CSE 421 Winter 2025
Lecture 9: Divide and Conquer

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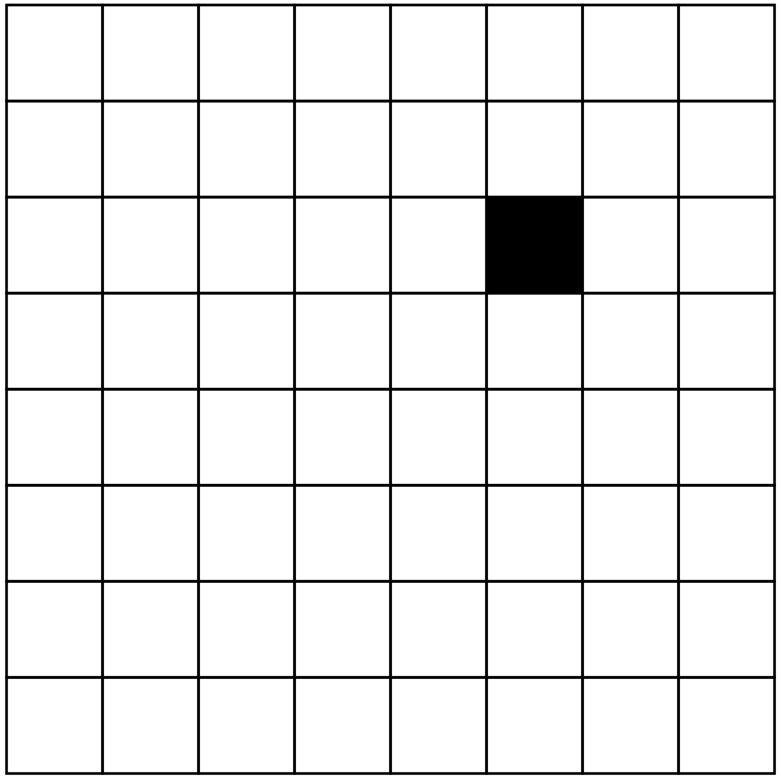
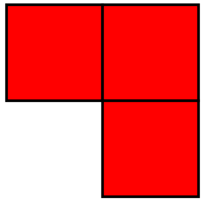
Trominos Tiling

- Given an 8x8 grid with 1 cell missing, can we exactly cover it with "trominoes"?

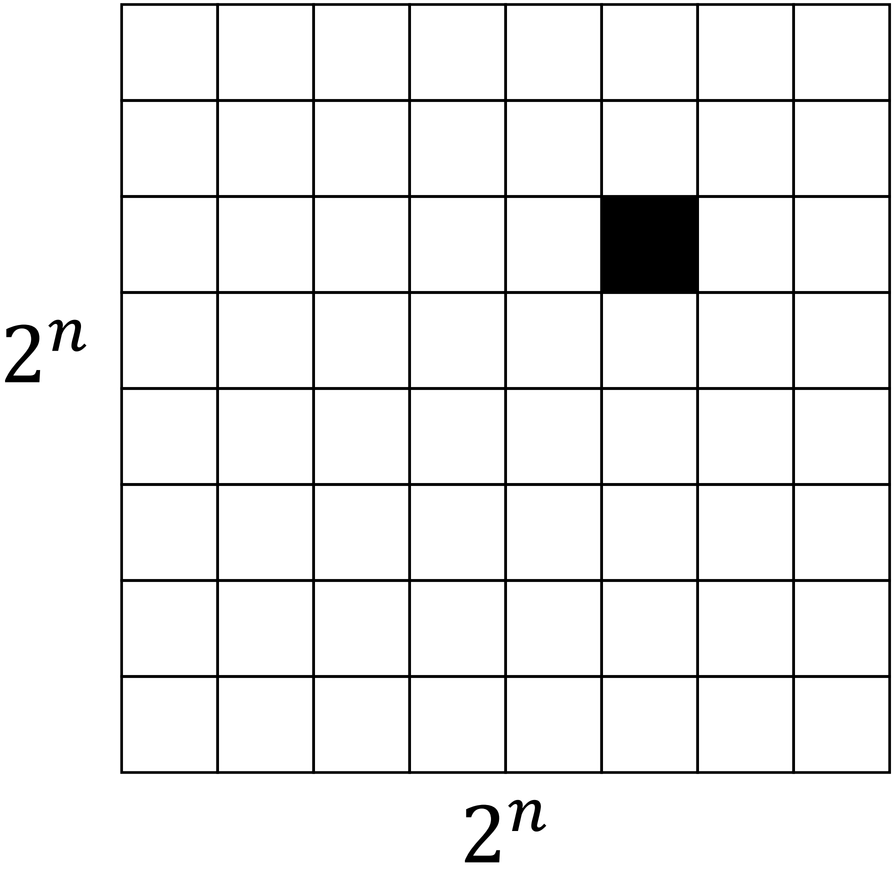
Can you cover this?



With these?

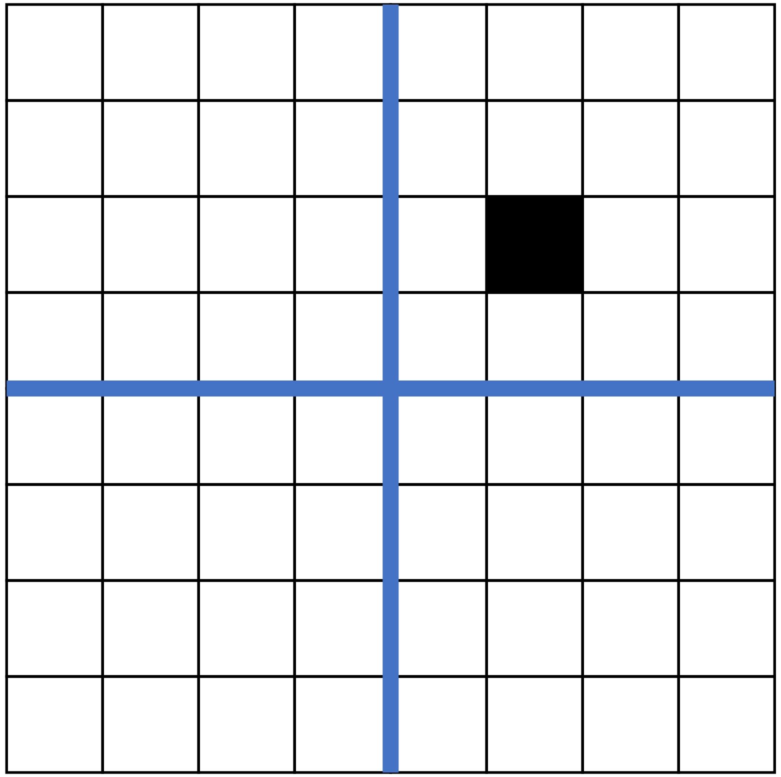


Trominoes Puzzle Solution



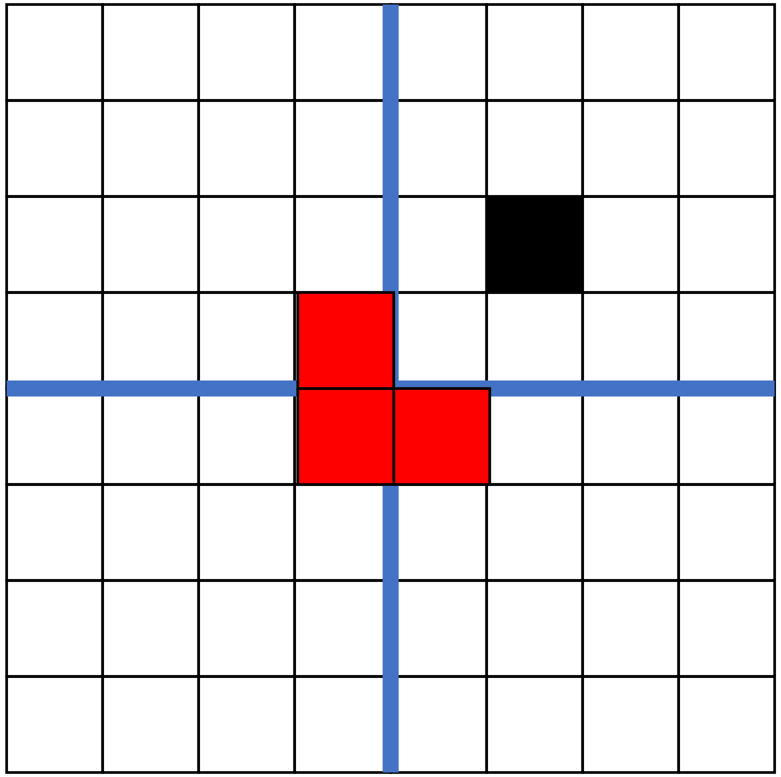
What about larger boards?

