CSE 421 Winter 2025
Lecture 6: Greedy

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Greedy Algorithms

Hard to define exactly but can give general properties

- Solution is built in small steps
- Decisions on how to build the solution are made to maximize some criterion without looking to the future
  - Want the ‘best’ current partial solution as if the current step were the last step

May be more than one greedy algorithm using different criteria to solve a given problem

- Not obvious which criteria will actually work

Greedy Algorithms

- Greedy algorithms
  - Easy to describe
  - Fast running times
  - Work only on certain classes of problems
    - Hard part is showing that they are correct

- Focus on methods for proving that greedy algorithms do work

Interval Scheduling

Interval Scheduling:

- Single resource
- Reservation requests of form:

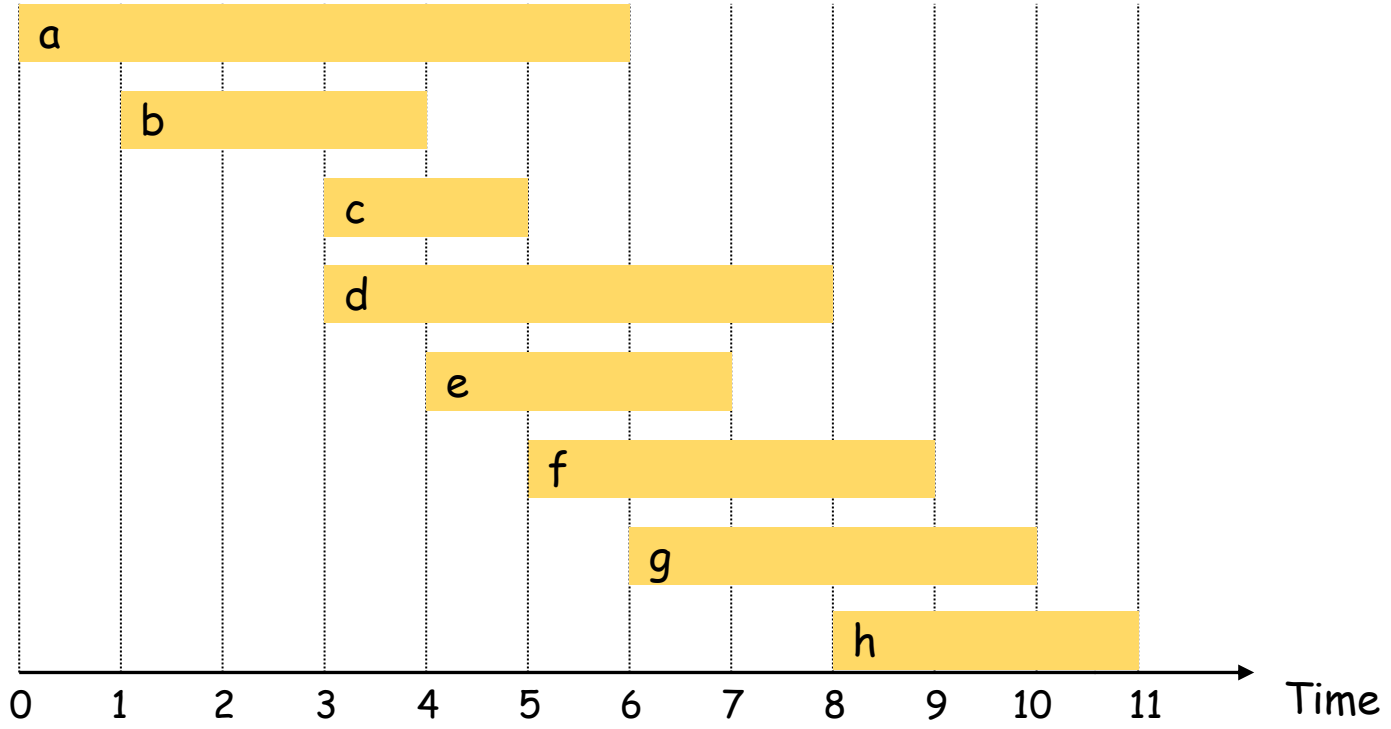
  "Can I reserve it from start time s to finish time f?"

