Wrap D&C
Dynamic Programming

CSE 421 26Sp
Lecture 11

Takeaways from Monday

Wow! That was an unexpected algorithm.

- You can implement quicksort with guaranteed $O(n \log n)$ running time!
  - Use QuickSelect to find the pivot.
  - Don't actually do this though. Median-of-3 or a uniformly random pivot are better in practice.
- Generalizing a problem can make it easier to solve
  - Instead of just the median, finding a general index is recursive.

Divide And Conquer Summary

Takeaways from D&C:

Recursive thinking can let us solve problems faster!

When solving a recursive problem, state precisely what the recursive call is giving back to you.

Use the values of the recursive call (or at least the fact you've made recursive calls) when designing the combine step

If your "combine" step isn't faster than baseline, your whole algorithm isn't better than the baseline!

How Were We Solving All Those Recurrences?

The techniques from CSE 332 (unrolling and/or recursion trees) still work!

But now that you've done them a bunch in 332, we'll give you a shortcut…

Master Theorem

Given a recurrence of the following form, where a, b, c, and d are constants:

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

Where $f(n)$ is $\Theta\left(n^c \log^k n\right)$ for $k \geq 0$, $a \in \mathbb{Z}^+$, $c \geq 1$

If $\log_b a < c$ then $T(n) \in \Theta\left(n^c \cdot \log^k n\right)$
If $\log_b a = c$ then $T(n) \in \Theta\left(n^c \log^{k+1} n\right)$
If $\log_b a > c$ then $T(n) \in \Theta\left(n^{\log_b a}\right)$

Theorem still holds even if there are ceilings/floors in the $T\left(\frac{n}{b}\right)$ term.

Proof?

The easiest way to understand is actually to make a bunch of recursion trees!

a is the "branching factor" you end up with a^i nodes at level i.

b is the amount you cut down the problem size by.

So at level i you do: $a^i \cdot f\left(\frac{n}{b^i}\right)$ work.

For example, when $c = \log_b(a)$, that simplifies to $\frac{a^i n^c}{b^{\log_b(a) \cdot i}} = \frac{a^i n^c}{a^i} = n^c$

And there will be $\log_b n$ levels, so the work is: $n^c \log n$.

Proof? (2)

When $\log_b a < c$

The total work decreases at each level, so the first level dominates, where you do $O(n^c)$ work.

When $\log_b a > c$

The total work increases at each level, so the last level (which is at $\log_b a$) dominates, where you have $O(a^{\log_b n})$ nodes, each doing $O(1)$ work. Some log tricks will rearrange to the more familiar $O(n^{\log_b a})$.

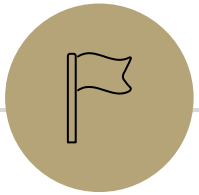
A stronger version

Also what to do when format doesn't quite fit (e.g. k negative).
The version on the last slide should suffice.

From Wikipedia

$$c_{\text{crit}} = \log_b a = \log(\#\text{subproblems}) / \log(\text{relative subproblem size})$$

Case	Description	Condition on $f(n)$ in relation to c_{crit}, i.e. $\log_b a$	Master Theorem bound
1	Work to split/recombine a problem is dwarfed by subproblems. i.e. the recursion tree is leaf-heavy	When $f(n) = O(n^c)$ where $c < c_{\text{crit}}$ (upper-bounded by a lesser exponent polynomial)	... then $T(n) = \Theta\left(n^{c_{\text{crit}}}\right)$ (The splitting term does not appear; the recursive tree structure dominates.)
2	Work to split/recombine a problem is comparable to subproblems.	When $f(n) = \Theta(n^{c_{\text{crit}}} \log^k n)$ for a $k \geq 0$ (rangebound by the critical-exponent polynomial, times zero or more optional logs)	... then $T(n) = \Theta\left(n^{c_{\text{crit}}} \log^{k+1} n\right)$ (The bound is the splitting term, where the log is augmented by a single power.)
3	Work to split/recombine a problem dominates subproblems. i.e. the recursion tree is root-heavy.	When $f(n) = \Omega(n^c)$ where $c > c_{\text{crit}}$ (lower-bounded by a greater-exponent polynomial)	... this doesn't necessarily yield anything. Furthermore, if $af\left(\frac{n}{b}\right) \leq kf(n)$ for some constant $k < 1$ and sufficiently large n (often called the *regularity condition*) then the total is dominated by the splitting term $f(n)$: $T(n) = \Theta(f(n))$



Dynamic Programming

Dynamic Programming (idea)

The most **robust** algorithm design paradigm we'll study this quarter.

Small changes in the problem usually lead to small changes in the algorithm.

Classic DP

This problem is going to **look** silly (and it is)