Trominoes Puzzle Solution



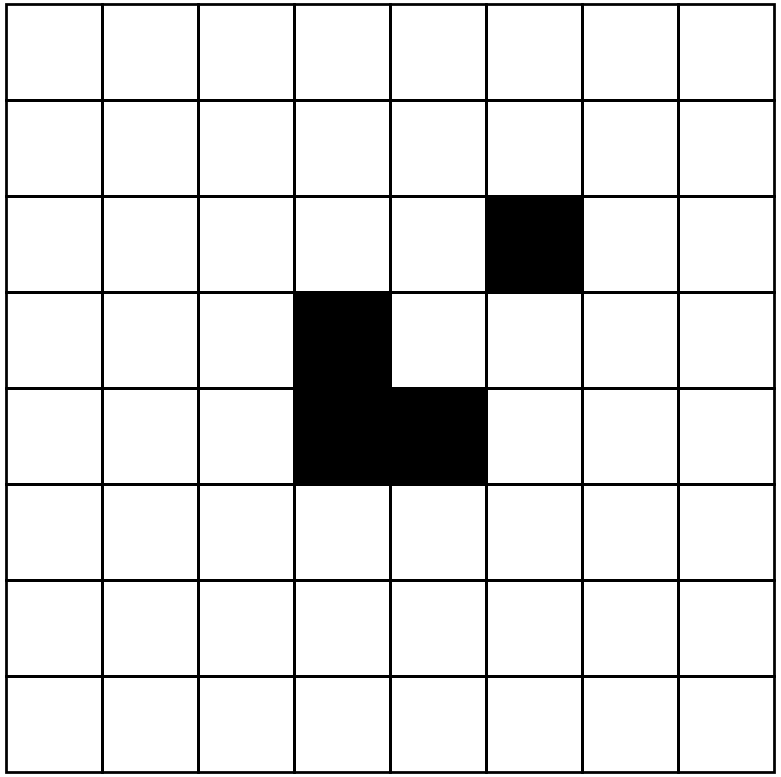
Divide the board into quadrants

Trominoes Puzzle Solution



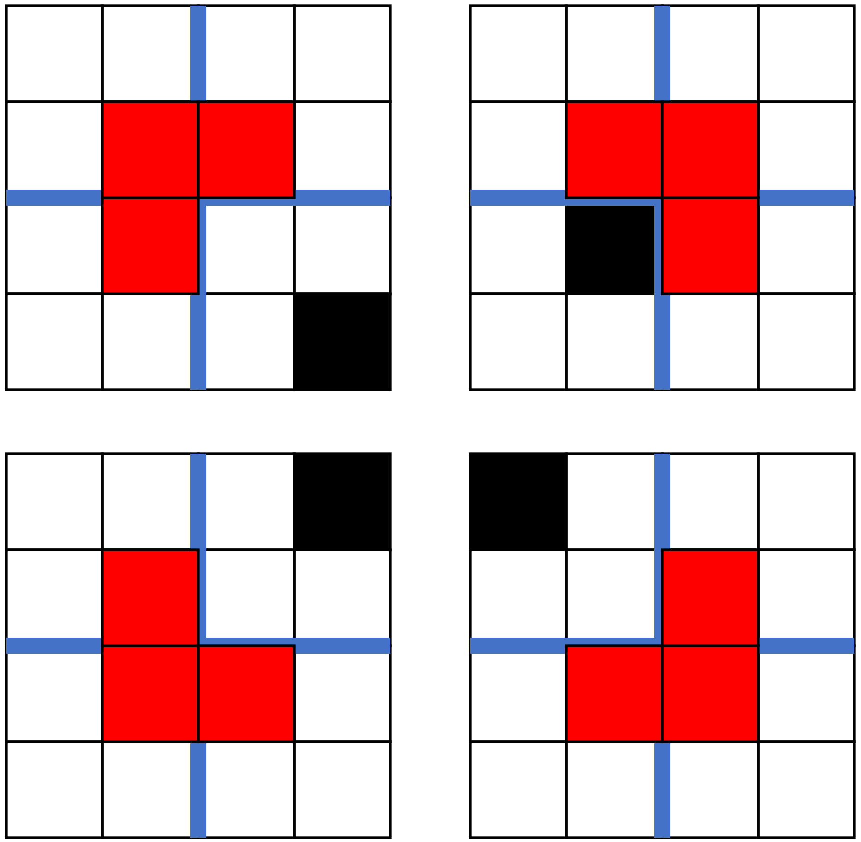
Place a tromino to occupy the three quadrants without the missing piece

Trominoes Puzzle Solution



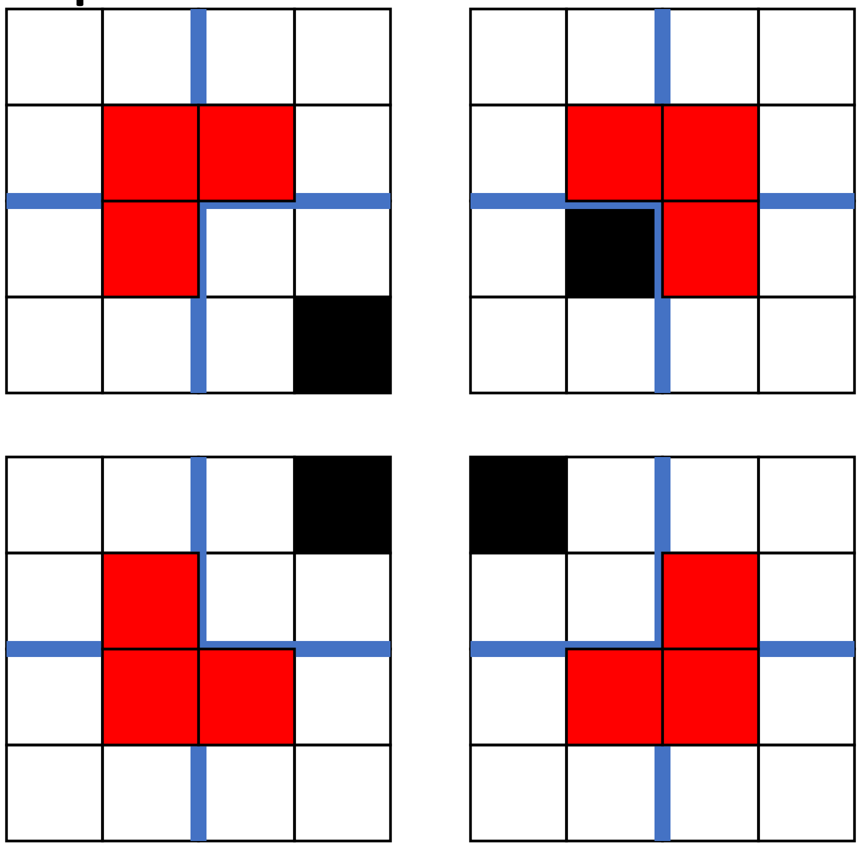
Each quadrant is now a smaller subproblem