$s < f$

Interval Scheduling

Interval scheduling:

- Job j starts at s_j and finishes at $f_j > s_j$.
- Two jobs i and j are compatible if they don't overlap: $f_i \leq s_j$ or $f_j \leq s_i$
- **Goal:** find maximum size subset of mutually compatible jobs.



Greedy Algorithms for Interval Scheduling

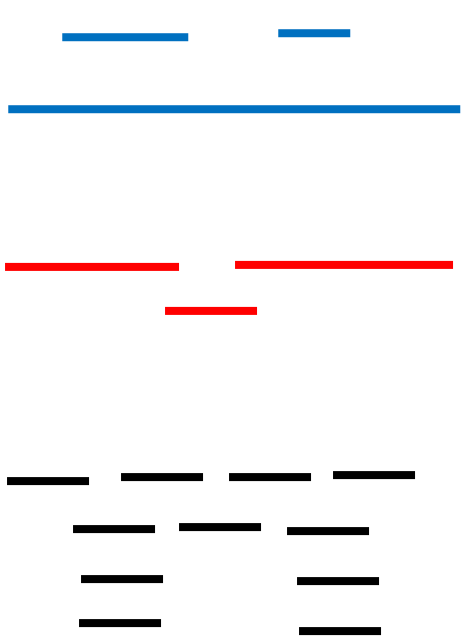
- What criterion should we try?

Greedy Algorithms for Interval Scheduling

- What criterion should we try?
  - Earliest start time s_i
  - Shortest request time $f_i - s_i$
  - Fewest conflicts

Greedy Algorithms for Interval Scheduling

- What criterion should we try?
  - Earliest start time s_i
    - Doesn't work
  - Shortest request time $f_i - s_i$
    - Doesn't work
  - Fewest conflicts
    - Doesn't work
  - Earliest finish time f_i
    - Works!



Greedy (by finish time) Algorithm for Interval Scheduling

$\boldsymbol{R}$ = set of all requests

$\boldsymbol{A} = \emptyset$

while $\boldsymbol{R} \neq \emptyset$ do:

Choose request $\boldsymbol{i} \in \boldsymbol{R}$ with smallest finish time $\boldsymbol{f_i}$

Add request $\boldsymbol{i}$ to $\boldsymbol{A}$

Delete all requests in $\boldsymbol{R}$ not compatible with request $\boldsymbol{i}$

return $\boldsymbol{A}$

Greedy Analysis Strategies

Greedy algorithm stays ahead: Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's

For interval scheduling: Show that after the greedy algorithm selects each interval, any alternative schedule's selection would have also been non-conflicting.

Conclusion: Each choice from the alternative selections can be swapped with a greedy choice, making greedy no worse off.

Interval Scheduling: Analysis

Claim: A is a compatible set of requests and
requests are added to A in order of finish time

- When we add a request to A we delete all incompatible ones from R

Name the finish times of requests in A as $a_1, a_2, \ldots, a_t$ in order.

Claim: Let $O \subseteq R$ be a set of compatible requests whose finish times in order are $o_1, o_2, \ldots, o_s$. Then for every integer $k \geq 1$ we have:

a) if O contains a k^{th} request then A does too, and

b) $a_k \leq o_k$ "A is ahead of O"

Note that a) alone implies that $t \geq s$ which means that A is optimal but we also need b) "stays ahead" to keep the induction going.