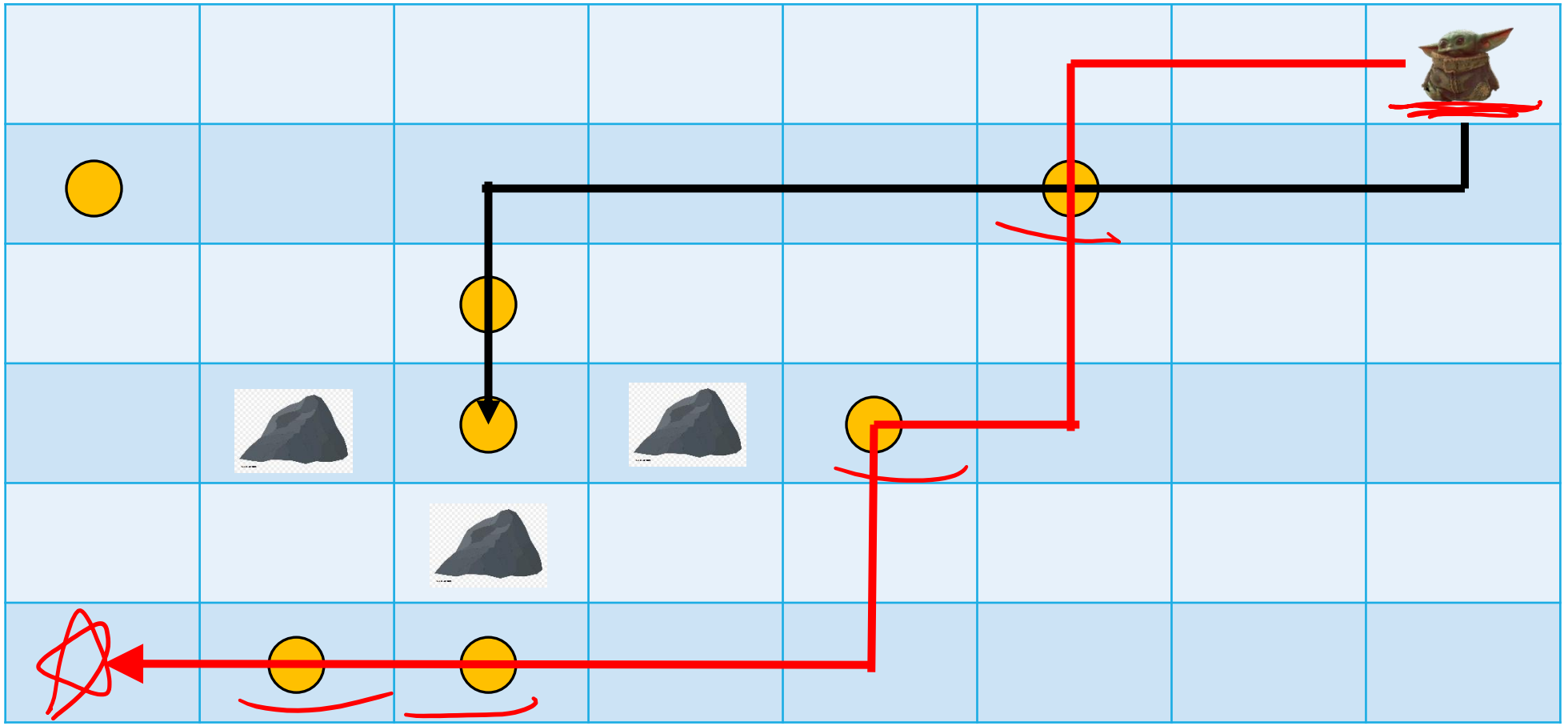
But it is going to make it much easier to do the hard DP problems next week.

Baby Yoda Searching

Baby Yoda has to get from the upper-right corner to the lower left.
His cradle will only let him go left and down.
He can't get past the rocks (too high) – he has to go around them (but still only going left and down)

As usual…he's hungry.
He wants to eat as many frog eggs as possible on the way.

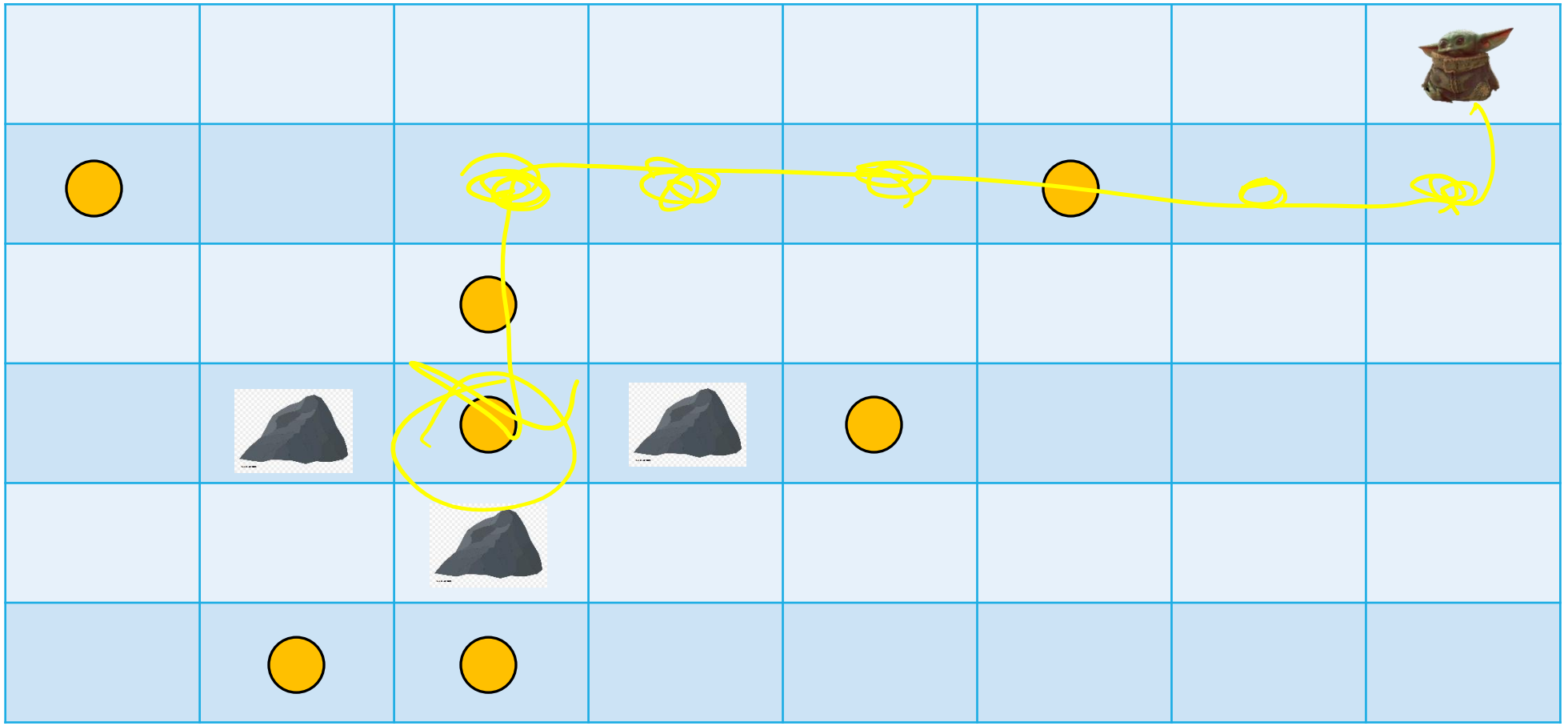
Baby Yoda Searching (example)



