Divide & Conquer

CSE 421 Spring 2026
Lecture 9

Up Next

Divide and Conquer. Using recursion to solve problems efficiently.

Sets us up for Dynamic Programming (a 2+ week adventure in using recursive thinking to solve problems efficiently).

Classic, beautiful algorithms.

Goal: see how far recursive thinking can take you.

Today:

What is D&C?

A classic D&C example

Divide & Conquer Paradigm

Algorithm Design Paradigm

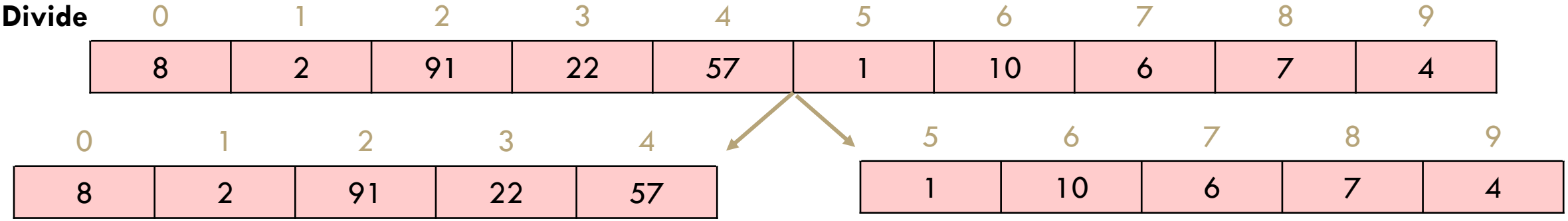
1. Divide instance into subparts.
2. Solve the parts recursively.
3. Conquer by combining the answers

You've seen Divide & Conquer before!

Merge Sort

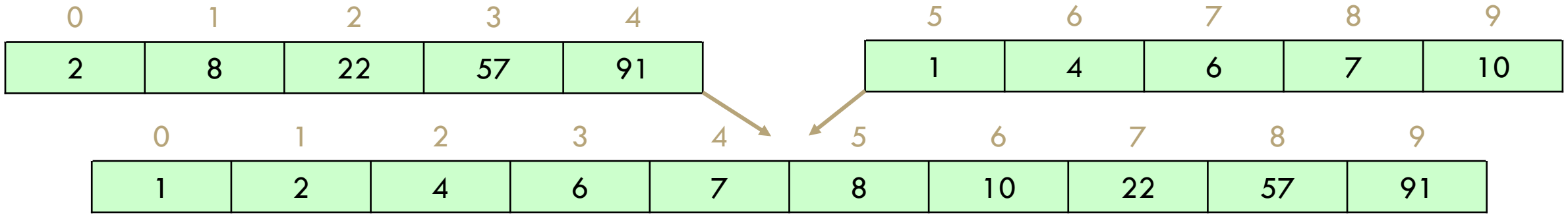
https://www.youtube.com/watch?v=XaqR3G_NVoo

Divide



Sort the pieces through the magic of recursion

Combine



Merge Sort (analysis)

```
mergeSort(input) {
    if (input.length == 1)
        return
    else
        smallerHalf = mergeSort(new [0, ..., mid])
        largerHalf = mergeSort(new [mid + 1, ...])
        return merge(smallerHalf, largerHalf)
}
```

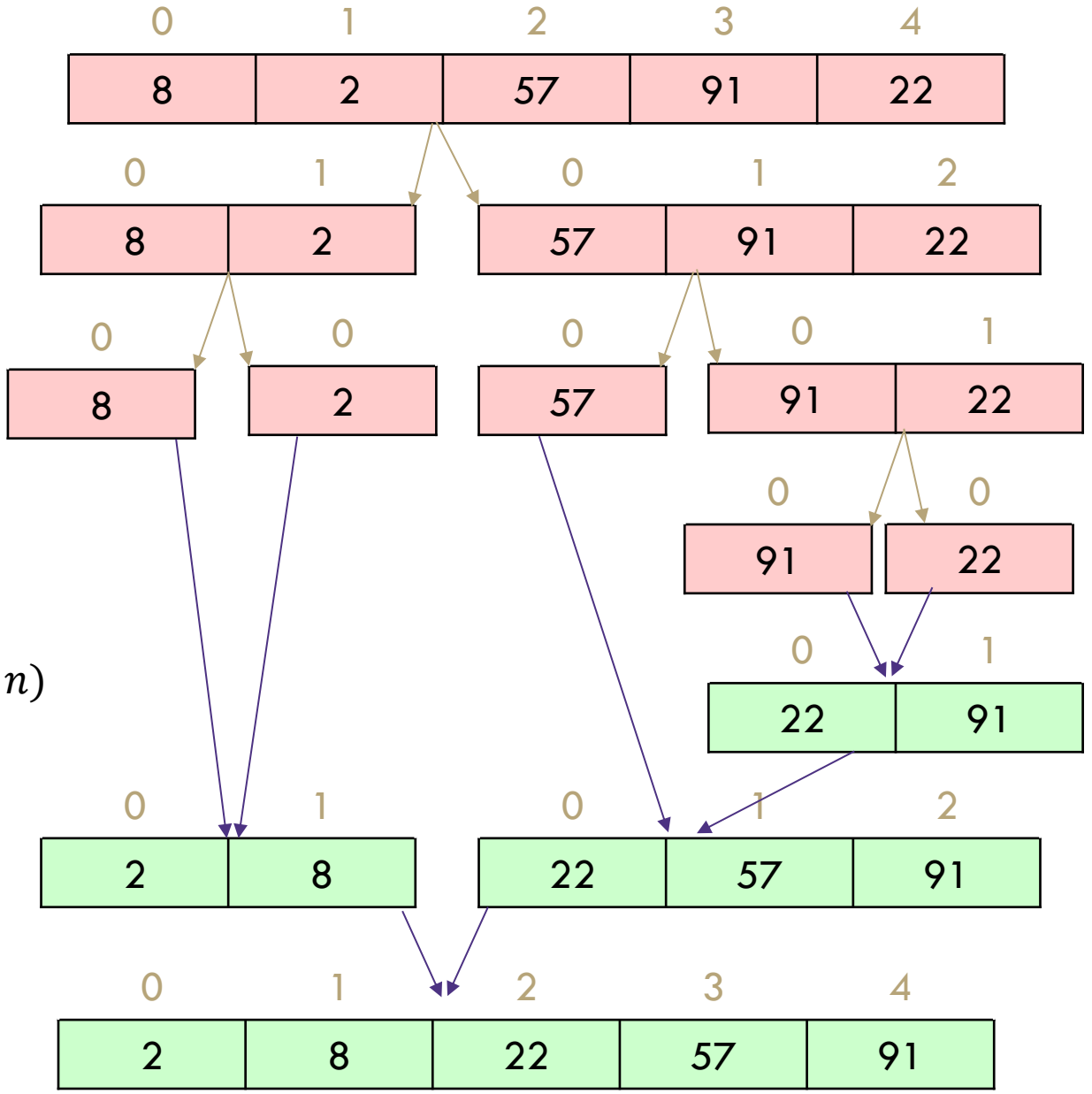
Worst case runtime? $T(n) = \begin{cases} 1 \text{ if } n \leq 1 \\ 2T\left(\frac{n}{2}\right) + n \text{ otherwise} \end{cases} = \Theta(n \log n)$

Best case runtime? Same; $\Theta(n \log n)$

Average runtime? Same; $\Theta(n \log n)$

Stable? Yes!

In-place? No.