Inductive Proof of Claim

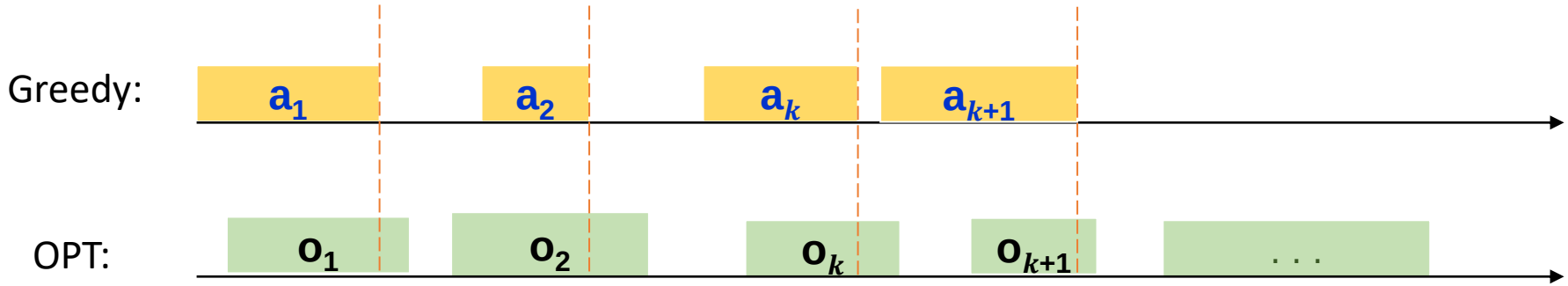
Base Case $k = 1$: A includes the request with smallest finish time, so if O is not empty then $a_1 \leq o_1$

Inductive Step: Suppose that $a_k \leq o_k$ and there is a $k+1^{st}$ request in O.

Then $k+1^{st}$ request in O is compatible with $a_1, a_2, \ldots, a_k$ since $a_k \leq o_k$ and $o_k \leq$ start time of $k+1^{st}$ request in O whose finish time is o_{k+1}

$\Rightarrow$ There is a $k+1^{st}$ request in A whose finish time is named a_{k+1}.

Also, since A would have considered both requests and chosen the one with the earlier finish time, $a_{k+1} \leq o_{k+1}$.



Interval Scheduling: Greedy Algorithm Implementation

```
Sort jobs by finish times so that 0 ≤ f1 ≤ f2 ≤ ... ≤ fn.

A = ϕ
last = 0
for j = 1 to n {
    if (last ≤ sj)
      A = A ∪ {j}
      last = fj
}
return A
```

$O(n \log n)$

$O(n)$

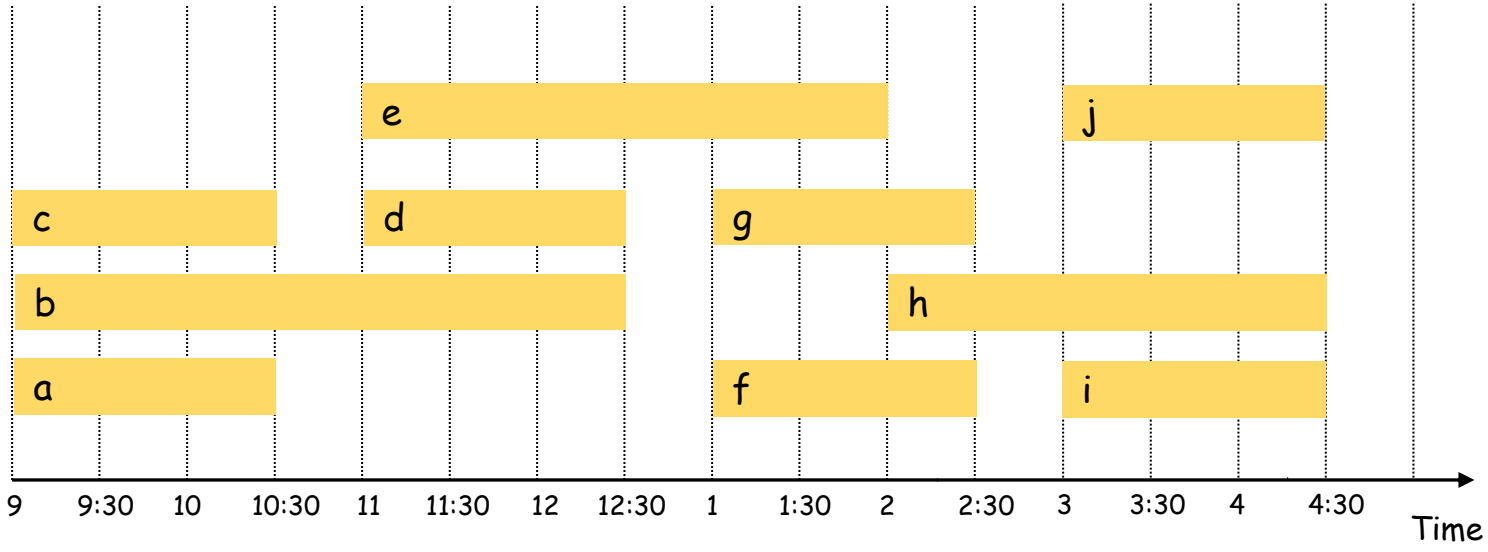
Scheduling All Intervals: Interval Partitioning

Interval Partitioning:

- Lecture j starts at s_j and finishes at f_j.