Black path: get stuck. Invalid.

Red path: valid! And optimal (no path collects more than 4 eggs.)

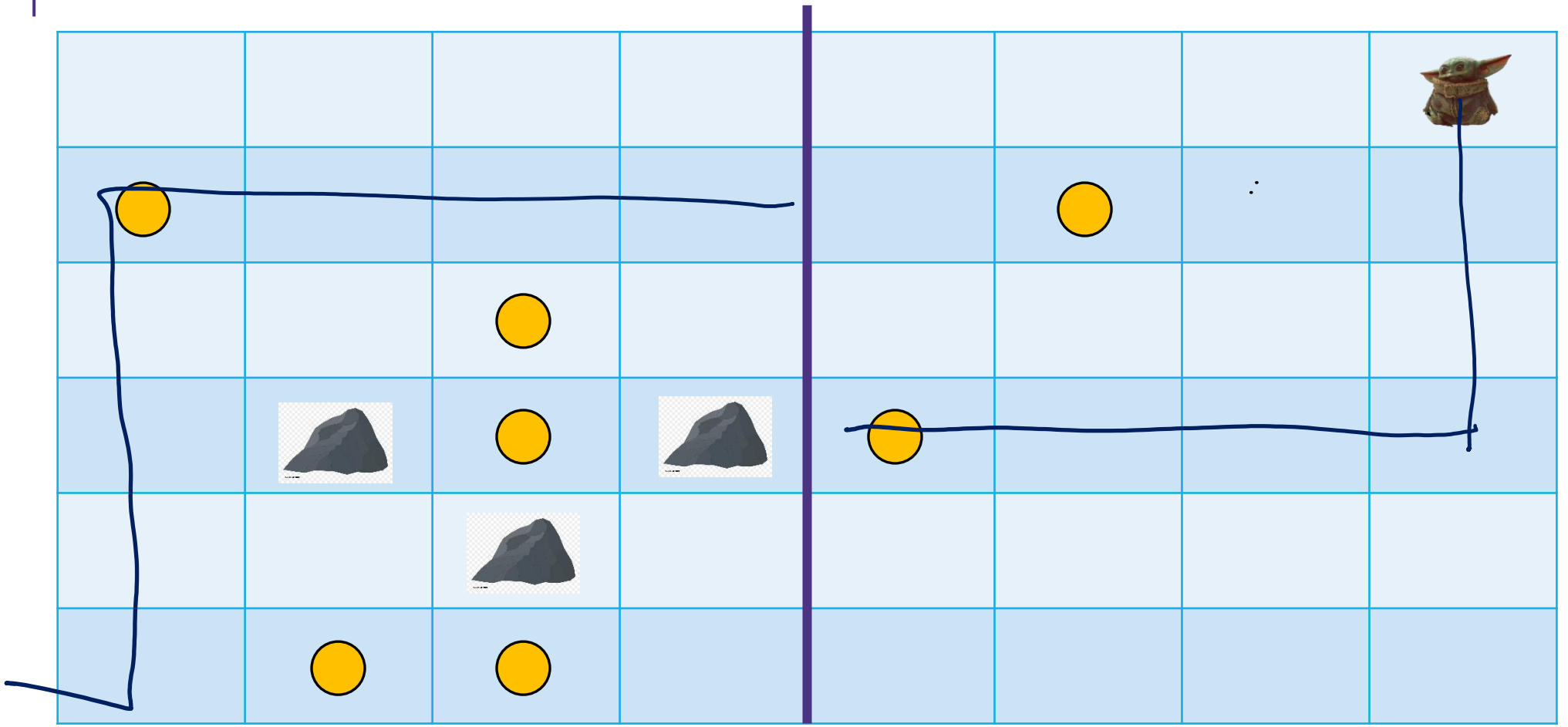
Baby Yoda Searching (greedy?)



Might get us stuck between rocks.
Or pass up a series of eggs we can't see.

Can we greedily head to the next accessible egg?

Baby Yoda Searching (D&C?)