Trominoes Puzzle Solution

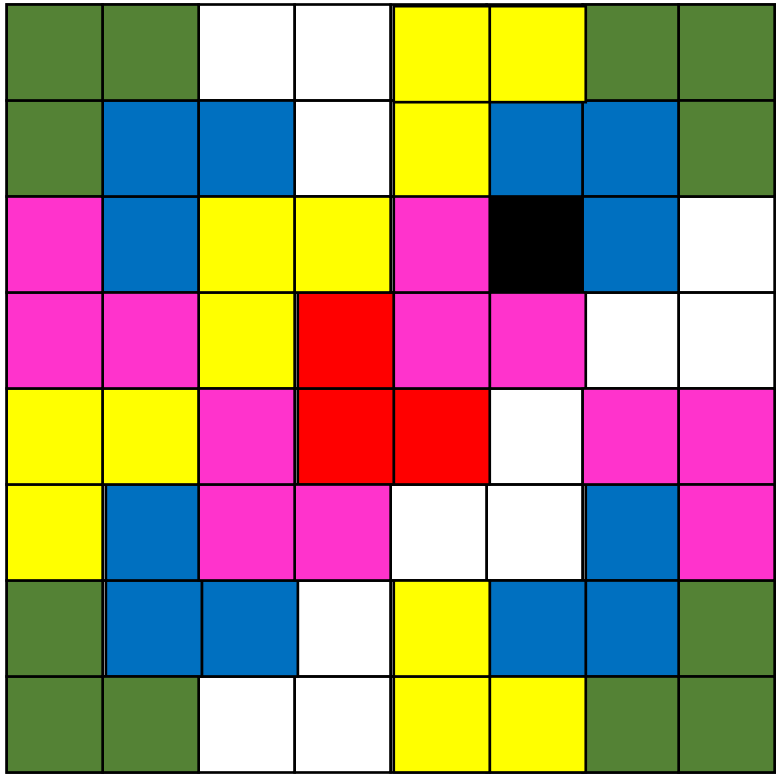


Solve **Recursively**

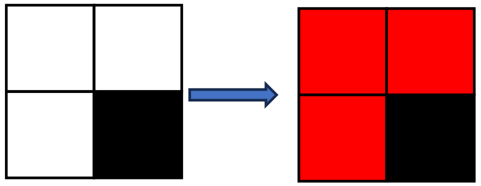
Divide and Conquer



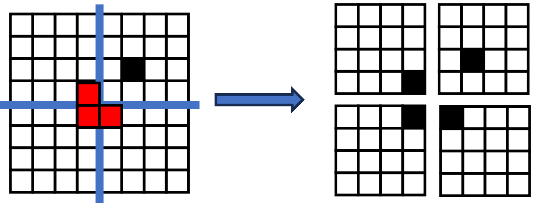
Trominoes Puzzle Solution



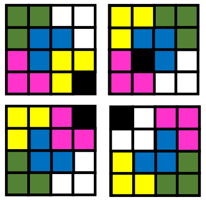
Divide and Conquer (Trominoes)



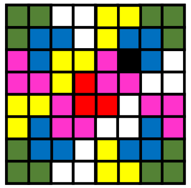
- **Base Case**:
  - For a 2×2 board, the empty cells will be exactly a tromino



- **Divide**:
  - Break of the board into quadrants of size $2^{n-1} \times 2^{n-1}$ each
  - Put a tromino at the intersection such that all quadrants have one occupied cell



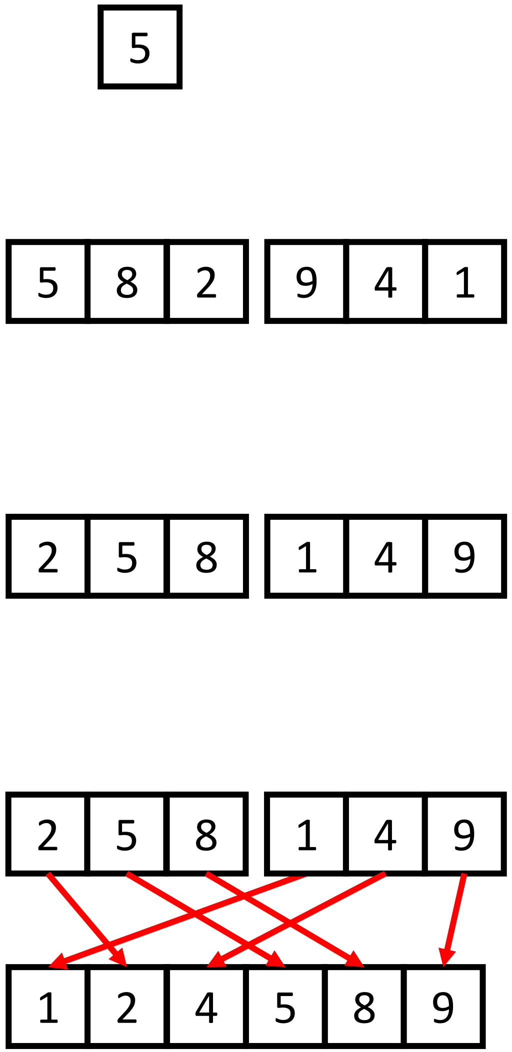
- **Conquer**:
  - Cover each quadrant



- **Combine**:
  - Reconnect quadrants

Divide and Conquer (Merge Sort)

$2T(\frac{n}{2}) + n$



- **Base Case**:
  - If the list is of length 1 or 0, it's already sorted, so just return it
  - (Alternative: when length is ≤ 15, use insertion sort)
- **Divide**:
  - Split the list into two "sublists" of (roughly) equal length

  $2, \frac{n}{2}$
- **Conquer**:
  - Sort both lists recursively

  $2T(\frac{n}{2})$
- **Combine**:
  - **Merge** sorted sublists into one sorted list

  n

Divide and Conquer (Running Time)

$T(c) = k$

$a =$ *number of subproblems*

$\frac{n}{b} =$ *size of each subproblem*

$f_d(n) =$ time to divide

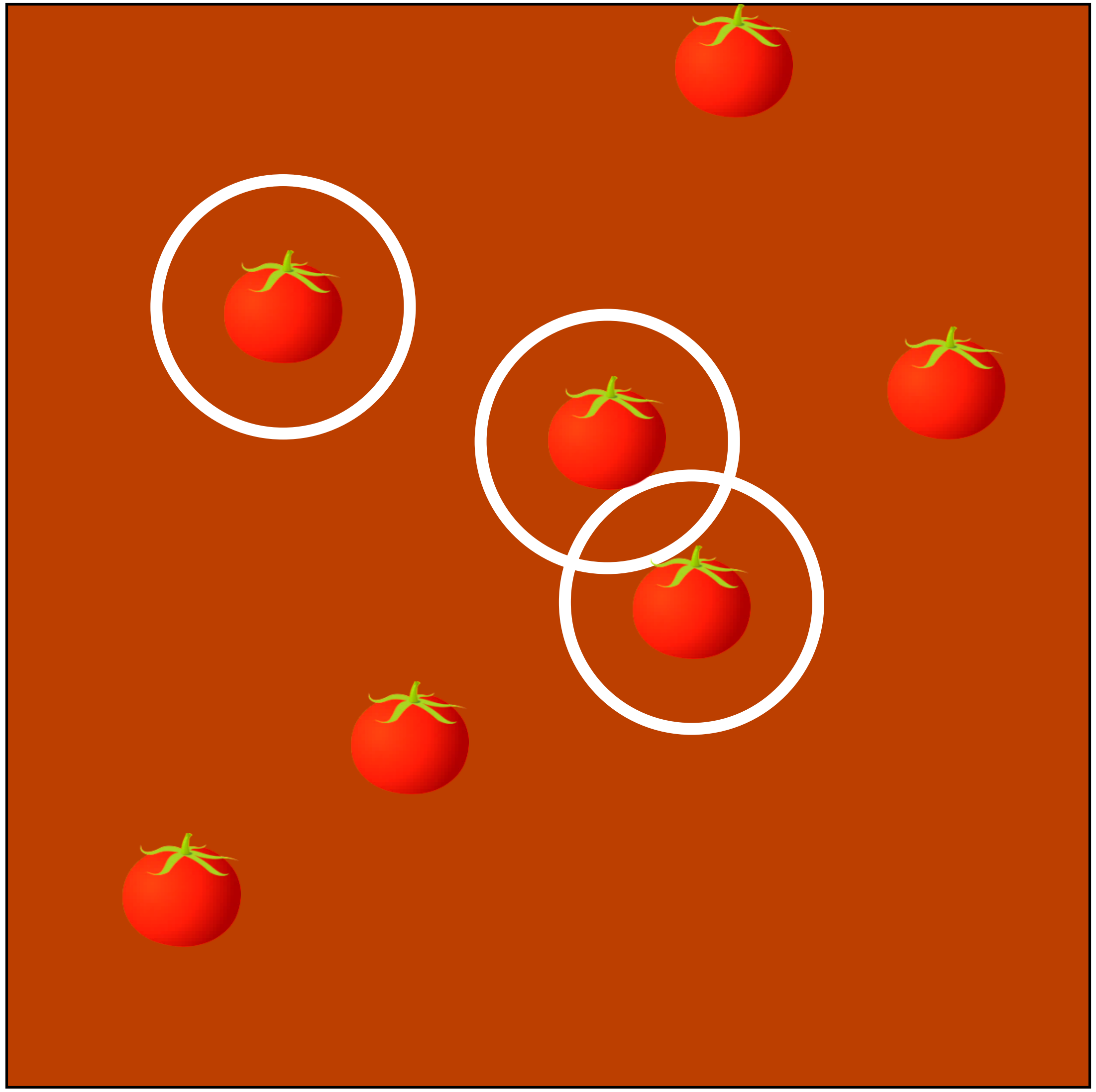
$a \cdot T\left(\frac{n}{b}\right)$

$f_c(n) =$ time to combine

- **Base Case**:
  - When the problem size is small ($\leq c$), solve non-recursively
- **Divide**:
  - When problem size is large, identify 1 or more smaller versions of exactly the same problem
- **Conquer**:
  - Recursively solve each smaller subproblem
- **Combine**:
  - Use the subproblems' solutions to solve to the original