Goal: find minimum number of rooms to schedule all lectures so that no two occur at the same time in the same room.

Example: This schedule uses 4 rooms to schedule 10 lectures.



Can you do better?

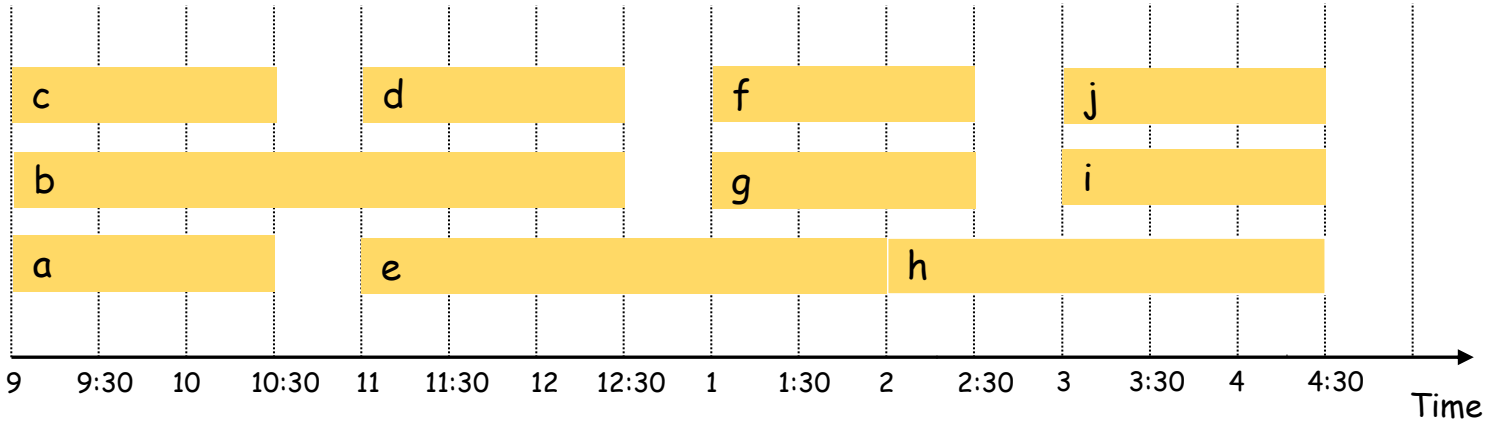
Scheduling All Intervals: Interval Partitioning

Interval Partitioning:

- Lecture j starts at s_j and finishes at f_j.

Goal: find minimum number of rooms to schedule all lectures so that no two occur at the same time in the same room.

Example: This schedule uses only 3 rooms.