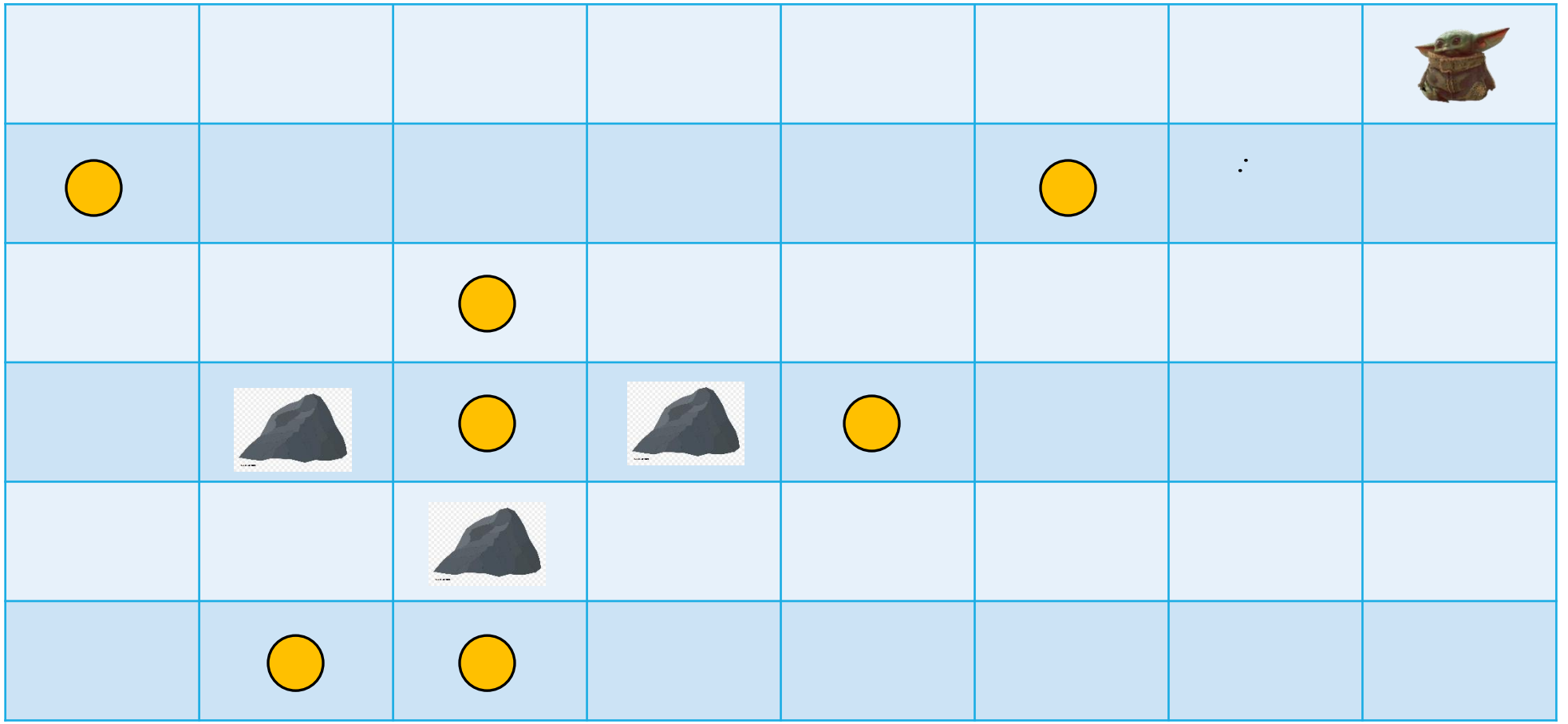


Best left-side path might start at a place inaccessible to end of best right-side path.

Could make a subproblem for each start and ending spot?

Can we divide and conquer?

Baby Yoda Searching (new idea?)



Baby Yoda Searching (new idea)

So, what should we do?

Let's try to use recursion.

What should our recursive calls be finding?
What recursive calls do we need?

Define the problem

Let `OPT(i,j)` be the maximum number of eggs we can get on a legal path from `(i,j)` to `(0,0)` (including the egg in `(i,j)` if there is one)

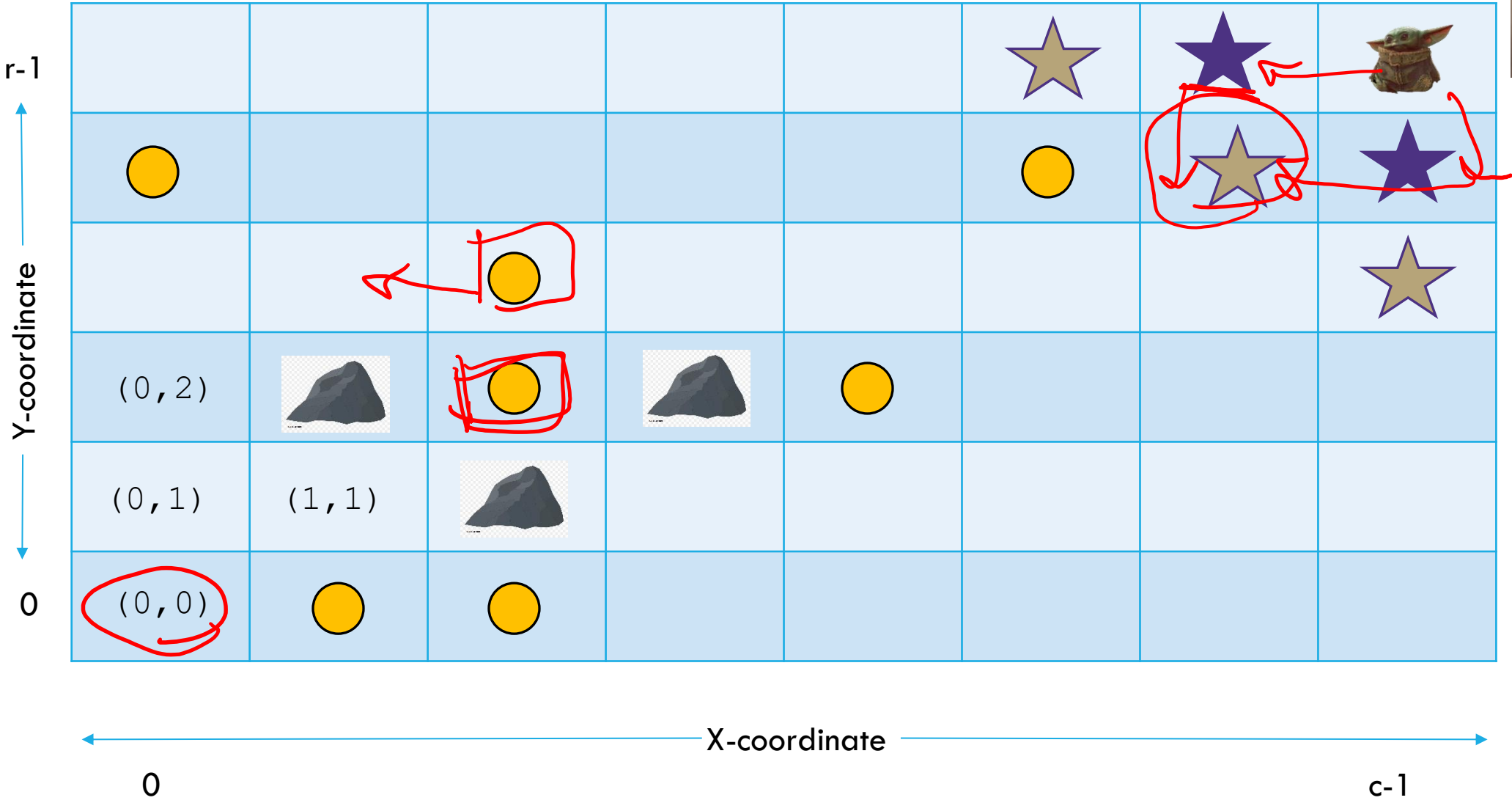
What recursive calls do we need?

