More DP

Hidden parameters/recurrences

CSE 421 Spring 2026
Lecture 13

Maximum Contiguous Subarray Sum

We saw an $O(n \log n)$ divide and conquer algorithm.

Can we do better with DP?

Given: Array $A[]$

Output: i, j such that $A[i] + A[i+1] + \cdots + A[j]$ is maximized.

Dynamic Programming Process

1. Define the object you're looking for

2. Write a recurrence to say how to find it

3. Design a memoization structure

4. Write an iterative algorithm

Maximum Contiguous Subarray Sum (define opt)

We saw an $O(n \log n)$ divide and conquer algorithm.

Can we do better with DP?

Given: Array $A[]$

Output: i, j such that $A[i] + A[i+1] + \cdots + A[j]$ is maximized.

For today: just output the value$A[i] + A[i+1] + \cdots + A[j]$.

Define `OPT(i)`.

Trying to Recurse

0	1	2	3	4	5	6	7
5	-6	3	4	-5	2	2	4

$OPT(3)$ would give $i = 2$, $j = 3$

$OPT(4)$ would give $i = 2, j = 3$ too

$OPT(7)$ would give $i = 2, j = 7$ – we need to suddenly backfill with a bunch of elements that weren't optimal…

How do we make a decision on index 7? What information do we need?

What do we need for recursion?

If index i IS going to be included

We need the best subarray **that includes index $i-1$**

If we include anything to the left, we'll definitely include index $i-1$ (because of the contiguous requirement)

If index i isn't included

We need the best subarray up to $i-1$, regardless of whether $i-1$ is included.

Two Values

Pollev.com/robbie

Need two recursive values:

$INCLUDE(i)$: sum of the maximum sum subarray among elements from 0 to i **that includes index i** in the sum

$OPT(i)$: sum of the maximum sum subarray among elements 0 to i (that might or might not include i)

How can you calculate these values? Try to write recurrence(s), then think about memoization and running time.

Recurrences

$$INCLUDE(i) = \begin{cases} \max\{A[i], A[i] + INLCUDE(i-1)\} & \text{if } i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$OPT(i) = \begin{cases} \max\{INCLUDE(i), OPT(i-1)\} & \text{if } i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

If we include i, the subarray must be either just i or also include $i-1$.

Overall, we might or might not include i. If we don't include i, we only have access to elements $i-1$ and before. If we do, we want $INCLUDE(i)$ by definition.

Example (1)

A

0	1	2	3	4	5	6	7
5	-6	3	4	-5	2	2	4

$OPT(i)$

0	1	2	3	4	5	6	7
5							

$INCLUDE(i)$

0	1	2	3	4	5	6	7
5							

Example (2)

A

0	1	2	3	4	5	6	7
5	-6	3	4	-5	2	2	4

$OPT(i)$

0	1	2	3	4	5	6	7
5	5						

$INCLUDE(i)$

0	1	2	3	4	5	6	7
5	-1						

Example (3)

A

0	1	2	3	4	5	6	7
5	-6	3	4	-5	2	2	4

$OPT(i)$

0	1	2	3	4	5	6	7
5	5	5	7	7	7	7	10

$INCLUDE(i)$

0	1	2	3	4	5	6	7
5	-1	3	7	2	4	6	10

Pseudocode (subarray sum)

```
int maxSubarraySum(int[] A)
  int n=A.length
  int[] OPT = new int[n]
  int[] Inc = new int[n]
  inc[0]=A[0]; OPT[0] = max{A[0],0}
  for(int i=0;i<n;i++)
    inc[i]=max{A[i], A[i]+inc[i-1]}
    OPT[i]=max{inc[i], opt[i-1]}
  endFor
return OPT[n-1]
```

Recursive Thinking In General (top-down)

As before, the hardest part is designing the recurrence.

It sometimes helps to think from multiple different angles.

Top-down: What's the first step to take?

Baby Yoda will first go left or down. Use recursion to find out which of left or down is better.

Should I include this element or not? Use recursion to see whether it should be included (then you must include i-1 if you have anything else) or not (just OPT(i-1)

Recursive Thinking In General (bottom-up)

Bottom-Up: What information could a recursive call give me that would help?

How does a path through most of the map help Baby Yoda?

Well we just need to know the values one left and one down.

The answers to which subarrays would let us solve the problem?

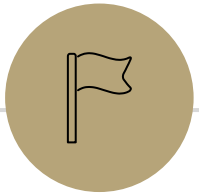
Well, if we know the best option to our left, we could insert one more element or not…but if we insert one more element, we need the best option to our left including the second-to-last element.