Overall: $T(n) = aT\left(\frac{n}{b}\right) + f(n)$ **where** $f(n) = f_d(n) + f_c(n)$

Closest Pair of Tomatoes



Closest Pair of Points

Given:

- A sequence of n points $p_1, \ldots, p_n$ with real coordinates in 2 dimensions ($\mathbb{R}^2$)

Find:

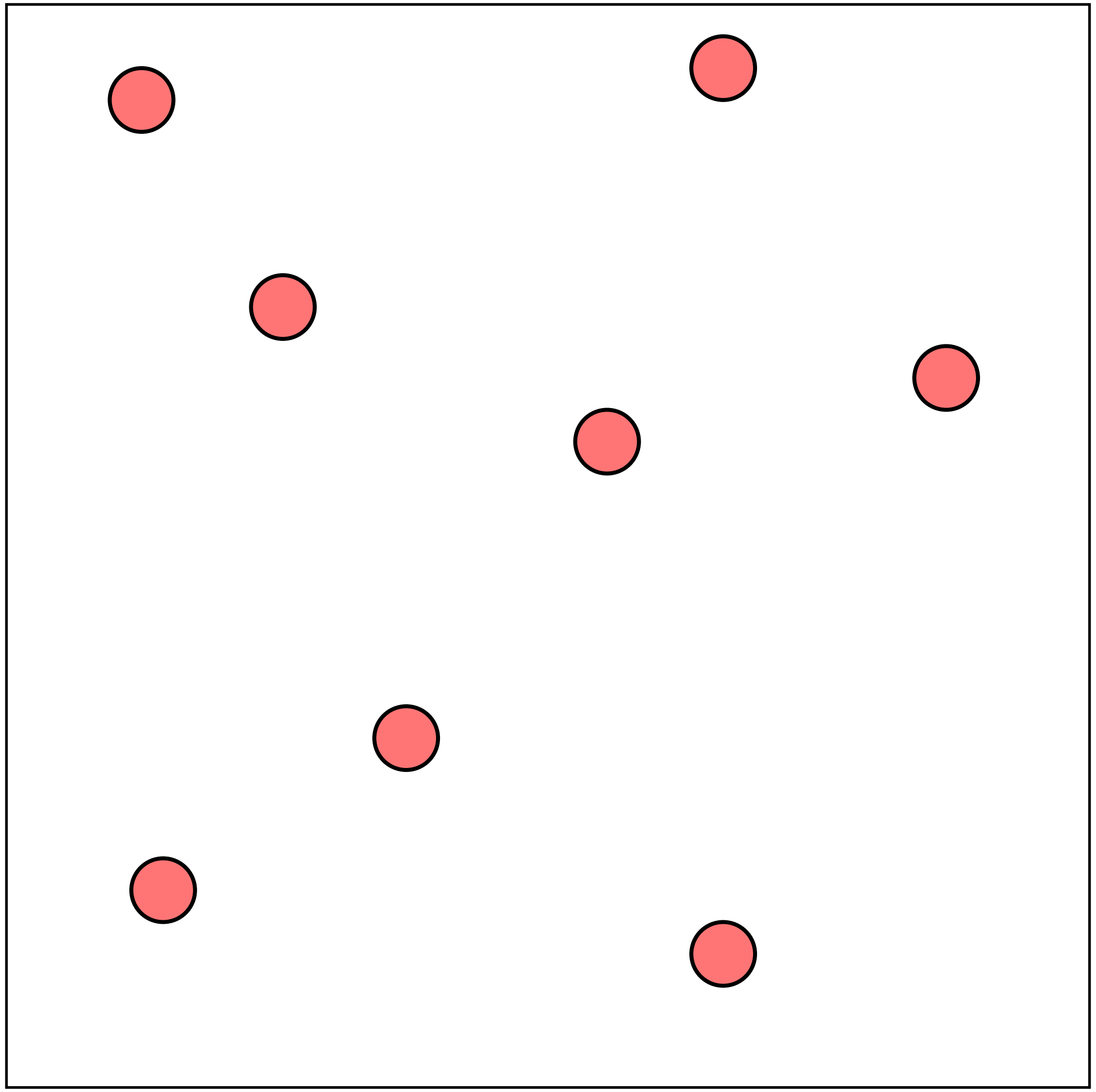
- A pair of points p_i, p_j s.t. the Euclidean distance $d(p_i, p_j)$ is minimized

How about a $\Theta(n^2)$ algorithm?

- Try all possible pairs, keeping the smallest

Our goal:

- Use D&C to create a $\Theta(n \log n)$ algorithm

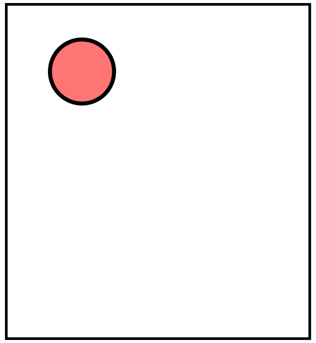


Closest Pair of Point D&C Idea

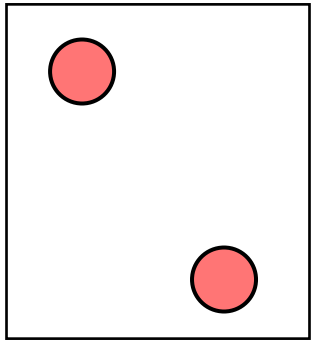
To get $\Theta(n \log n)$, we will aim for $T(n) = 2T\left(\frac{n}{2}\right) + n$

- **Base Case:**
  - If the number of points is small, do use a naïve solution
- **Divide:**
  - Otherwise partition the points into 2 subsets
  - Running time "budget" $O(n)$
- **Conquer:**
  - Find the closest pair of points in each subset
- **Combine:**
  - Use those closest pairs of points to find the closest overall
  - Running time "budget" $O(n)$