Don't try to divide & conquer, think closer to home…

We have to decide whether to go down or left…

"How could we take one step toward the solution?"

Baby Yoda Searching (calls) 5



Recursive Baby Yoda

Let `OPT(i,j)` be the maximum number of eggs we can get on a legal path from `(i,j)` to `(0,0)` (including the egg in `(i,j)` if there is one)

Base Case?

- At `(0,0)`, nowhere to go, return `eggs[0][0]`

Recursive case?

- Find best path to left `OPT(i-1, j)`, and down `OPT(i,j-1)`
  Take `max` of those, add in `eggs[i][j]`
  Need some error handling (can't go off the edge)
  And if we're on rocks, we can't get to the end `(return -∞)`

A Recursive Function

```
FindOPT(int i,int j, bool[][] rocks, bool[][] eggs)
    if(i<0 || j < 0) return -∞
    if(rocks[i][j]) return -∞
    if(i==0 && j==0) return eggs[0][0]
    int left = FindOPT(i-1,j,rocks,eggs)
    int down = FindOPT(i,j-1,rocks,eggs)
    return Max(left,down) + eggs[i][j]
```

Recurrence Form

$$OPT(i,j) = \begin{cases} -\infty & \text{if } rocks(i,j) \text{ is true} \\ -\infty & \text{if } i < 0 \text{ or } j < 0 \\ eggs(0,0) & \text{if } i = 0 \text{ and } j = 0 \\ \max\{OPT(i-1,j), OPT(i,j-1)\} + eggs(i,j) & \text{otherwise} \end{cases}$$

Recurrences can also be used for outputs of a recursive function (not just their running times!)

This definition is a little more compact than code.

And you could write a recursive function for a recurrence like this.

Analyzing the recursive function

So…how does the code work? What's its running time?

$$T(c,r) = \begin{cases} T(c-1,r) + T(c,r-1) + \Theta(1) & \text{if } r \geq 0 \text{ and } c \geq 0 \\ \Theta(1) & \text{otherwise} \end{cases}$$

Master Theorem says…

Analyzing the recursive function (ans)

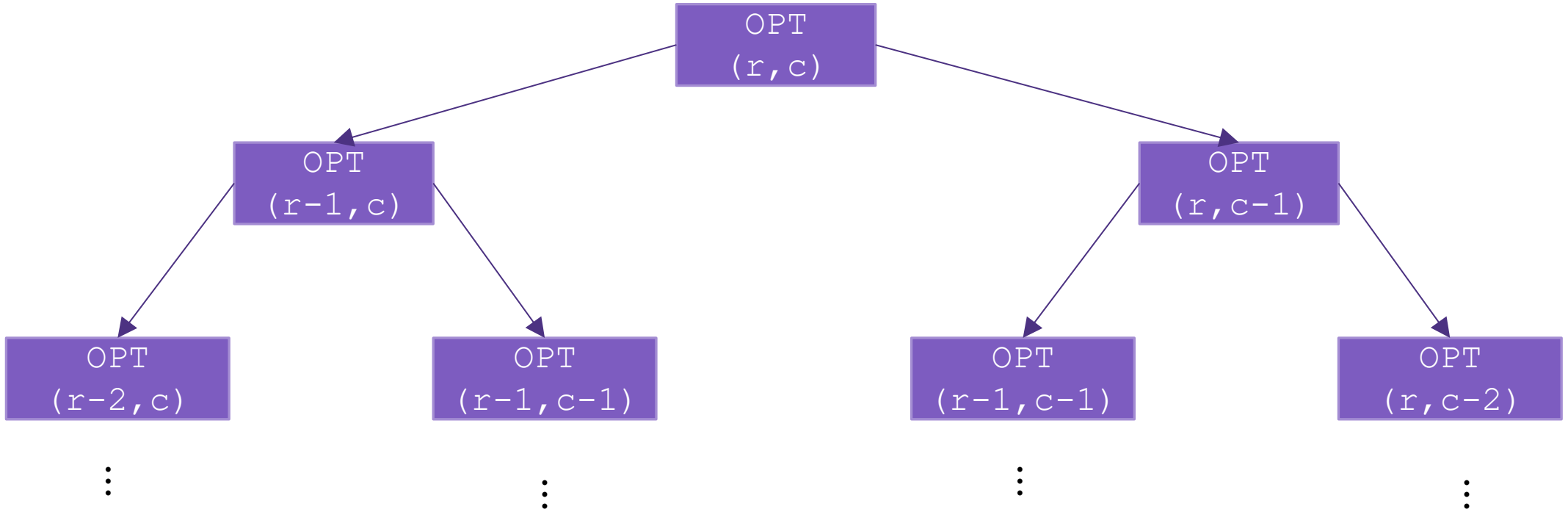
So…how does the code work? What's its running time?