Recursive Thinking In General (optimal substructure)

Some people refer to the “Optimal Substructure Property”

From the optimum (most eggs, greatest sum) for a slightly smaller problem (Baby Yoda starting closer to the end, slightly smaller array), we need to be able to build up the optimum for the full problem.

This property is true of INCLUDE, but not (directly) of OPT---that’s why we added INCLUDE.



Hidden parameter: Longest Increasing Subsequence

Longest Increasing Subsequence

0	1	2	3	4	5	6	7
5	-6	3	6	-5	2	8	10

Longest set of (not necessarily consecutive) elements that are increasing

5 is optimal for the array above

(indices $1,2,3,6,7$; elements $-6,3,6,8,10$)

For simplicity – assume all array elements are distinct.

LIS, recursively

0	1	2	3	4	5	6	7
5	-6	3	6	-5	2	8	10

We're going to think recursively. To get the LIS for the whole array, it would help to have the LIS for indices [0..6]

Imagine you're a recursive call: [0..3]. The call that made you might have decided that the 2 at index 5 should be taken…you really want to know that! You can't include the 6 at index 3 if that's there…

We're going to need to do our recursive calls a favor: warn them if their value might be illegal.

Thinking Recursively (1)

Phrasing our question as "do we take element i or not?" What do we need to know to decide on element i?

1. Is it allowed?

Will the sequence still be increasing if it's included?

2. Will it help?

Is it part of the best subsequence?

Still recursing right-to-left: make a recursive call on a smaller array.

What do you need to keep track of?

Thinking Recursively (2)

Phrasing our question as "do we take element i or not?" What do we need to know to decide on element i?

1. Is it allowed?

Will the sequence still be increasing if it's included?

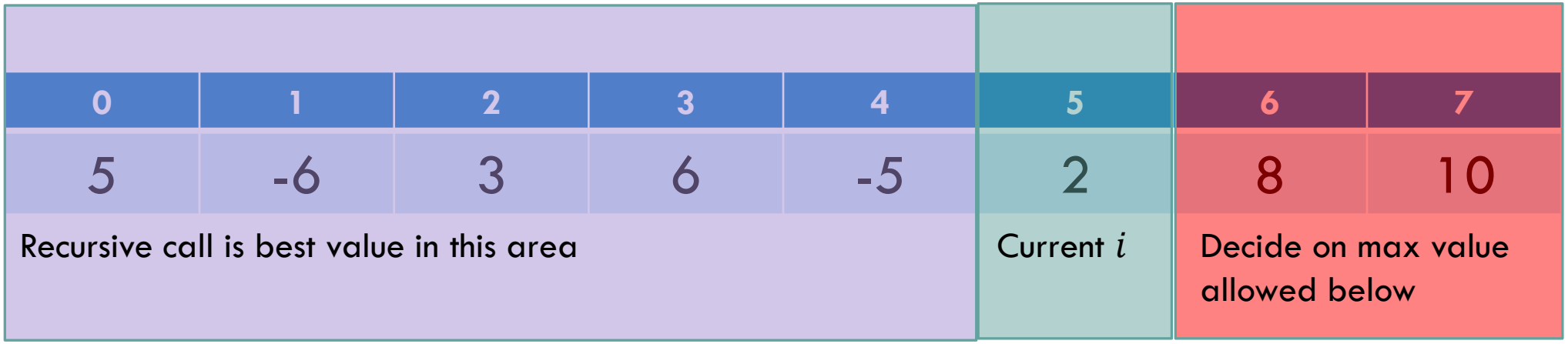
2. Will it help?

Is it part of the best subsequence?

Still recursing right-to-left: make a recursive call on a smaller array.

Two indices: index we're looking at, and index of upper bound on elements (i.e. the value we need to decide if we're still increasing).

LIS, visualized



Need recursive answer to the left

Currently processing i

Recursive calls to the left are needed to know optimum from $1 \ldots i-1$

LIS Recurrence

$LIS(i, j)$ is "Number of elements of the maximum increasing subsequence from $0, \ldots, i$ where every element of the sequence is at most $A[j]$"

Need a recurrence

$$LIS(i,j) = \begin{cases} ? & \text{if } i < 0 \\ ? & \text{if } i = 0 \\ ? & \text{if } A[i] > A[j] \\ ? & \text{otherwise} \end{cases}$$

LIS recurrence, key ideas

$LIS(i, j)$ is "Number of elements of the maximum increasing subsequence from $0, \ldots, i$ where every element of the sequence is at most $A[j]$"

Need a recurrence

$$LIS(i,j) = \begin{cases} ? & \text{if } i < 0 \\ ? & \text{if } i = 0 \\ ? & \text{if } A[i] > A[j] \\ ? & \text{otherwise} \end{cases}$$

If $A[i] > A[j]$ element i cannot be included in an increasing subsequence where every element is at most $A[j]$. So taking the largest among the first $i - 1$ suffices.