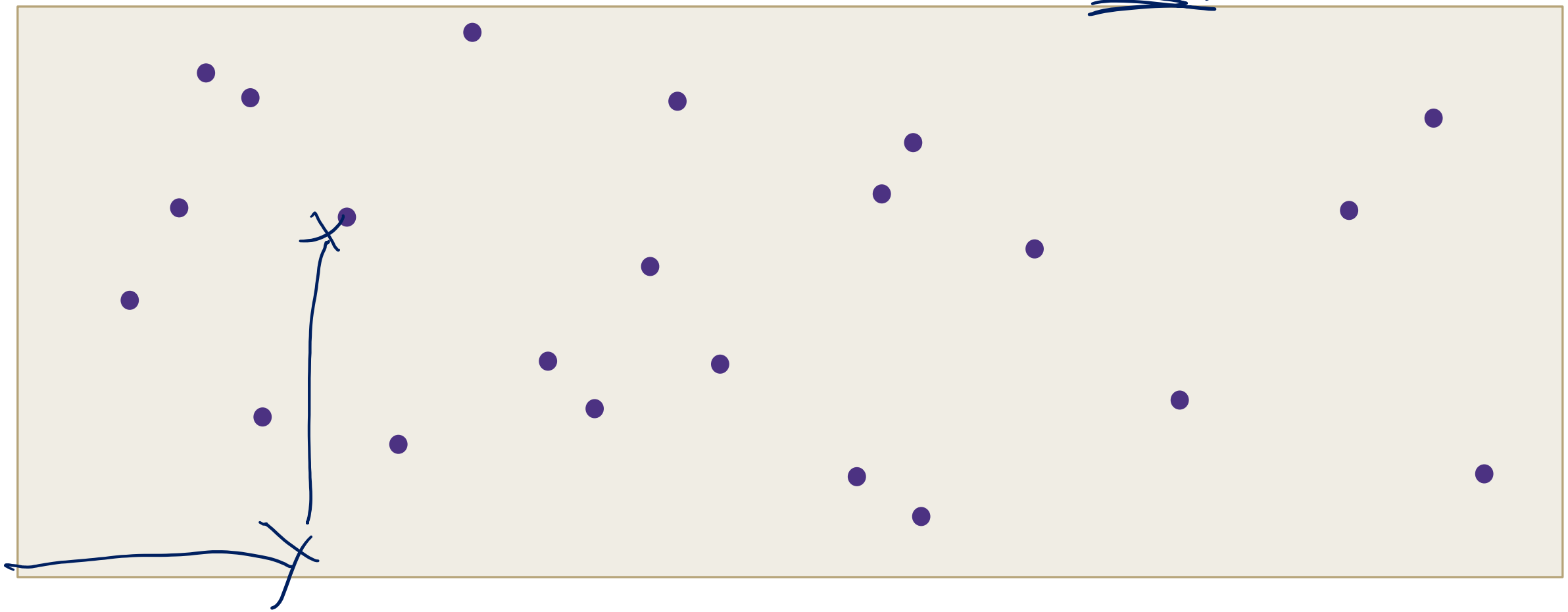


Our Standard Problem-Solving Process

1. Read and understand the problem
2. Generate examples
3. **Come up with a baseline** *<particularly important for D&C*
4. Brainstorm and analyze possible algorithms
5. Write the algorithm
6. Show the algorithm is correct
7. Optimize and analyze the running time

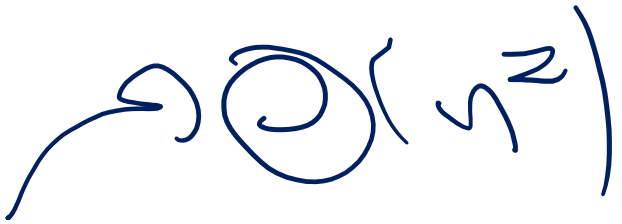
Our First Problem

Given: A list of points in 2-dimensions
Return: The distance between the two points that are closest to each other.



Our First Problem – Baseline

Given: A list of points in 2-dimensions
Return: The distance between the two points that are closest to each other.



What's the first algorithm you can think of? What's its running time?

Our First Problem – Baseline (ans)

Given: A list of points in 2-dimensions
Return: The distance between the two points that are closest to each other.

What's the first algorithm you can think of? What's its running time?

Just calculate the distance between all pairs.

$\Theta(n^2)$.

Keep in the back of your minds: if we aren't doing better than $\Theta(n^2)$ we haven't made any progress.

Let’s make the problem easier

A common problem-solving technique:

Make the problem easier somehow (add some extra assumptions, make it smaller, etc.).

Solve that version

See if you can extend the idea to your original problem

An Easier Version

Given: A list of points in 1-dimension
Return: The distance between the two points that are closest to each other.

Your input will be a list of `doubles` (in no particular order).

What's your algorithm?

An Easier Version (2)

Given: A list of points in 1-dimension
Return: The distance between the two points that are closest to each other.

Your input will be a list of `doubles` (in no particular order).

What's your algorithm?

Sort the list! Find the minimum `dist(A[i], A[i+1])`

Correctness:

Running time:

An Easier Version (3)

Given: A list of points in 1-dimension
Return: The distance between the two points that are closest to each other.

Your input will be a list of `doubles` (in no particular order).

What's your algorithm?

Sort the list! Find the minimum `dist(A[i], A[i+1])`

Correctness: After sorting, closest elements must be consecutive.

Running time: $O(n \log n) + O(n) = O(n \log n)$.

Let’s make the problem harder again

A common problem-solving technique:

Make the problem easier somehow (add some extra assumptions, make it smaller, etc.).

Solve that version

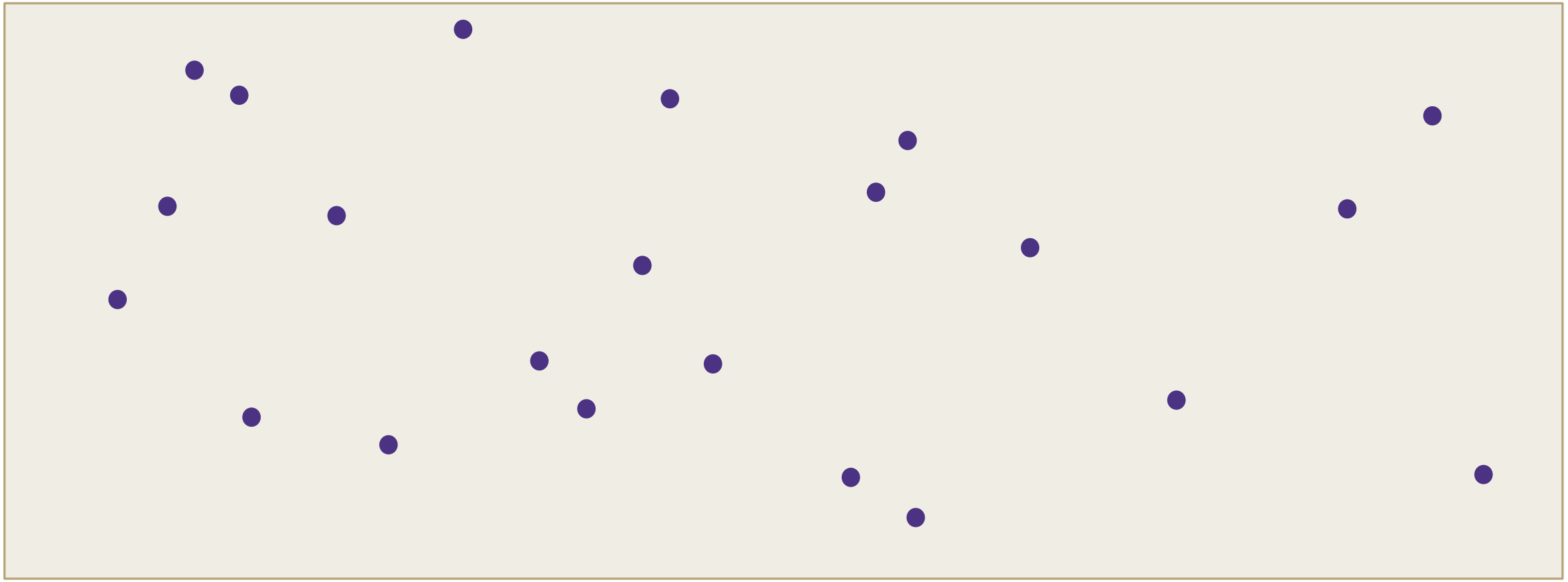
See if you can extend the idea to your original problem

Keep that idea in the back of your mind…it’ll come back (but not for a while).