$$T(c,r) = \begin{cases} T(c-1,r) + T(c,r-1) + \Theta(1) & \text{if } r \geq 0 \text{ and } c \geq 0 \\ \Theta(1) & \text{otherwise} \end{cases}$$

Master Theorem doesn't help.

Not even the fancy version on Wikipedia. It's only for "divide and conquer style" recurrences (**dividing** the problem)

Tree Method, Maybe…



When do we hit the base case?
Sometime between $\min(r, c)$ and $r + c$ levels.

Tree Method

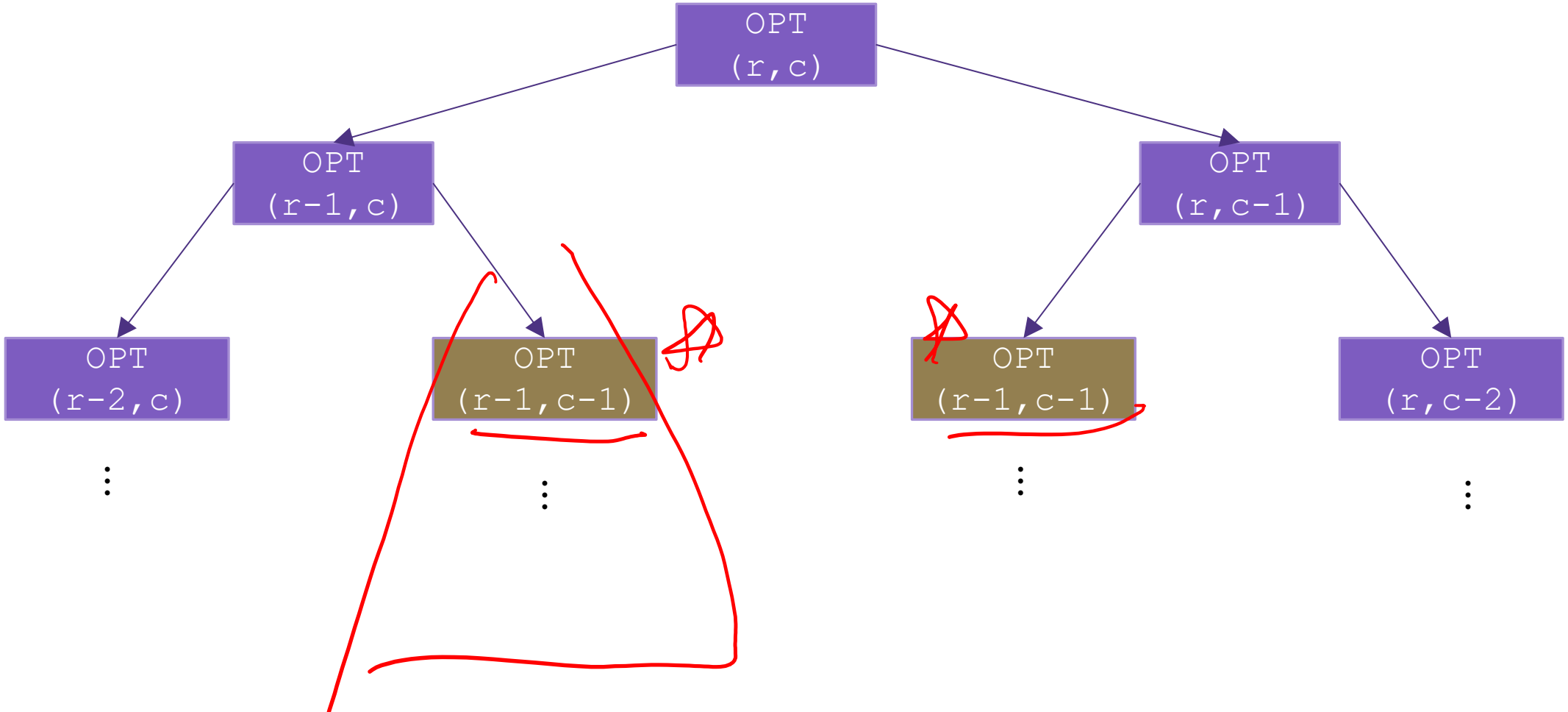
Analysis	minimum	maximum
Nodes at level i	2^i	
Work/node	$\Theta(1)$	
Work at level i	$\Theta(2^i)$	
Base Case level	At least $\min(r, c)$	At most $r + c$
Work at base case	$\Omega(2^{\min(r,c)})$	$O(2^{r+c})$
Total work	$\Omega(2^{\min(r,c)})$	$O(2^{r+c})$

Overall work is sum over all levels – each level has twice the work as the last, so the last level is about half the total work.

Tight big-O depends on relationship between r and c…but regardless – it's slow.

Speedup

That's way too slow…but it doesn't have to be.