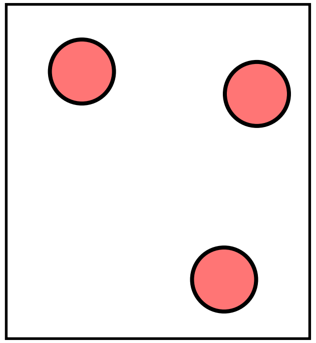
Closest Pair: Base Cases



If $n = 1$
return ∞

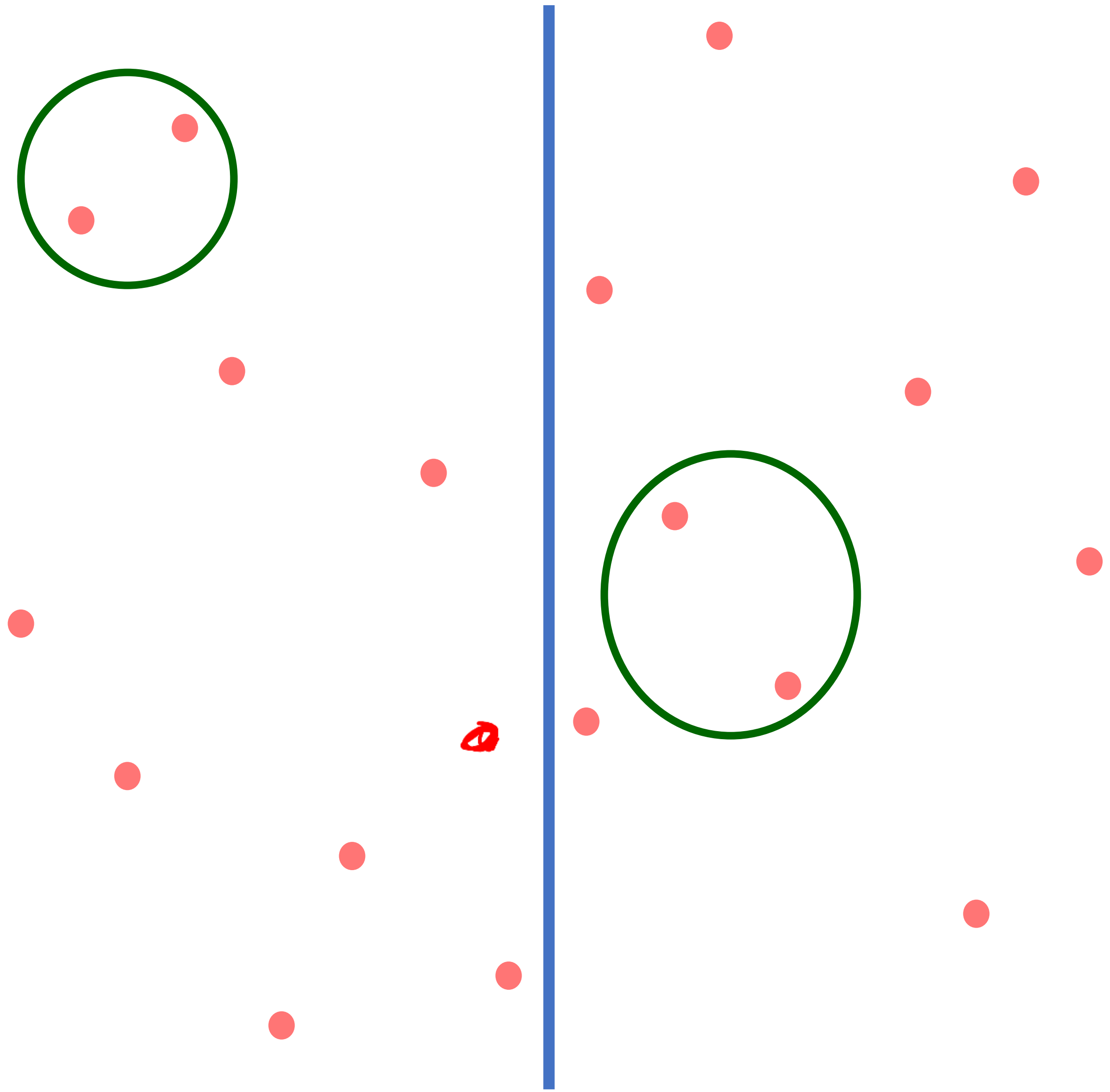


If $n = 2$
return the distance



If $n = 3$
check all 3 pairs
return the closest

Closest Pair: First Idea



Divide:

- Split using **median** x-coordinate
- each subpart has size $n/2$.

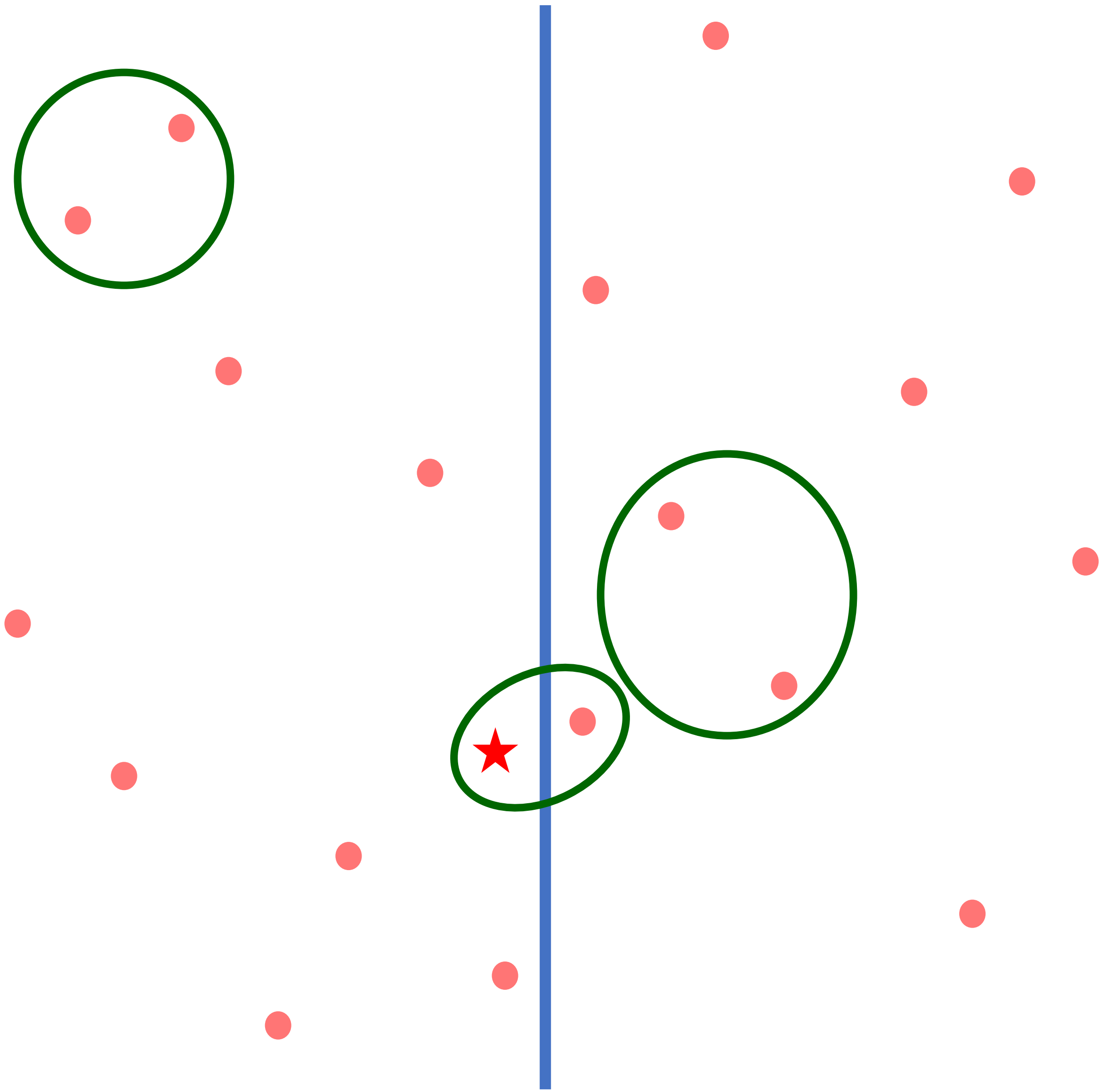
Conquer:

- Solve both size $n/2$ subproblems
- We now have the closest pair from the left and from the right

Combine:

- Return the closer of the left pair and the right pair

Closest Pair: First Idea - Problem



Divide:

- Split using **median** x-coordinate
- each subpart has size $n/2$.