2D Closest Points

Back to the main problem.
Baseline is $\Theta(n^2)$
1D version is $\Theta(n \log n)$

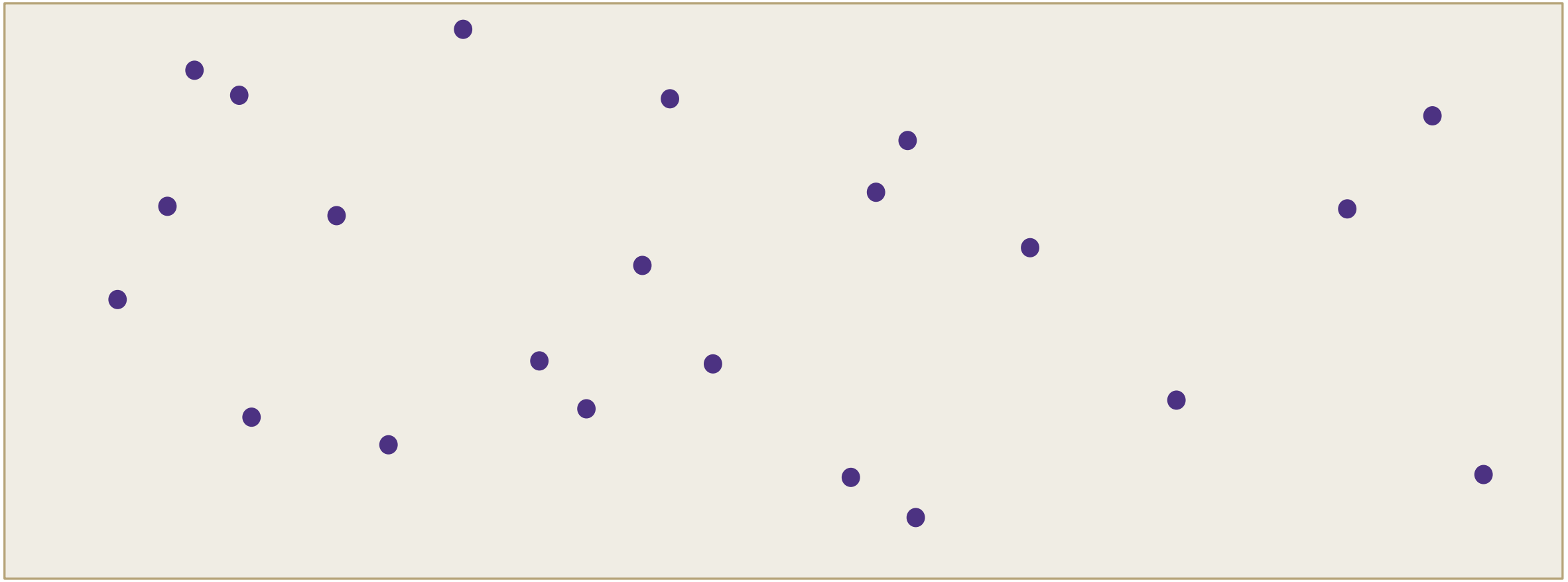
Given: A list of points in 2-dimensions
Return: The distance between the two points that are closest to each other.



2D Closest Points (d&c)

Back to the main problem.
Baseline is $\Theta(n^2)$
1D version is $\Theta(n \log n)$

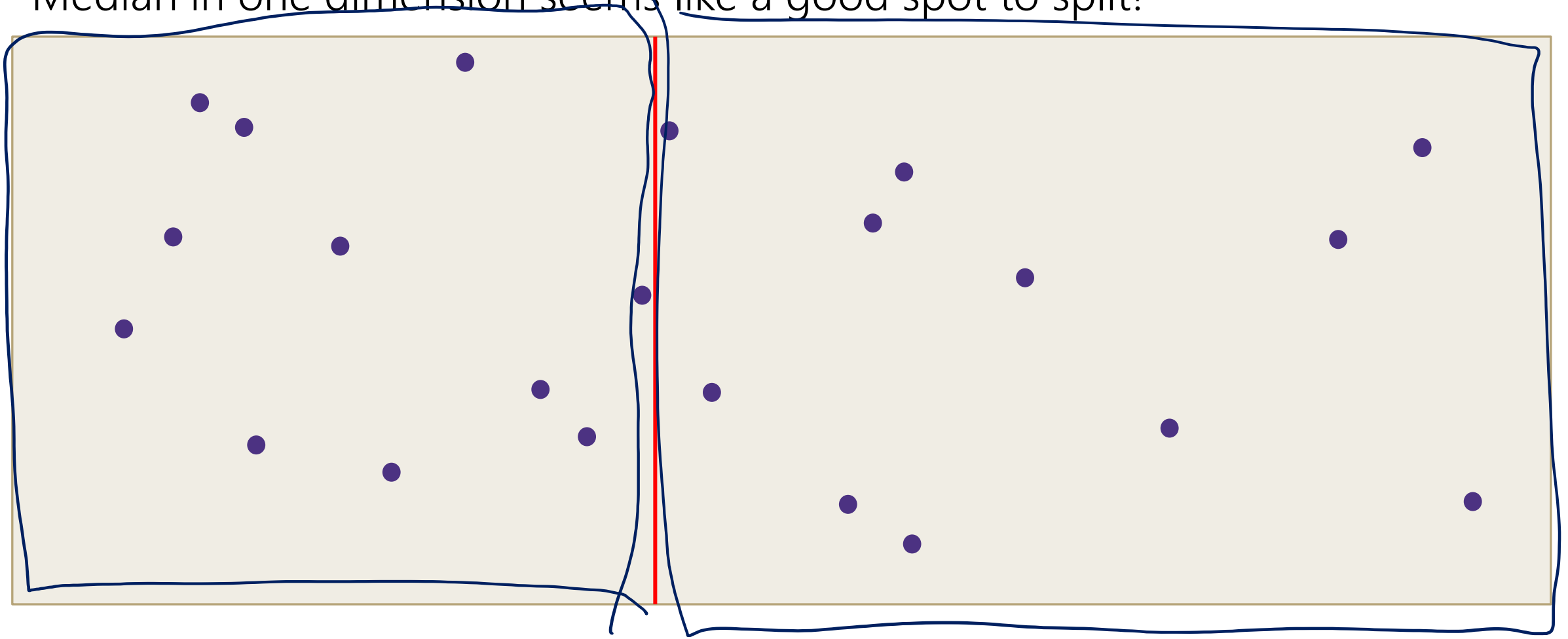
Need to Divide and Conquer. How do we divide?



2D Closest Points (median)

Back to the main problem.
Baseline is $\Theta(n^2)$
1D version is $\Theta(n \log n)$

Need to Divide and Conquer. How do we divide?
Median in one dimension seems like a good spot to split!