If $A[i] \leq A[j]$, then if we include i, we may include elements to the left only if they are less than $A[i]$ (since $A[i]$ will now be the last, and therefore largest, of elements $1 \ldots i$. If we don't include i we want the maximum increasing subsequence among $1 \ldots i - 1$.

LIS Recurrence (finished)

$LIS(i,j)$ is "Number of elements of the maximum increasing subsequence from $0, \ldots, i$ where every element of the sequence is at most $A[j]$"

Need a recurrence

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \leq A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

If $A[i] > A[j]$ element i cannot be included in an increasing subsequence where every element is at most $A[j]$. So taking the largest among the first $i-1$ suffices.

If $A[i] \leq A[j]$, then if we include i, we may include elements to the left only if they are less than $A[i]$ (since $A[i]$ will now be the last, and therefore largest, of elements $0 \ldots i$. If we don't include i we want the maximum increasing subsequence among $0 \ldots i-1$.

Finishing LIS

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \leq A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

Memoization structure? $n \times n$ array.

Filling order?

pseudocode

```
//real code snippet that actually generated the table on the last slide
for(int j=0; j < n; j++){
     vals[0][j] = (A[0] <= A[j]) ? 1 : 0;
}
for(int i = 1; i < 8; i++){
     for(int j = 0; j < n; j++){
          if(A[i] > A[j])
               vals[i][j] = vals[i-1][j];
          else{
               vals[i][j] = Math.max(1+vals[i-1][i], vals[i-1][j]);
          }
     }
}
```

LIS (1)

0	1	2	3
5	−6	3	6

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \le A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

j

i	0, 5	1, −6	2, 3	3, 6	4, −5	5, 2	6, 8	7, 10
0, 5								
1, −6								
2, 3				LIS(2,3)			LIS(2,6)	
3, 6							LIS(3,6)	
4, −5								
5, 2								
6, 8								
7, 10								

The recursive call that made us included $A[6] = 8$ in the sequence.
We now need to decide on $A[3] = 6$.
$A[3] = 6 < 8 = A[6]$. We are allowed to include $A[3]$.
Try that: then need LIS(2,3) [LIS among A[0,…,2] where all less than A[3]
And try LIS(2,6) [don't include] LIS among A[0,..,2] all less than A[6]

LIS (2)

0	1	2	3
5	−6	3	6

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \le A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

j →

	0, 5	1, −6	2, 3	3, 6	4, −5	5, 2	6, 8	7, 10
0, 5	1	0	0	1	0	0	1	1
1, −6	1	1	1	1	1	1	1	1
2, 3	2	1	2	2	1	1	2	2
3, 6	2	1	2	3	1	1	3	3
4, −5	2	1	2	3	2	2	3	3
5, 2	3	1	3	3	2	3	3	3
6, 8	3	1	3	3	2	3	4	4
7, 10	3	1	3	3	2	3	4	5

i ↓

LIS (3)

0	1	2	3
5	−6	3	6

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \le A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

j

	0, 5	1, −6	2, 3	3, 6	4, −5	5, 2	6, 8	7, 10
0, 5								
1, −6								
2, 3						LIS(2,5)		
3, 6						LIS(3,5)		
4, −5								
5, 2								
6, 8								
7, 10								

i

The recursive call that made us included $A[5]$ is the sequence.
We now need to decide on $A[3] = 6$.
$A[3] = 6 > 2 = A[5]$. We **cannot** include $A[3]$.
Need LIS among A[0,…,2] with

LIS (4)

$LIS(1,0)$ $A[1] < A[0]$ not allowed:
Take $LIS(0,0)$

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \leq A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

j →

i ↓	0, 5	1, −6	2, 3	3, 6	4, −5	5, 2	6, 8	7, 10
0, 5	1	0	0	1	0	0	1	1
1, −6	1							
2, 3								
3, 6								
4, −5								
5, 2								
6, 8								
7, 10								

LIS (5)