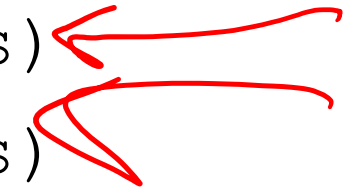


Activity

Fill out the question at pollev.com/robbie

Figure out how to take advantage of the repeated calculation.
What do you think the running time will be of your new algorithm?

```
FindOPT(int i,int j, bool[][] rocks, bool[][] eggs)
    if(i<0 || j < 0) return -∞
    if(rocks[i][j]) return -∞
    if(i==0 && j==0) return eggs[0][0]
    int left = FindOPT(i-1,j,rocks,eggs)
    int down = FindOPT(i,j-1,rocks,eggs)
    return Max(left,down) + eggs[i][j]
```



Speedup Strategy

Not a typo! We're leaving ourselves a memo.

How do we go faster? Don't recalculate! **memoize**

Once you know $OPT(i, j)$ put it in an array `OPT[i][j]`

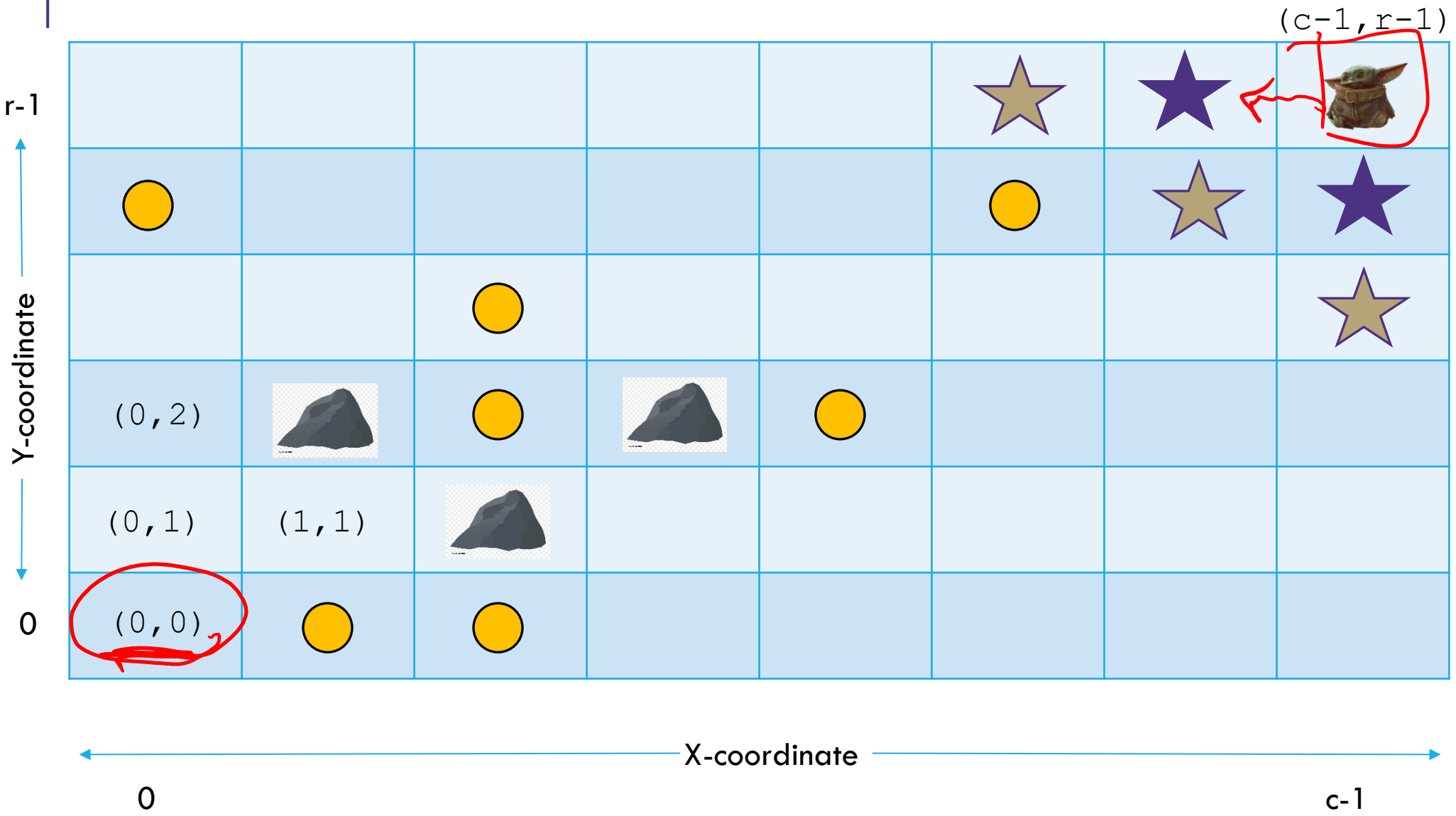
Have some initial value (null?) to mark as uninitialized

If initialized, return that.

Otherwise do the algorithm from the last slide.

How fast? Now $\Theta(rc)$.

Baby Yoda Searching (recursive call location)



Going Bottom-up

So how does that recursion work?