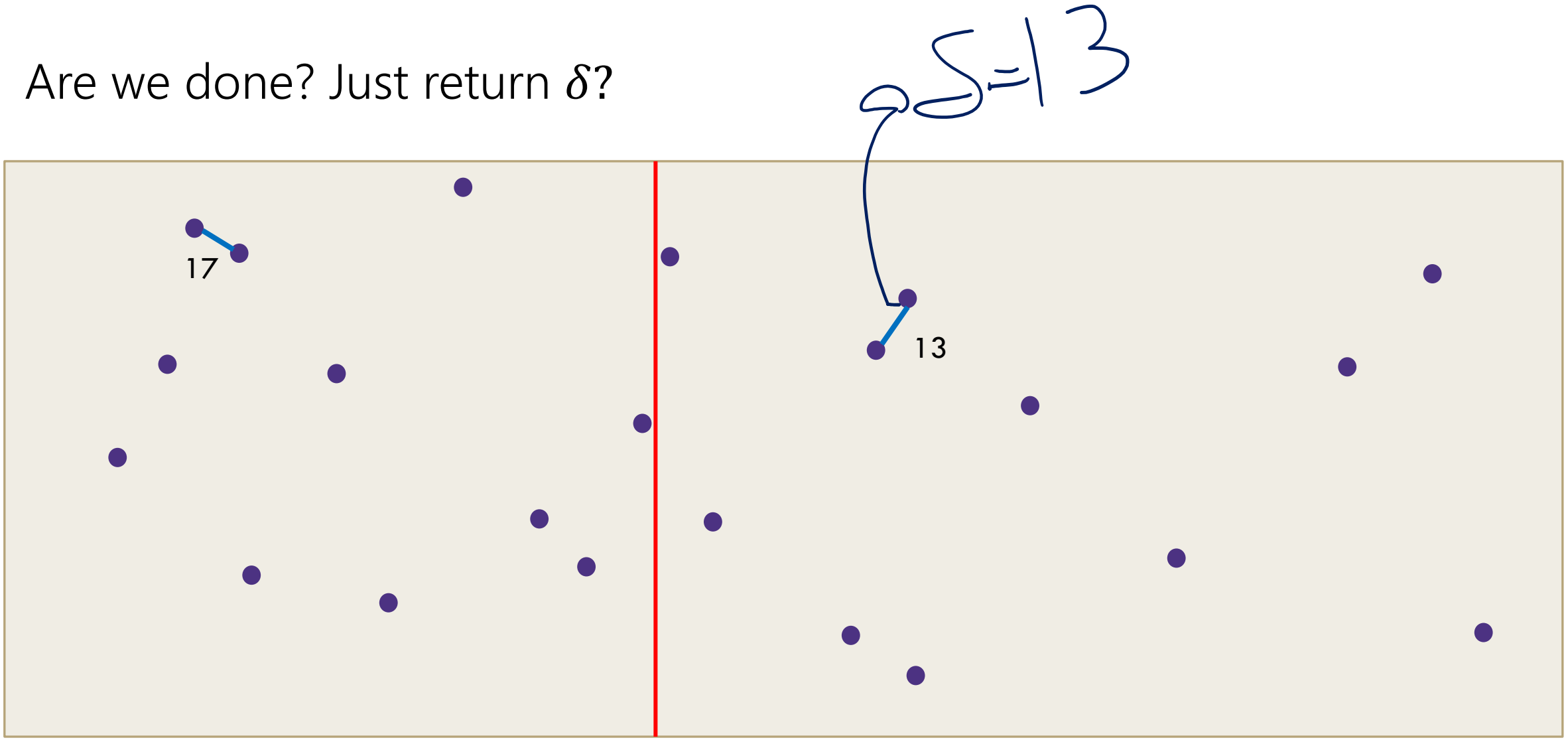


Pseudocode

```
double 2DClosestPoints(P[1..n])
        if(n ≤ 100)  //pick a cutoff you like; doesn't matter for big-O
                check all possible pairs, return the smallest distance.
        Sort P[] by x-coordinate
        δ ←min{2DClosestPoints(P[1..n/2]), 2DClosestPoints(P[n/2+1,n])}
        //TODO: conquer
```

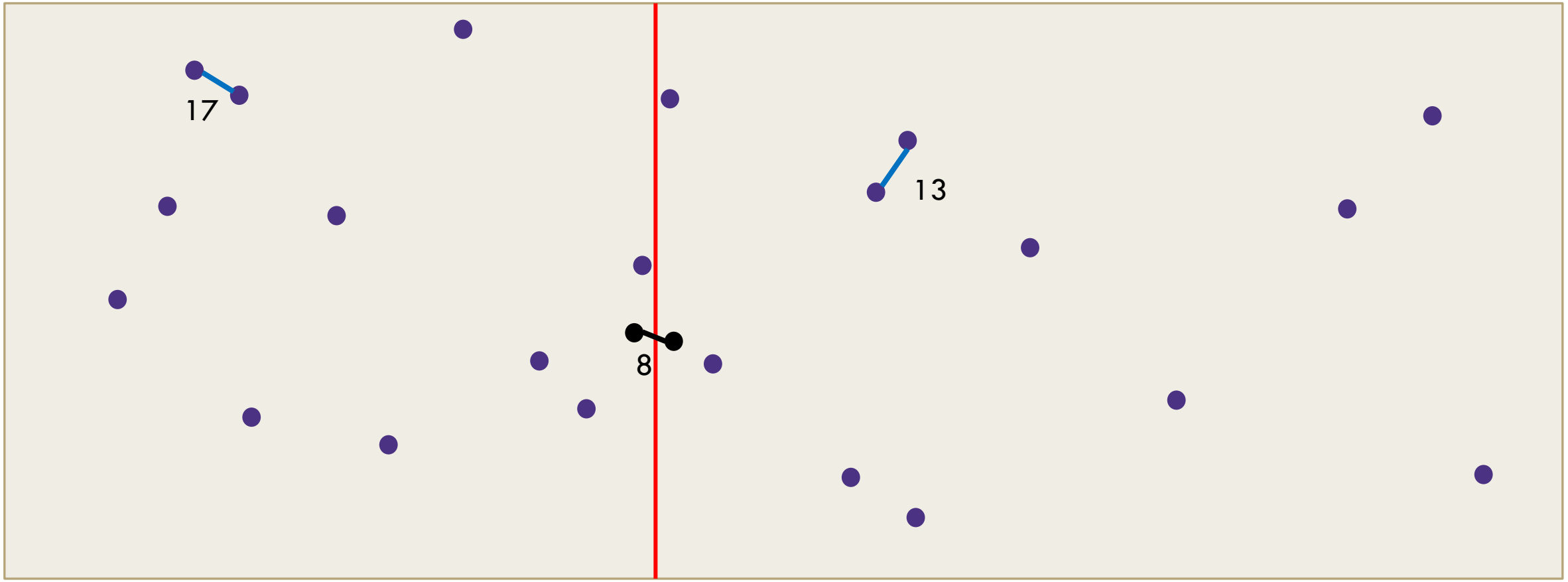
Combine Step

Are we done? Just return δ?



Combine Step (2)

Are we done? Just return δ?
No! What if the closest pair goes from the left to the right?



Conquer

Can we just check every pair that goes from the left to the right?

Some Questionable Pseudocode

```
double 2DClosestPoints(P[1..n])
    if(n ≤ 100)  //pick a cutoff you like; doesn't matter for big-O
        check all possible pairs, return the smallest distance.
    Sort P[] by x-coordinate
    δ ←min{2DClosestPoints(P[1..n/2]), 2DClosestPoints(P[n/2+1,n])}
    for(i from 1 to n/2)
        for(j from n/2 +1 to n)
            δ ← min{δ, dist(P[i],P[j])}
    return δ
```

Conquer Requires Improvement

Can we just check every pair that goes from the left to the right?

Well…it'd, give us the right answer, but…

The key to divide & conquer is making sure the conquer step is a faster calculation than brute force.