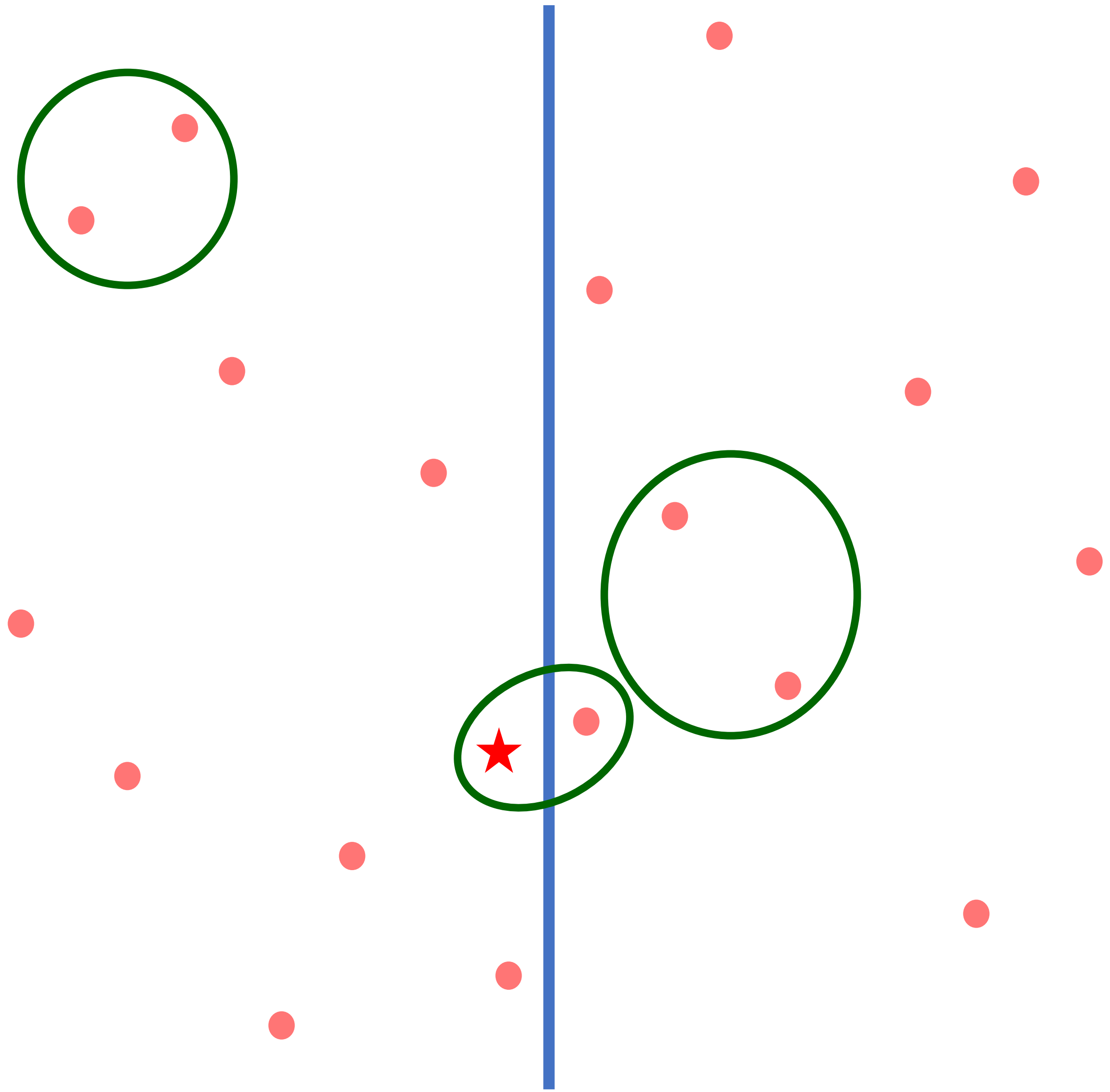
Conquer:

- Solve both size $n/2$ subproblems
- We now have the closest pair from the left and from the right

Combine:

- Find the closest pair crossing the middle
- Return the closest of the left, right, and crossing pairs

Finding the Closest Crossing Pair – 1st Idea



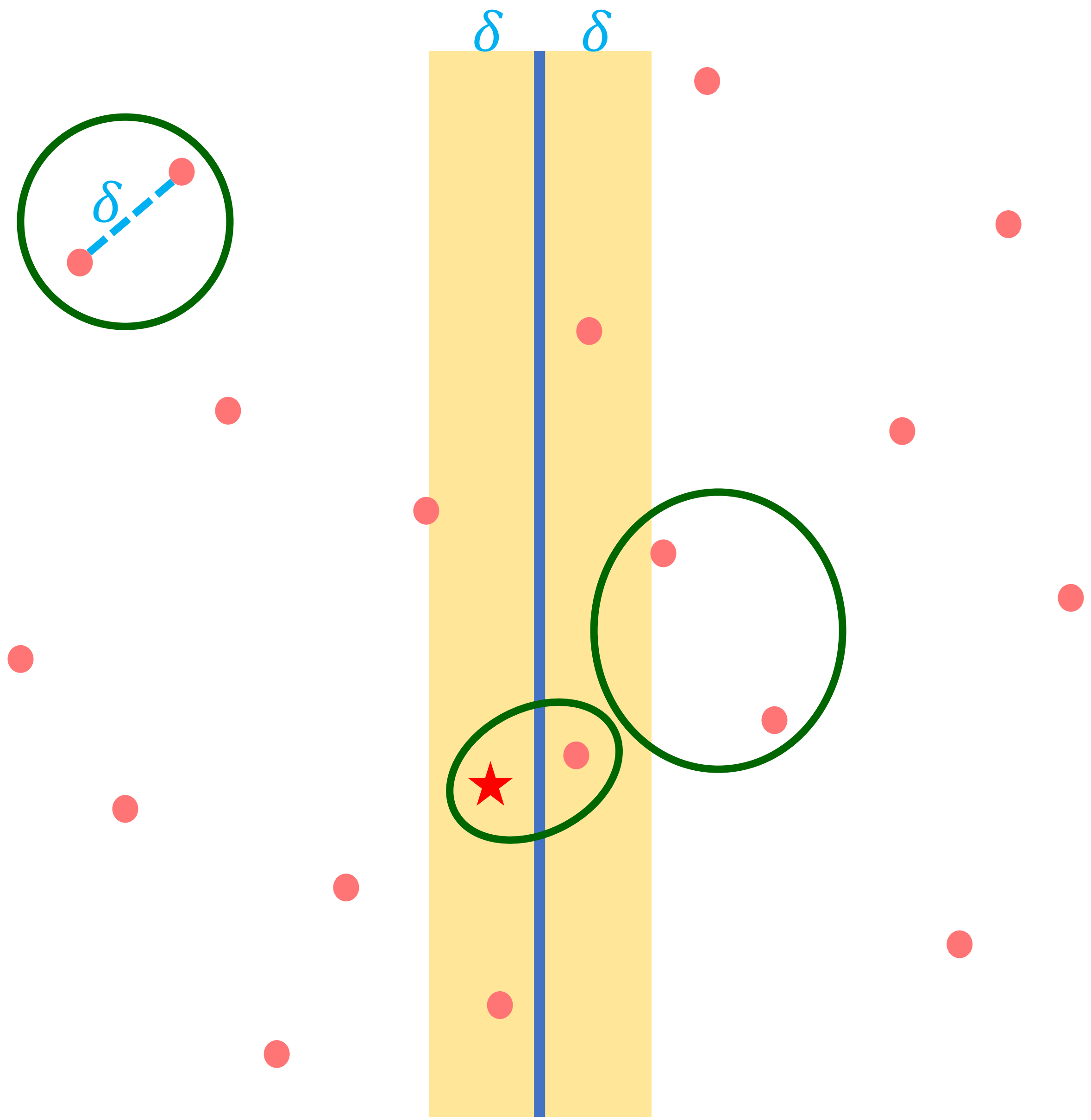
Combine:
- Find the closest pair crossing the middle
- Return the closest of the left, right, and crossing pairs

Procedure:
- For each point on the left, find its closest point on the right
- Save the closest seen as the crossing pair

Problem?

Running time is $\left(\frac{n}{2}\right)^2$

Finding the Closest Crossing Pair – 2nd Idea



Combine:

- Find the closest pair crossing the middle
- Return the closest of the left, right, and crossing pairs