$LIS(1,1)$ $A[1] \leq A[1]$ can add,
$1 + LIS(0,1)$ or $LIS(0,1)$

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \leq A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

j →

	0, 5	1, −6	2, 3	3, 6	4, −5	5, 2	6, 8	7, 10
0, 5	1	0	0	1	0	0	1	1
1, −6	1	1						
2, 3								
3, 6								
4, −5								
5, 2								
6, 8								
7, 10								

i ↓

LIS (6)

$LIS(1,2)$ $A[1] \leq A[2]$ allowed to add:
$1 + LIS(0,1)$ or $LIS(0,2)$

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \leq A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

j →

i ↓

	0, 5	1, −6	2, 3	3, 6	4, −5	5, 2	6, 8	7, 10
0, 5	1	0	0	1	0	0	1	1
1, −6	1	1	1					
2, 3								
3, 6								
4, −5								
5, 2								
6, 8								
7, 10								

LIS (7)

$LIS(2,0)$ $A[2] \leq A[0]$ allowed to add
$1 + LIS(1,2)$ or $LIS(1,0)$

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \leq A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

j →

i ↓	0, 5	1, −6	2, 3	3, 6	4, −5	5, 2	6, 8	7, 10
0, 5	1	0	0	1	0	0	1	1
1, −6	1	1	1	1	1	1	1	1
2, 3	2							
3, 6								
4, −5								
5, 2								
6, 8								
7, 10								

LIS (8)

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \le A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

j →

	0, 5	1, −6	2, 3	3, 6	4, −5	5, 2	6, 8	7, 10
0, 5	1	0	0	1	0	0	1	1
1, −6	1	1	1	1	1	1	1	1
2, 3	2	1	2	2	1	1	2	2
3, 6	2	1	2	3	1	1	3	3
4, −5	2	1	2	3	2	2	3	3
5, 2	3	1	3	3	2	3	3	3
6, 8	3	1	3	3	2	3	4	4
7, 10	3	1	3	3	2	3	4	5

i ↓

Finishing LIS (iterative order)

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \leq A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

Memoization structure? $n \times n$ array.

Filling order?

Outer loop: increasing i

Inner loop: increasing j

Getting the Final Answer

One more thing….what's the final answer?

We want the longest increasing sequence in the whole array.

$LIS(i, j)$ is "Number of elements of the maximum increasing subsequence from $0, \ldots, i$ where every element of the sequence is at most $A[j]$"

What do we want?

Final Answer (version 1)

The principled approach:

What does j mean? It's an already added element to our right.

There's nothing already on our right at the start. In recursive code, we'd probably call $LIS(n, \texttt{null})$. So do that…just tweak the recurrence

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ 1 & \text{if } i = 0, j = \text{null} \\ \mathbb{I}\left[A[i] \leq A[j]\right] & \text{if } i = 0, j \neq \text{null} \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

LIS, final answer

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \le A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

j →

	0, 5	1, −6	2, 3	3, 6	4, −5	5, 2	6, 8	7, 10	Null
0, 5	1	0	0	1	0	0	1	1	1
1, −6	1	1	1	1	1	1	1	1	1
2, 3	2	1	2	2	1	1	2	2	2
3, 6	2	1	2	3	1	1	3	3	3
4, −5	2	1	2	3	2	2	3	3	3
5, 2	3	1	3	3	2	3	3	3	3
6, 8	3	1	3	3	2	3	4	4	4
7, 10	3	1	3	3	2	3	4	5	5

i ↕

Final Answer (option 2)

The clever hack:

What does j mean? $LIS(i, j)$ is "Number of elements of the maximum increasing subsequence from $0, \ldots, i$ where every element of the sequence is at most $A[j]$"

So, if j is the last element of the true sequence, that's what we want!

Just return $\max_j \mathrm{LIS}(n, j)$

LIS

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \le A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

j →

	0, 5	1, −6	2, 3	3, 6	4, −5	5, 2	6, 8	7, 10
0, 5	1	0	0	1	0	0	1	1
1, −6	1	1	1	1	1	1	1	1
2, 3	2	1	2	2	1	1	2	2
3, 6	2	1	2	3	1	1	3	3
4, −5	2	1	2	3	2	2	3	3
5, 2	3	1	3	3	2	3	3	3
6, 8	3	1	3	3	2	3	4	4
7, 10	3	1	3	3	2	3	4	5

i ↓

Final Answer (option 2, explained)

One more thing….what's the final answer?

We want the longest increasing sequence in the whole array.

$LIS(i, j)$ is "Number of elements of the maximum increasing subsequence from $0, \ldots, i$ where every element of the sequence is at most $A[j]$"

$\max_j LIS(n, j)$. Intuitively, j represents "the last element" in the array. Anything could be the last one! Take the maximum.