Otherwise it's brute force all the way down!

It's very Questionable Pseudocode

The Questionable Code on the last slide is equivalent to

$\delta \leftarrow \infty$

for(i from 1 to n)

    for(j from $i+1$ to n)

        $\delta \leftarrow$min{ `dist(P[i],P[j])` }

Literally. They both check every possible i, j. The questionable code just does the checks in a different order!

We need to use the results of our recursive calls to narrow down the problem! How does δ help?

Deep Breath

Where were we?

We found the median, made two recursive calls.

We need to see if there is a pair with one point on the left, the other on the right, with a closer distance than the recursive calls gave.

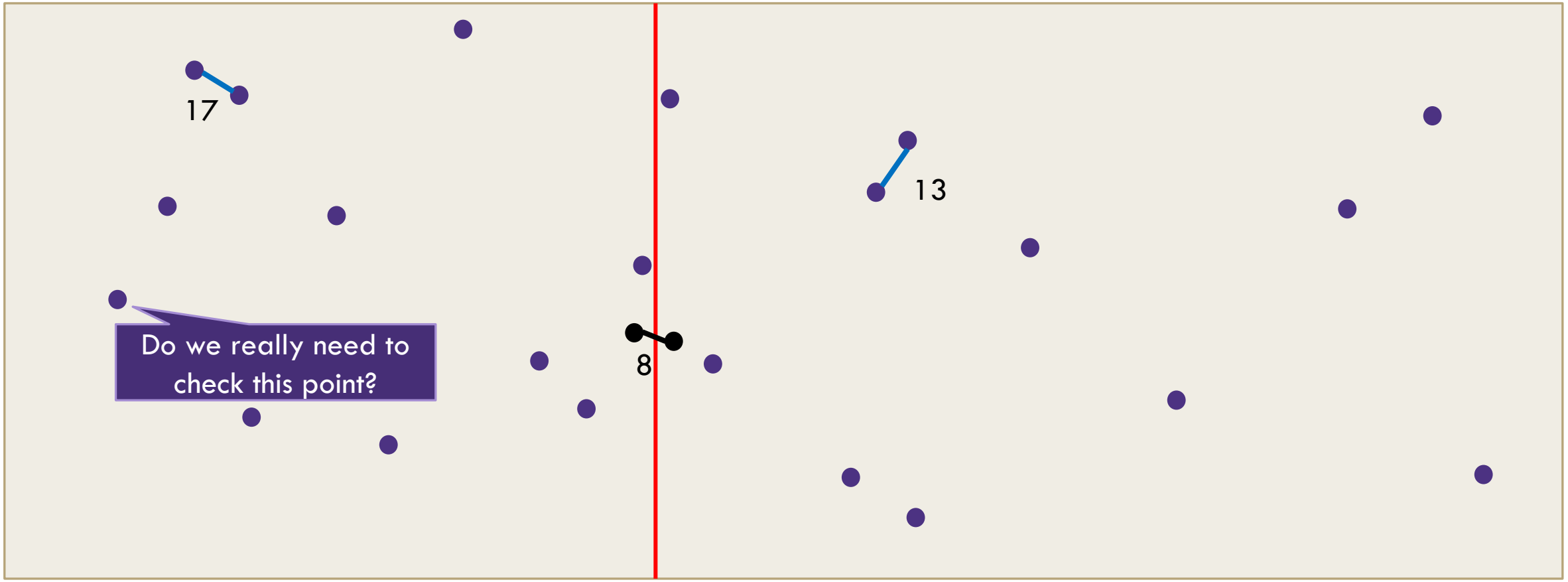
And we need to do it without looking at all possible pairs.

Pseudocode: Need the the conquer

```
double 2DClosestPoints(P[1..n])
        if(n ≤ 100)  //pick a cutoff you like; doesn't matter for big-O
                check all possible pairs, return the smallest distance.
        Sort P[] by x-coordinate
        δ ←min{2DClosestPoints(P[1..n/2]), 2DClosestPoints(P[n/2+1,n])}
        //TODO: conquer
```

Speeding up the combine step

Conquer: Find the closest "crossing pair."



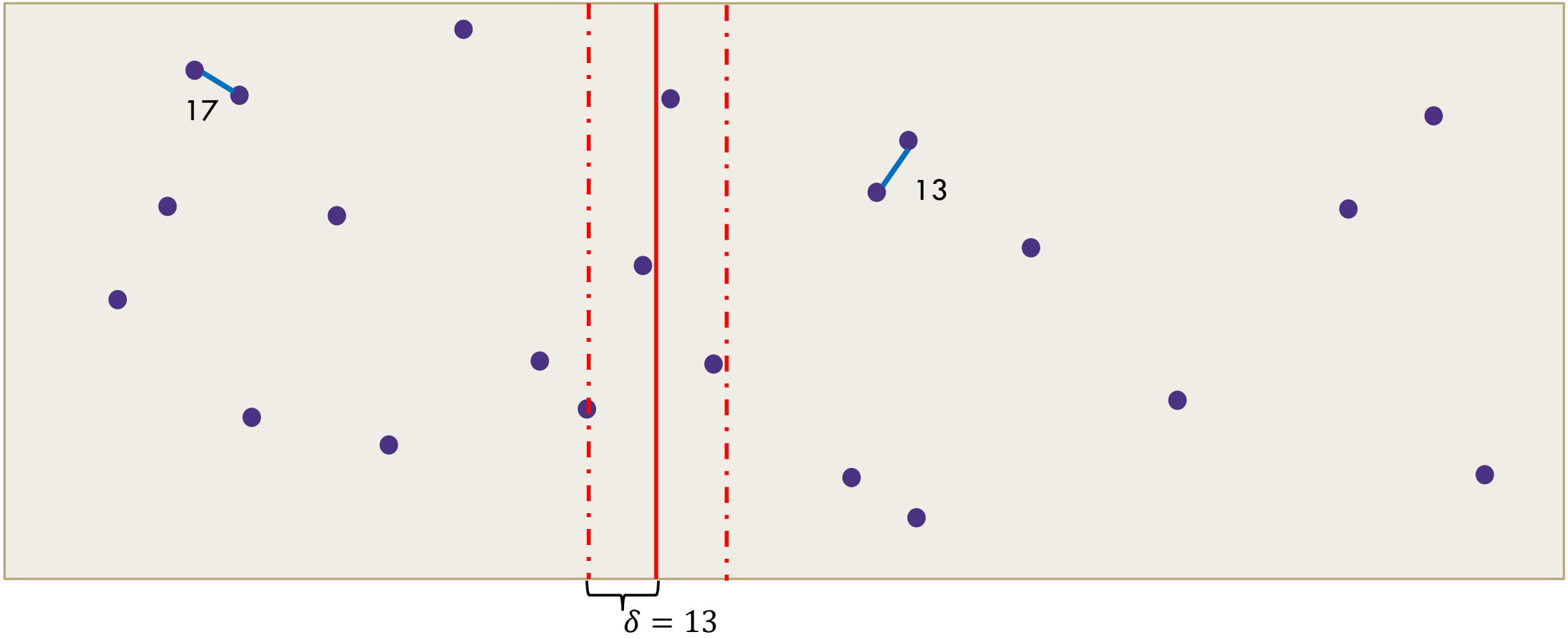
Pseudocode: updating conquer

```
double 2DClosestPoints(P[1..n])
        if(n ≤ 100)  //pick a cutoff you like; doesn't matter for big-O
                check all possible pairs, return the smallest distance.
        Sort P[] by x-coordinate
        δ ←min{2DClosestPoints(P[1..n/2]), 2DClosestPoints(P[n/2+1,n])}
         //TODO: conquer
```

Idea: We only care about a particular point if it is distance δ or less from the dividing line. Otherwise it can't possibly be in a "crossing" pair.

