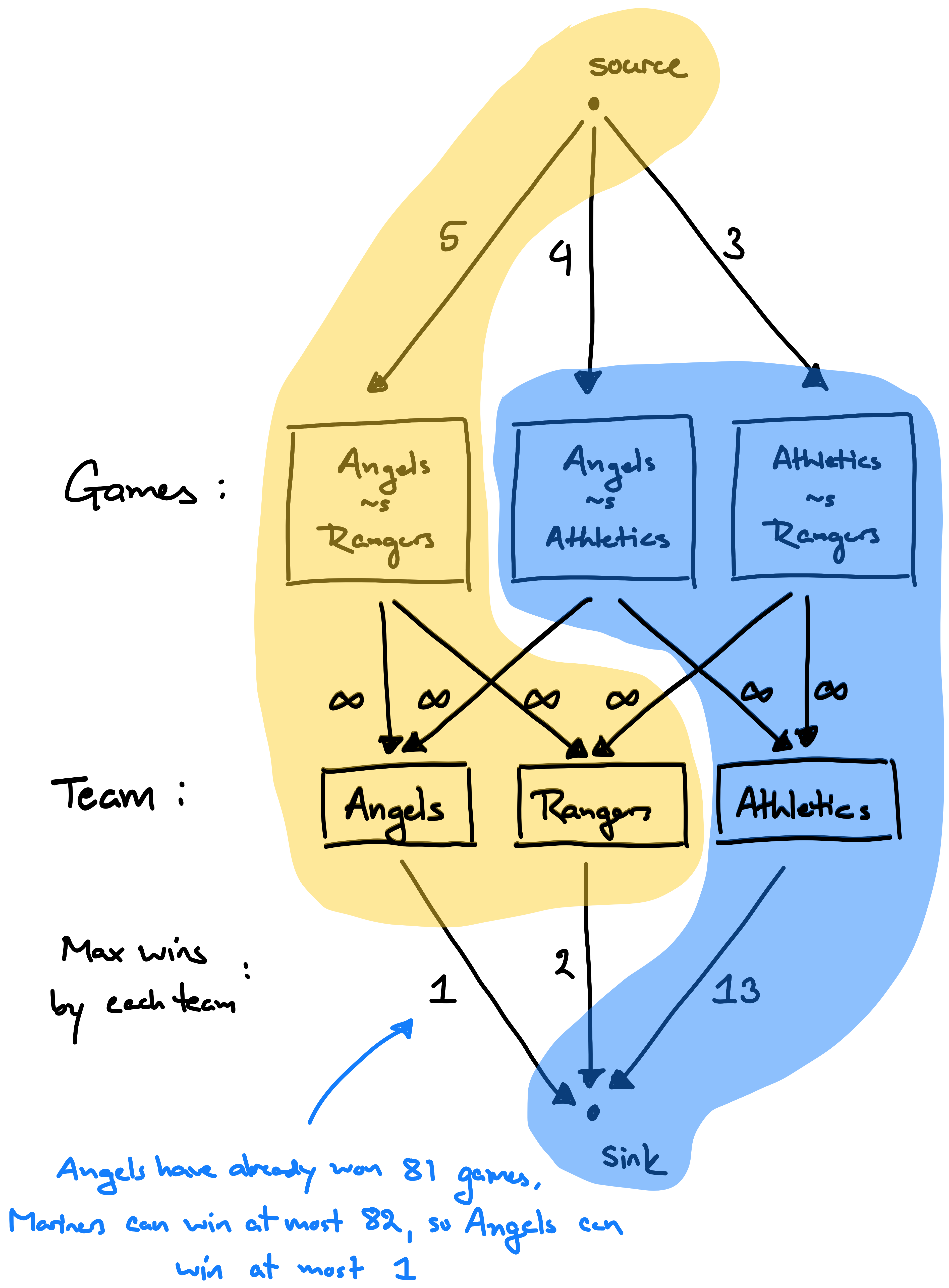

Baseball winner

- Even though no team has won > 82 games yet, this mathematically proves that the Mariners cannot win/tie for 1st.
- A clever way to consider all possible scenarios without exploring all the remaining games.

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