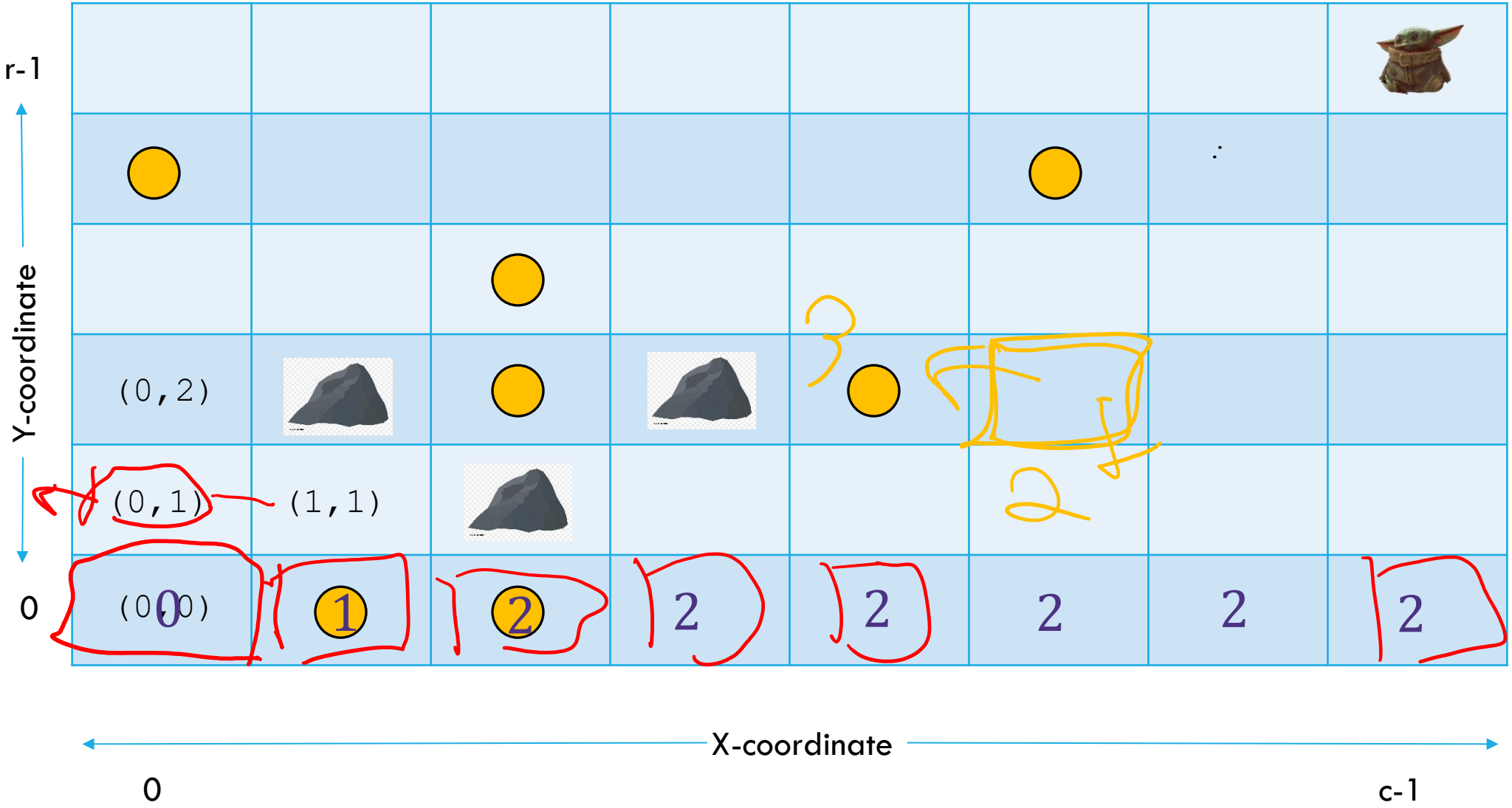
What's the first entry of the table that we fill?

```
OPT[0][0]
```

Why not just start filling in there?

Baby Yoda Searching (row 1 filled)

What else can we fill in?



Baby Yoda Searching (row 2 filled)

What else can we fill in?

(c-1,r-1)

r-1

Y-coordinate

0

(0,2)							
(0,1)	(1,1)		2	2	2	2	2
0	1	2	2	2	2	2	2

X-coordinate

0

c-1

Baby Yoda Searching (start row 3)

(c-1,r-1)

r-1

Y-coordinate

0

							
						∴	
							
0	−∞						
0	1	−∞	2	2	2	2	2
0	1	2	2	2	2	2	2

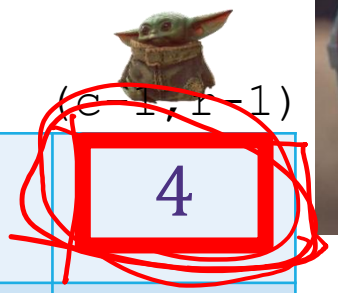
X-coordinate

0

c-1

What else can we fill in?

Baby Yoda Searching (filled-in!)



1	1	1	1	3	4	4	4
1	1	1	1	3	4	4	4
0	0	1	1	3	3	3	3
0	−∞	−∞	−∞	3	3	3	3
0	1	−∞	2	2	2	2	2
0	1	2	2	2	2	2	2

r-1

Y-coordinate

0

X-coordinate

0

c-1

Where's the final answer?

In the top right. Where Baby Yoda starts.

What order?

- Fill in a row at a time (left to right)
- Going up to the next row once a level is done.

In actual code, probably easier to handle edges first

Avoid the index-out-of-bound exceptions.

Pseudocode

```
int eggsSoFar=0;
Boolean rocksInWay=false
for(int x=0; x<c; x++)
   if(rocks[x][0]) rocksInWay = true
   eggsSoFar+=eggs[x][0]
   OPT[x][0]= rocksInWay ? −∞ : eggsSoFar

eggsSoFar=0
rocksInWay=false
for(int y=0; y<r; y++)
   if(rocks[0][y]) rocksInWay = true
   eggsSoFar+=eggs[0][y]
   OPT[0][y]= rocksInWay ? −∞ : eggsSoFar

for(int y=0;y<r;y++)
  for(int x=0;x<c;x++)
    if(rocks[x][y])
        OPT[x][y]=-∞
    else
        OPT[x][y]=max(OPT[x-1][y], OPT[x][y-1])+eggs[x][y]
```