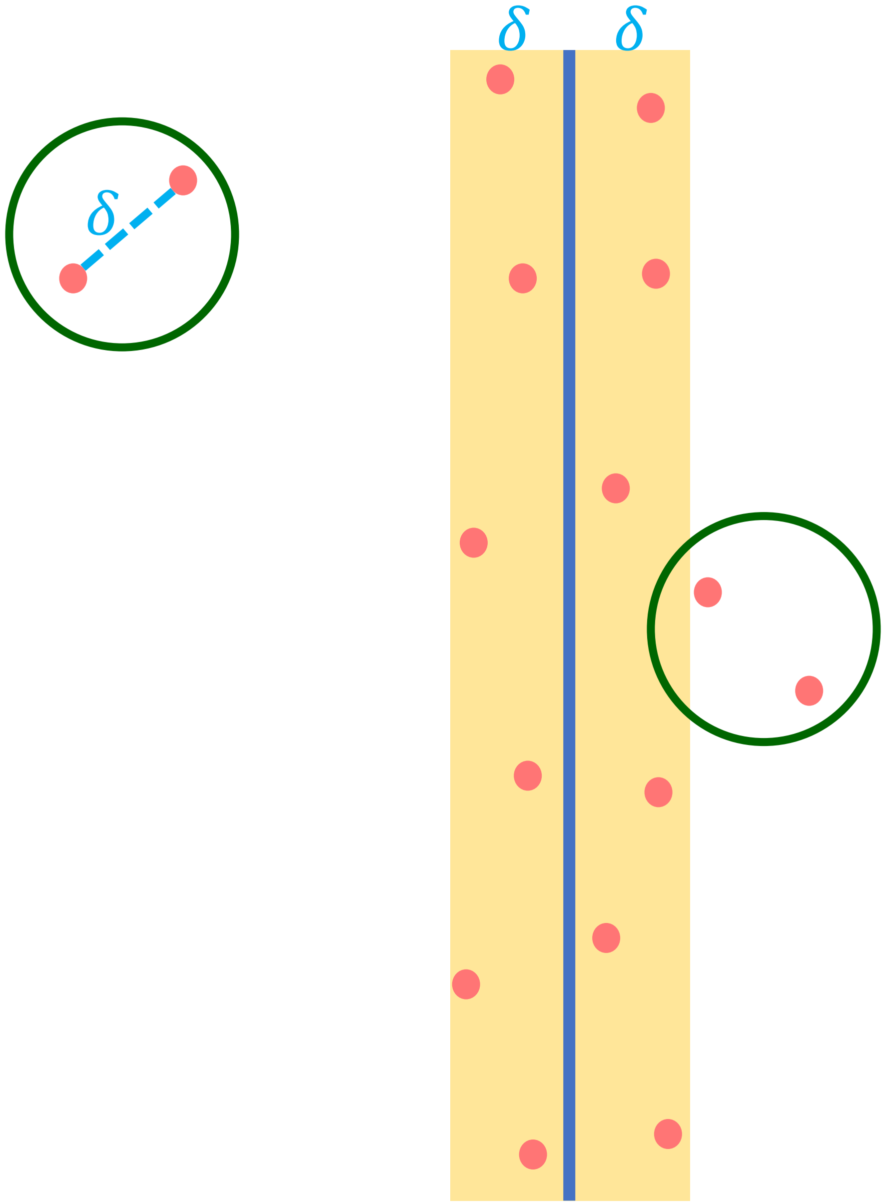
Observation:

- We only care about crossing pairs that might be closer than left and right
- Ignore points too far from the divide

Procedure:

- Let δ be the closest distance from left and right
- For each point on the left that's within δ of the divide, find its closest match from among points within δ on the right

Problem with the 2nd Idea



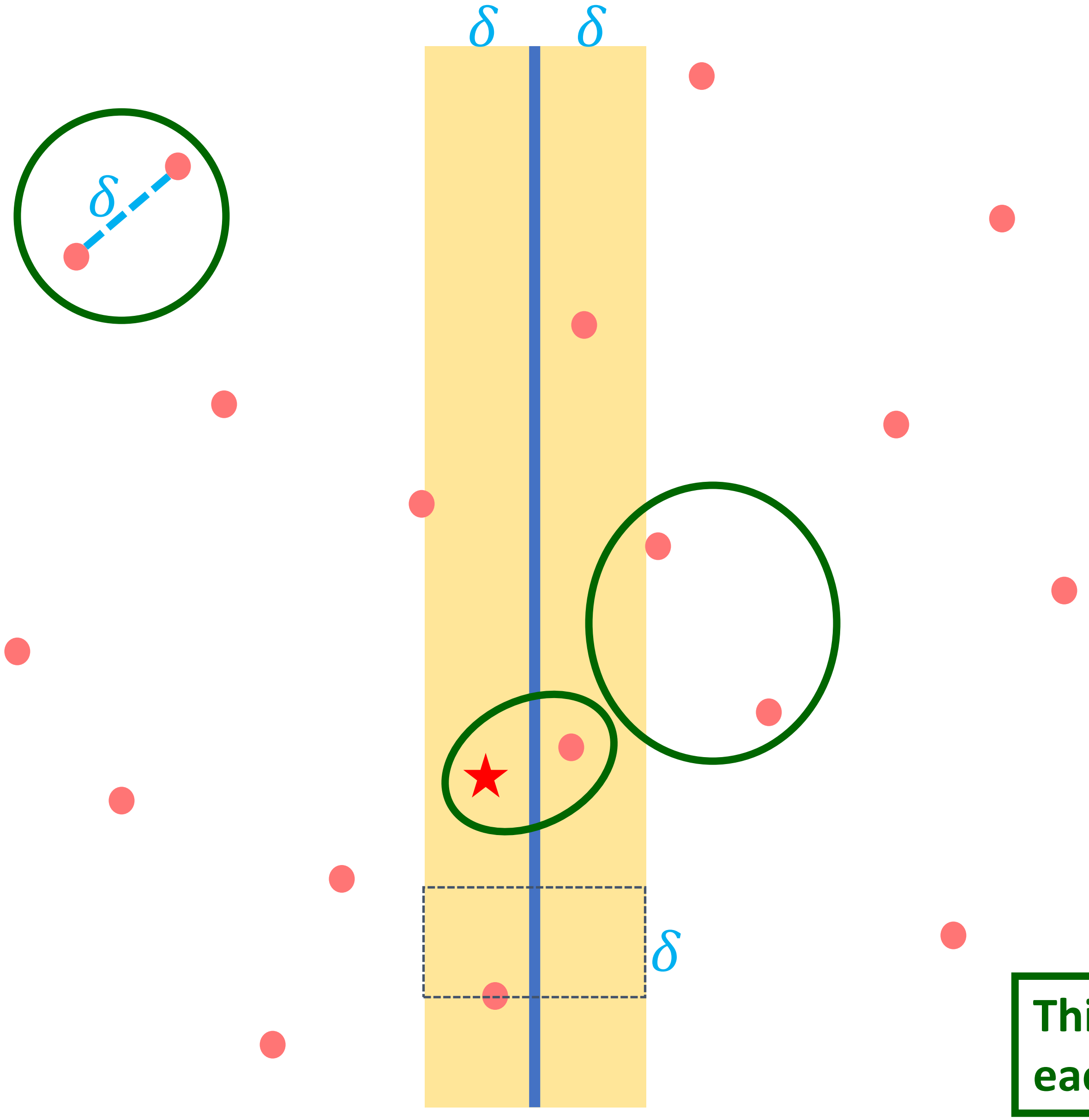
Combine:
- Find the closest pair crossing the middle
- Return the closest of the left, right, and crossing pairs

Observation:
- We only care about crossing pairs that might be closer than left and right
- Ignore points too far from the divide

Problem:
- We could still exceed our budget!

Solution:
- Re-apply the observation vertically!
- We only need to consider points within δ above the current point as well!

Finding the Closest Crossing Pair – 3rd Idea



Combine:

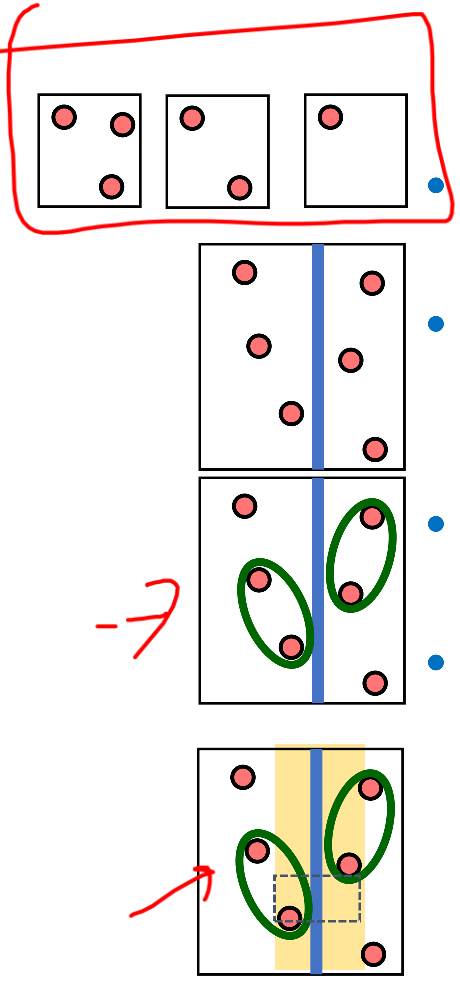
- Find the closest pair crossing the middle
- Return the closest of the left, right, and crossing pairs