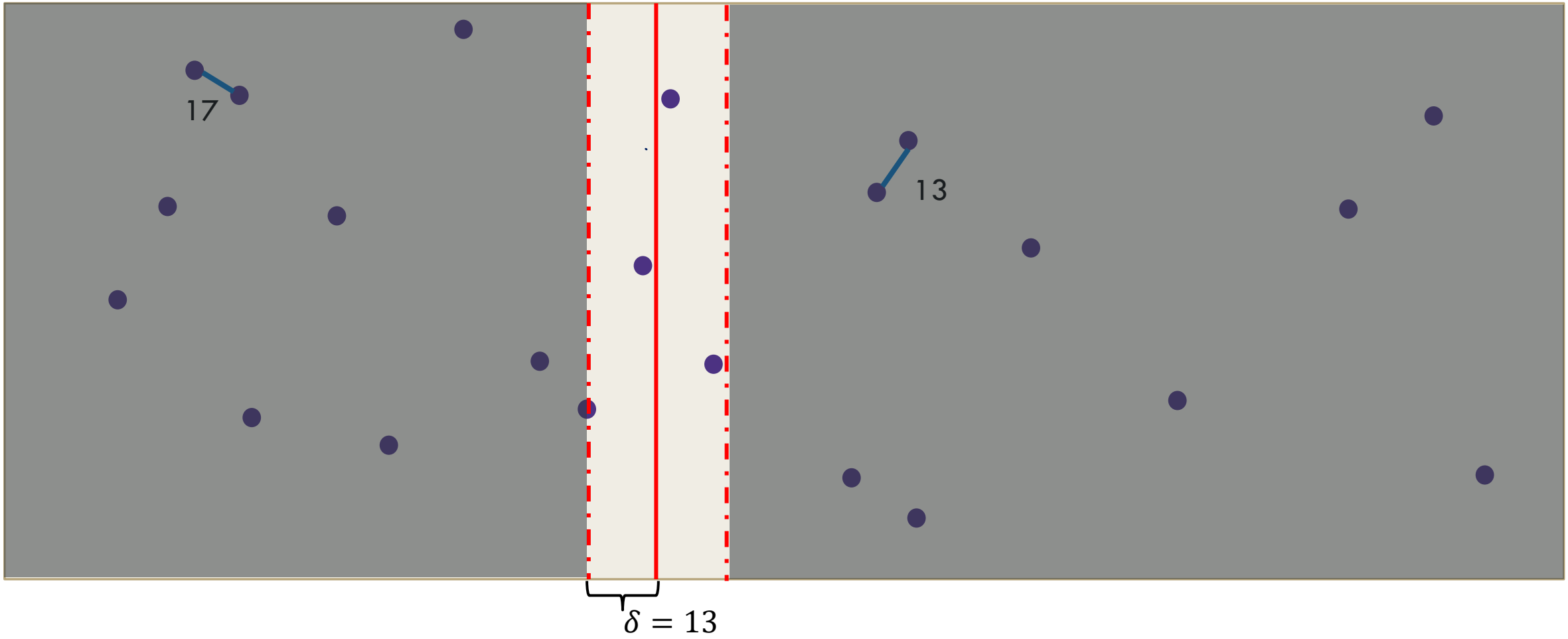
Key Area for Combine

Do we need to check every pair?
Only care if distance is less than δ. (Won't be right answer otherwise)



Key Area for Combine (2)

Can ignore the shaded regions. Only candidates are in the narrow strip.



Conquering

In that example, we made the problem smaller because there were fewer points…

…That doesn’t always happen.

But we’ve still made our problem easier!

How? Are problem is “almost” one-dimensional.

Almost One-Dimensional

If the problem were **really** one-dimensional, we could just sort and check adjacent elements.

I.e., if `P[i], P[j]` were the closest points, then $|i - j| = 1$.

We can get pretty close.

If `dist(P[i],P[j])` $< \delta$ and `P[i], P[j]` are both in the middle strip, then $|i - j| \leq 11$

11 is a constant. Independent of δ

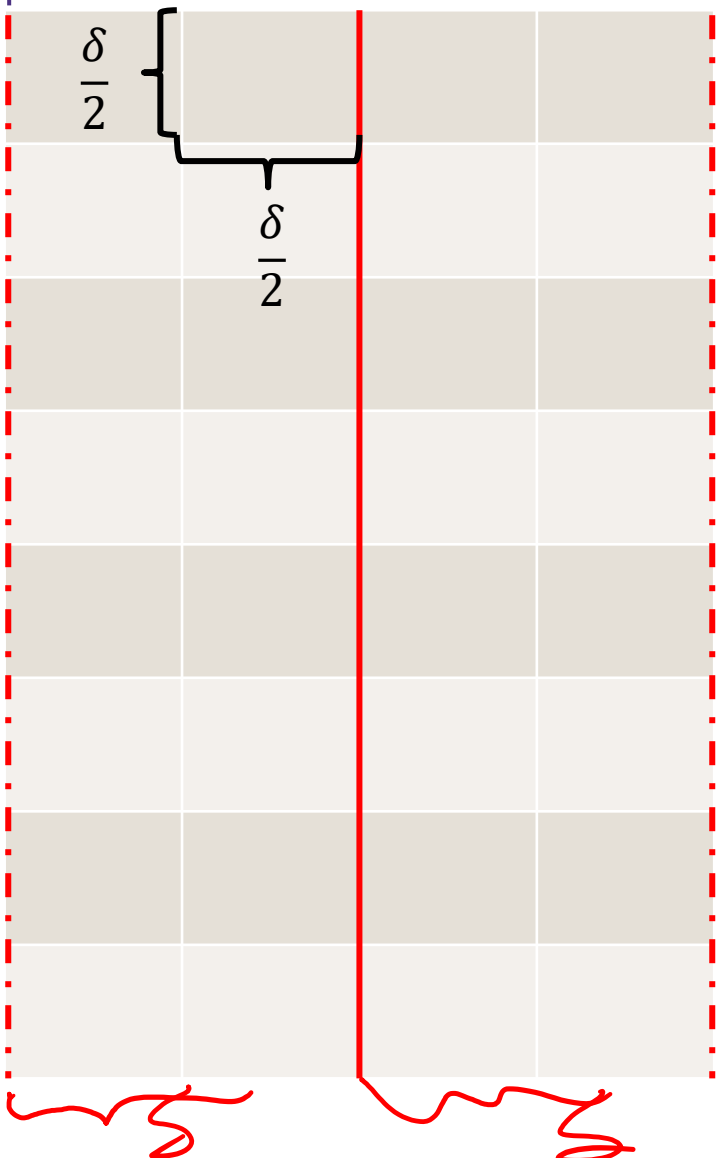
Intuition: two points on the left must be at least δ from each other. Only so much room in a 2δ wide-strip to fit points in δ height too.