Takeaways

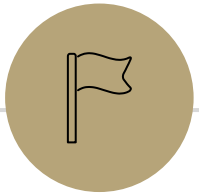
Sometimes you need to add an extra recurrence

Or add an extra parameter.

You need to write a recursive function!

That's all DP is.

But the recursion itself can be tricky.



More Practice: Edit Distance

Edit Distance

The edit distance between two strings is:

The minimum number of **deletions, insertions,** and **substitutions** to transform string x into string y.

Deletion: removing one character

Insertion: inserting one character (at any point in the string)

Substitution: replacing one character with one other.

Example

What's the distance between `babyyodas` and `tastysoda`?

x	B	A	B		Y	Y	O	D	A	S
op	Sub		sub	ins		sub				del
y	T	A	S	T	Y	S	O	D	A	

Distance: 5, one point for each colored box

Quick Checks – can you explain these?
If x has length n and y has length m, the edit distance is at most $\max(x, y)$

The distance from x to y is the same as from y to x (i.e. transforming x to y and y to x are the same)

Finding a recurrence

What information would let us simplify the problem?

What would let us "take one step" toward the solution?

"Handling" one character of x or y

i.e. choosing one of insert, delete, or substitution and increasing the "distance" by 1

OR realizing the characters are the same and matching for free.

$OPT(i, j)$ is the minimum number of insertions, deletions, and substitutions to transform $x_1 x_2 \cdots x_i$ into $y_1 y_2 \cdots y_j$. (we're indexing strings from 1, it'll make things a little prettier).

The recurrence

"Handling" one character of x or y

i.e. choosing one of insert, delete, or substitution and increasing the "distance" by 1

OR realizing the characters are the same and matching for free.

What does delete look like? $OPT(i-1, j)$ (delete character from x match the rest)

Insert $OPT(i, j-1)$ Substitution: $OPT(i-1, j-1)$

Matching characters? Also $OPT(i-1, j-1)$ but only if $x_i = y_j$

The recurrence (starting)

"Handling" one character of x or y

i.e. choosing one of insert, delete, or substitution and increasing the "distance" by 1

OR realizing the characters are the same and matching for free.

$$OPT(i,j) = \begin{cases} \min\{\ 1 + OPT(i-1,j), 1 + OPT(i,j-1),\ 1 + OPT(i-1,j-1), \} \\ j & \text{if } i = 0 \\ i & \text{if } j = 0 \end{cases}$$

Delete

Insert

Substitution

TODO: Just Match

The recurrence (v1, we'll improve soon)

"Handling" one character of x or y

i.e. choosing one of insert, delete, or substitution and increasing the "distance" by 1

OR realizing the characters are the same and matching for free.

$$OPT(i,j) = \begin{cases} \min\{\underbrace{1 + OPT(i-1,j)}_{\text{Delete}}, \underbrace{1 + OPT(i,j-1)}_{\text{Insert}}, \underbrace{1 + OPT(i-1,j-1)}_{\text{Substitution}}, \underbrace{OPT(i-1,j-1) + \infty \cdot \mathbb{I}\{x_i \neq y_j\}}_{\text{Just Match}}\} \\ j \quad \text{if } i = 0 \\ i \quad \text{if } j = 0 \end{cases}$$

Idea: only allow "just match" when you can just match.
Otherwise make it ∞ (will never be the min).
In code: if/else branch, probably. This is a math notation trick.

"Indicator" like from 312

The recurrence (finalized)

"Handling" one character of x or y

i.e. choosing one of insert, delete, or substitution and increasing the "distance" by 1

OR realizing the characters are the same and matching for free.

Delete Insert Sub and matching

$$OPT(i,j) = \begin{cases} \min\{\, 1 + OPT(i-1,j), 1 + OPT(i,j-1),\ \mathbb{I}[x_i \neq y_j] + OPT(i-1,j-1)\} \\ j & \text{if } i = 0 \\ i & \text{if } j = 0 \end{cases}$$

When we could match, we will never substitute; matching will always give us a better score! Still have to check delete, insert (those could be better).

LIS (fill in yourself)

j →

	0, 5	1, −6	2, 3	3, 6	4, −5	5, 2	6, 8	7, 10
0, 5	1	0	0	1	0	0	1	1
1, −6	1	1	1	1	1	1	1	1
2, 3	2							
3, 6								
4, −5								
5, 2								
6, 8								
7, 10								

i ↓