Procedure:

- Let δ be the closest distance from left and right
- From bottom to top, for each point p_l on the left that's within δ of the divide on the left:
    - compare it to each point on the right that is within δ of the divide and no more than δ above p_l

This will only fit within our budget if we compare each p_l to a constant number of other points

Divide and Conquer (Closest Pair of Points)

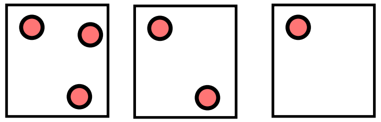


- **Preprocessing:**
  - Sort the points by x coordinate (call this list L_x)
  - Make a copy of the points and sort by y coordinate (call this list L_y)
- **Base Case:**
  - If there's 1 point then return ∞, If there's 2 or 3 points, solve naively
- **Divide:**
  - Find the median x coordinate
  - Partition L_x and L_y into the points on the left vs. right of the median
- **Conquer:**
  - Recursively find the closest pair from among the left and right of the median
- **Combine:**
  - Let δ be the closest from the left and the right solutions
  - Filter L_y to include only the points within δ of the median x
  - For each point p still in L_y:
    - For each point within δ of p vertically:
      - Compare p with that point and save if the distance is less than δ
  - Return minimum of the saved pair and the one used for δ

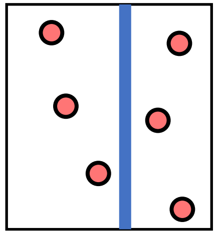
$n \log n$

Surprisingly, This works!

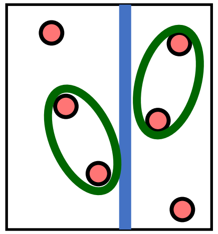
- **Preprocessing:**
  - Sort the points by x coordinate (call this list L_x)
  - Make a copy of the points and sort by y coordinate (call this list L_y)



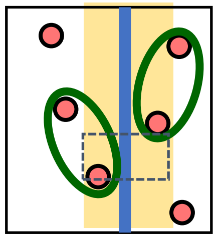
- **Base Case:**
  - If there's 1 point then return ∞, If there's 2 or 3 points, solve naively



- **Divide:**
  - Find the median x coordinate
  - Partition L_x and L_y into the points on the left vs. right of the median

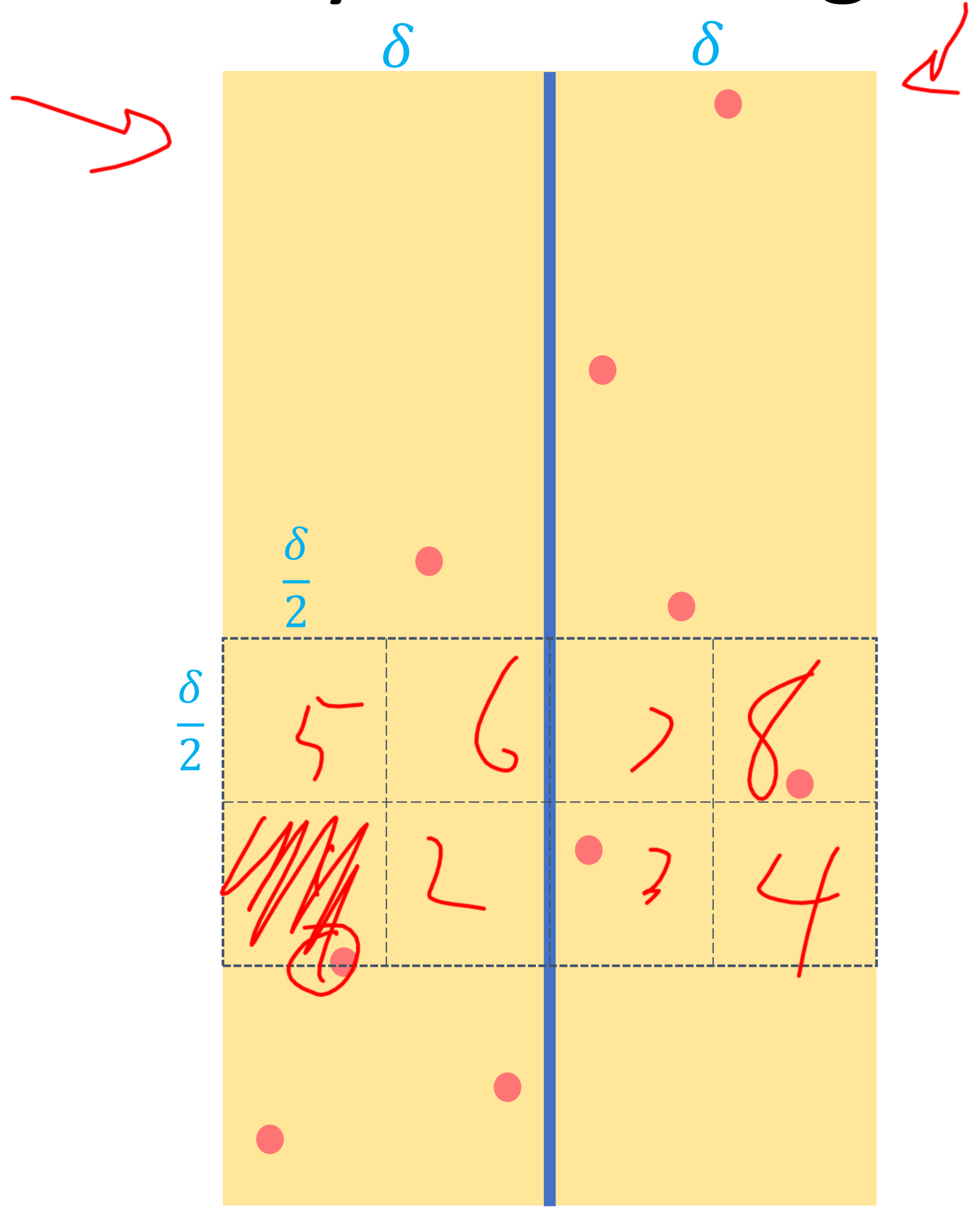


- **Conquer:**
  - Recursively find the closest pair from among the left and right of the median



- **Combine:**
  - Let δ be the closest from the left and the right solutions
  - Filter L_y to include only the points within δ of the median x
  - For each point p still in L_y:
    - **For the next 7 points vertically:**
      - Compare p with that point and save if the distance is less than δ
  - Return minimum of the saved pair and the one used for δ

Why is 7 enough?

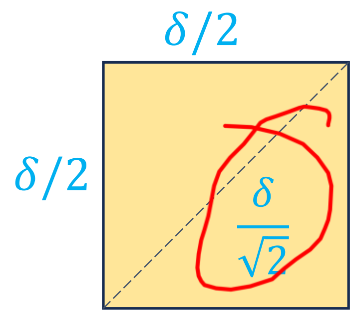


Claim:

- For any point p in the "strip", the 8th point above it is guaranteed to be more than δ away.

Proof:

- Consider a grid of $\frac{\delta}{2} \times \frac{\delta}{2}$ squares starting from p
- Any two points within the same square are at most $\frac{\delta}{\sqrt{2}}$ apart.



- Because $\sqrt{2} > 1$, we know that $\frac{\delta}{\sqrt{2}} < \delta$
- Therefore, there is at most one point per square
- Besides the one which contains p there are only 7 other squares within range δ

Full Algorithm

ClosestPair(L):
    $L_x = L$ sorted by x coordinate
    $L_y = L$ sorted by y coordinate
    return ClosestPairRec(L_x, L_y)

```
ClosestPairRec(L_x, L_y):
    # Base cases omitted
    m = median x coordinate
    P_x1 = the points from L_x to the left of the median
    P_y1 = the points from L_y to the left of the median
    P_x2 = the points from L_x to the right of the median
    P_y2 = the points from L_y to the right of the median
    a_1 = ClosestPairRec(P_x1, P_y1)
    a_2 = ClosestPairRec(P_x2, P_y2)
    a = closer of a_1 and a_2
    δ = distance(a)
    for each p in L_y:
        if p's x coordinate is more than δ from m:
            remove p from L_y
    for each p in L_y:
        for each of the next 7 points q in L_y:
            if distance(p, q):
                a = (p, q)
    return a
```