Pseudocode (finished!)

```
double 2DClosestPoints(P[1..n])
    if(n ≤ 100)  //pick a cutoff you like; doesn't matter for big-O
        check all possible pairs, return the smallest distance.
    Sort P[] by x-coordinate
    δ ←min{2DClosestPoints(P[1..n/2]), 2DClosestPoints(P[n/2+1,n])}
    Let C be all points within δ of the median in x-coordinate.
    Sort C by y-coordinate
    for(i from 1 to C.length)
        for(j from i + 1 to i + 11)
            δ ←min{dist(C[i], C[j]), δ}
    return δ
```

Prove the Lemma (1)

If `dist(P[i], P[j])` $\leq \delta$ and `P[i], P[j]` are both in the middle strip, then $|i - j| \leq 11$

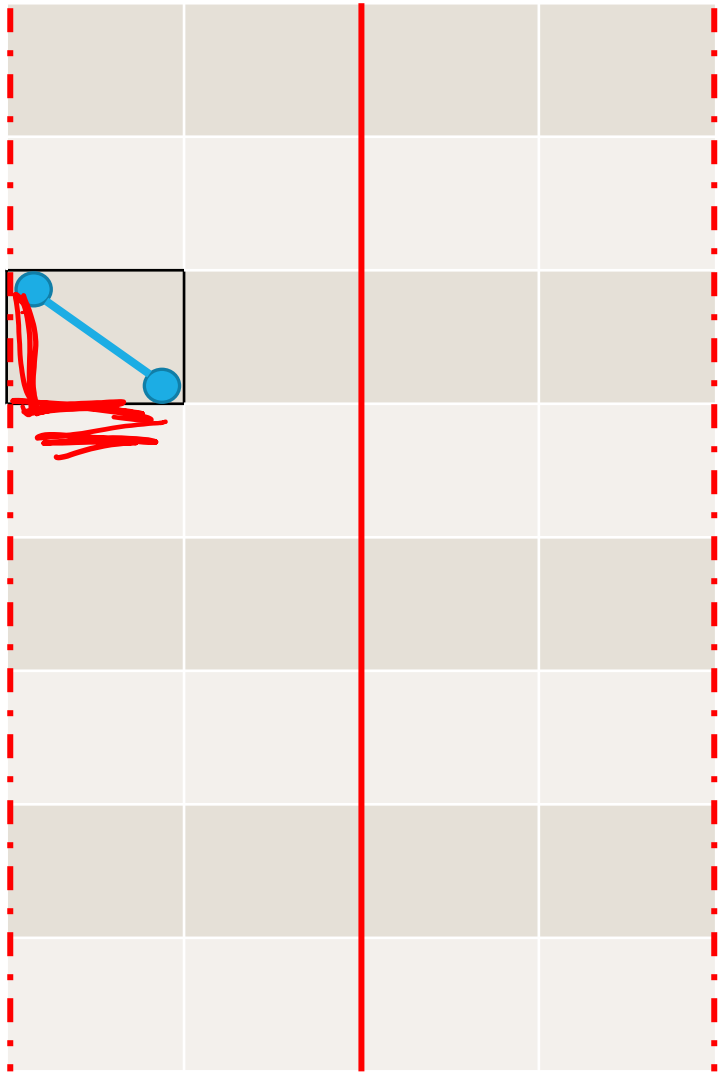


Place a grid of $\delta/2$x$\delta/2$ squares on the strip.

Strip is 4 squares wide.

Prove the Lemma (2)

If `dist(P[i], P[j])` $\leq \delta$ and `P[i], P[j]` are both in the middle strip, then $|i - j| \leq 11$



Place a grid of $\delta/2 \times \delta/2$ squares on the strip.

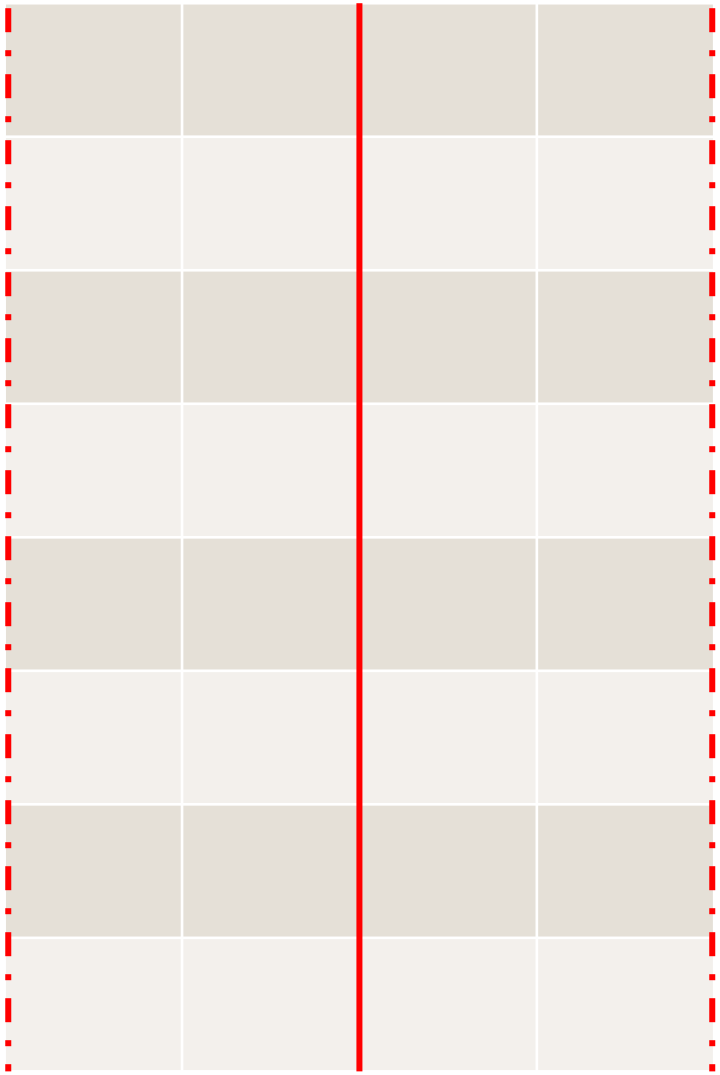
Only one point in each cell.

$$\texttt{dist} \leq \sqrt{\left(\frac{\delta}{2}\right)^2 + \left(\frac{\delta}{2}\right)^2} = \sqrt{2 \cdot \delta^2/4} < \sqrt{\delta^2} = \delta.$$

But δ is the smallest distance between points on the same side (found via recursive calls).

Prove the Lemma (3)

If `dist(P[i],P[j])` $\leq \delta$ and `P[i], P[j]` are both in the middle strip, then $|i - j| \leq 11$



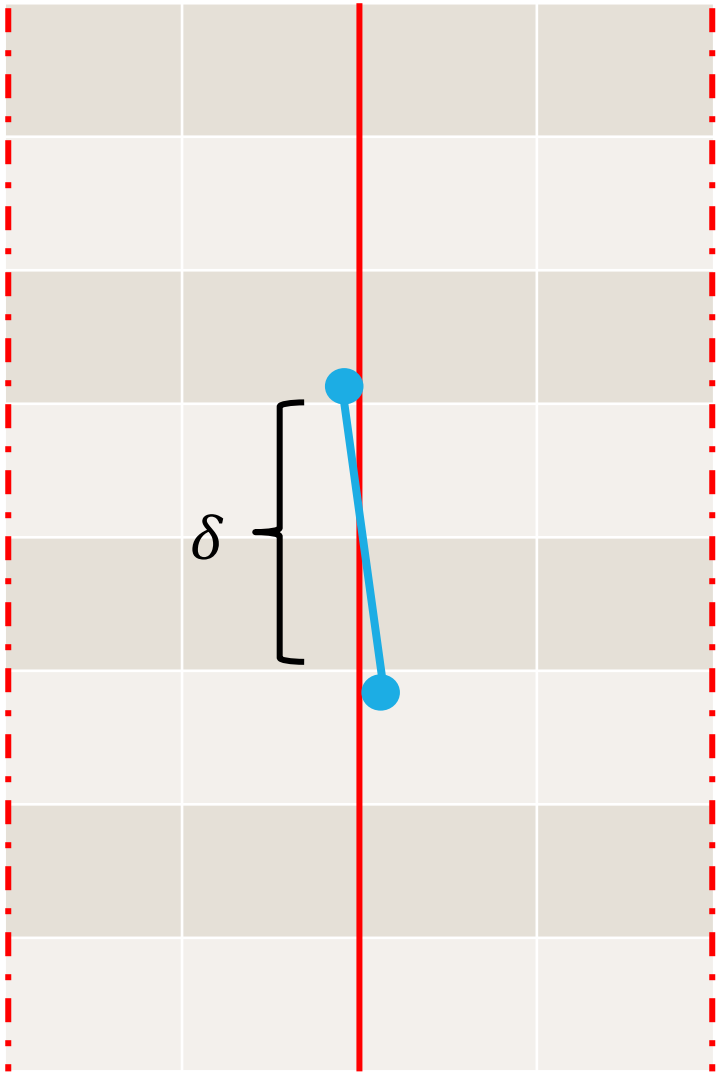
Place a grid of $\delta/2$x$\delta/2$ squares on the strip.

Only one point in each cell.

Two points in the same cell are on the same side and distance $< \delta$.

Prove the Lemma (4)

If `dist(P[i], P[j])` $\leq \delta$ and `P[i], P[j]` are both in the middle strip, then $|i - j| \leq 11$



Place a grid of $\delta/2 \times \delta/2$ squares on the strip.

Only one point in each cell.

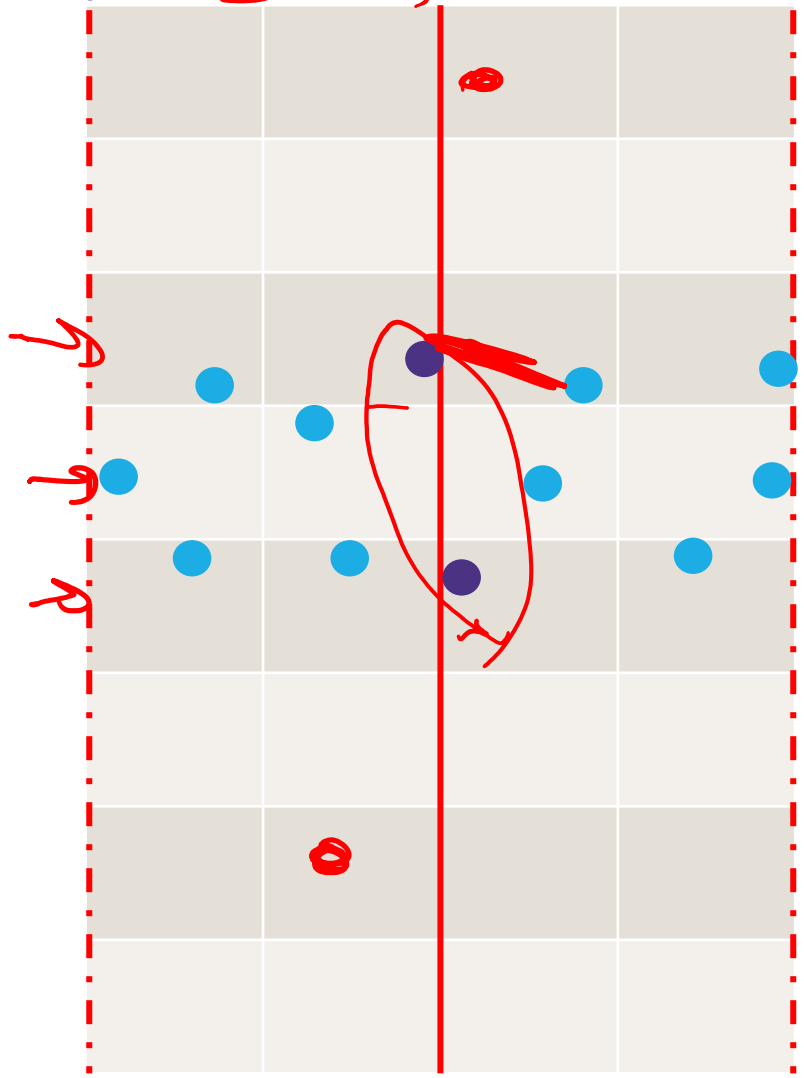
Two points in the same cell are on the same side and distance $< \delta$.

If two points have distance $< \delta$, they are in the same row, adjacent rows, or have one row between them.

Two rows between gives δ distance in the y-coordinate.

Prove the Lemma (5)

If `dist(P[i], P[j])` $\leq \delta$ and `P[i], P[j]` are both in the middle strip, then $|i - j| \leq 11$



Place a grid of $\delta/2 \times \delta/2$ squares on the strip.

Only one point in each cell.

Two points in the same cell are on the same side and distance $< \delta$.

If two points have distance $< \delta$, they are in the same row, adjacent rows, or have one row between them.

How far apart can those points be in the array?

Only 12 elements can be in those rows, so when ordered by y-coordinate, difference is at most 11.

It works! (a bit more formally)

A formal proof of correctness would be by induction.

Idea for IS:

Closest pair in overall instance is in one half, the other, or split. By IH, recursive calls give distance between closest pairs in each half.

Case 1: closest pair is in a half,

Recursive call gives true shortest distance, by IH. δ will never be overwritten because we only compare to other distances between points.

Case 2: closest pair crosses,

Then its distance is less than the (correct) distances found by the recursive calls. By lemma, we will check the closest pair, and since we take the min, δ will be set to that value and (because we only compare to other distances) never overwritten.

In both cases we return the true shortest distance.

Write a recurrence for the running time

double 2DClosestPoints(P[1..n])

if($n \leq 100$) //pick a cutoff you like; doesn't matter for big-O

check all possible pairs, return the smallest distance.

Sort P[] by x-coordinate

$\delta \leftarrow$min{2DClosestPoints(P[1..n/2]), 2DClosestPoints(P[n/2+1,n])}

Let C be all points within δ of the median in x-coordinate.

Sort C by y-coordinate

for(i from 1 to C.length)

for(j from $i+1$ to $i+10$)

$\delta \leftarrow$min{`dist`($C[i]$, $C[j]$), δ}

return δ

Write a recurrence for the running time (2)

```
double 2DClosestPoints(P[1..n])
    if(n ≤ 100)  //pick a cutoff you like; doesn't matter for big-O
        check all possible pairs, return the smallest distance.
    Sort P[] by x-coordinate
    δ ←min{2DClosestPoints(P[1..n/2]), 2DClosestPoints(P[n/2+1,n])}
    Let C be all points within δ of the median in x-coordinate.
    Sort C by y-coordinate
    for(i from 1 to C.length)
        for(j from i + 1 to i + 10)
            δ ←min{dist(C[i], C[j]), δ}
    return δ
```

$\Theta(n \log n)$

Recursive calls size $n/2$

$\Theta(n \log n)$

$O(1)$ per iteration of **outer**-loop

$O(n)$ iterations of outer-loop.

Running Time

$$T(n) = \begin{cases} O(1) & \text{if } n \leq 100 \\ 2T\left(\frac{n}{2}\right) + O(n \log n) & \text{otherwise} \end{cases}$$

$O(n \log^2 n)$

How did I find this closed form? For today, just believe it, we'll give you a tool next week.

Small optimization: just sort by x and y once; store the sorted versions.

$$T(n) = \begin{cases} O(1) & \text{if } n \leq 100 \\ 2T\left(\frac{n}{2}\right) + O(n) & \text{otherwise} \end{cases}$$

$O(n \log n)$ for sorting, $O(n \log n)$ for everything else; total $O(n \log n)$.

There’s no grid in the code

Note that $\delta/2$ grid thing was ***just*** for the proof. Look back at the pseudocode, the grid isn’t there.

Takeaways

A clever algorithm for finding closest points.

To make divide & conquer efficient, your conquer step must be faster than brute forcing the “crossing” pairs.

Look for how the problem becomes simpler.