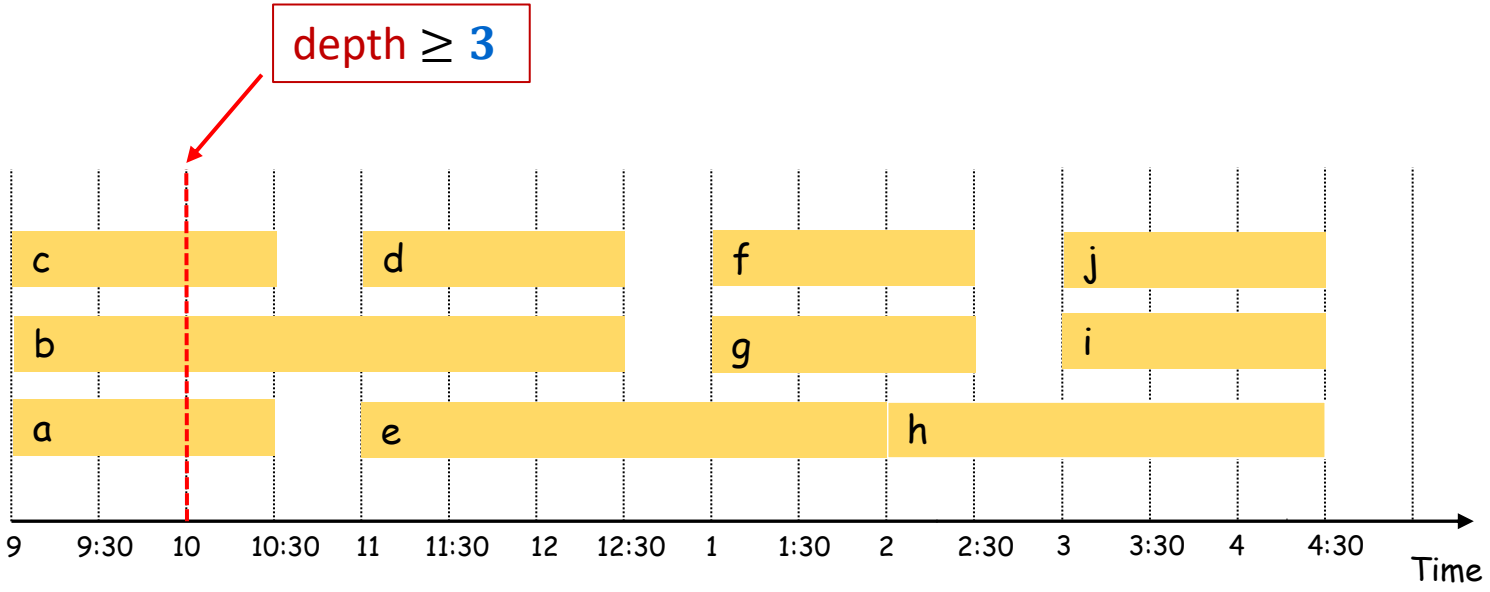


Scheduling All Intervals: Interval Partitioning

Defn: The depth of a set of open intervals is the maximum number that contain any given time.

Key observation: # of rooms needed $\geq$ depth.

Example: This schedule uses only $\mathbf{3}$ rooms. Since depth $\geq \mathbf{3}$ this is optimal.



A simple greedy algorithm

Sort requests in increasing order of start times $(s_1, f_1), \ldots, (s_n, f_n)$

$last_1 = 0$ // finish time of last request currently scheduled in room 1

for $i = 1$ to n {

  $j = 1$

  while (request i not scheduled) {

    if $s_i \geq last_j$ then

      schedule request i in room j

      $last_j = f_i$

    $j = j + 1$

    if $last_j$ undefined then $last_j = 0$

  }

}

Look for the first room where the request will fit, opening a new room if all the others used so far are full.

Interval Partitioning: Greedy Analysis

Observation: Greedy algorithm never schedules two incompatible lectures in the same room

- Only schedules request i in room j if $s_i \geq last_j$

Theorem: Greedy algorithm is optimal.

Proof:

Let d = number of rooms that the greedy algorithm allocates.

- Room d is allocated because we needed to schedule a request, say j, that is incompatible with some request in each of the other $d - 1$ rooms.
- Since we sorted by start time, these incompatibilities are caused by requests that start no later than s_j and finish after s_j.

So… we have d requests overlapping at time $s_j + \varepsilon$ for some (maybe tiny) $\varepsilon > 0$.

Key observation $\Rightarrow$ all schedules use $\geq d$ rooms.

A simple greedy algorithm

Runtime analysis

Sort requests in increasing order of start times $(s_1, f_1), \ldots, (s_n, f_n)$

$O(n \log n)$

$last_1 = 0$ // finish time of last request currently scheduled in room 1

for $i = 1$ to n {

  $j \leftarrow = 1$

  while (request i not scheduled) {

    if $s_i \geq last_j$ then

      schedule request i in room j

      $last_j = f_i$

    $j = j + 1$

    if $last_j$ undefined then $last_j = 0$

  }

}

Might need to try all d rooms to schedule a request

$O(n\,d)$

d might be as big as n

Worst case $\Theta(n^2)$

A more efficient implementation: Priority queue

$O(n \log n)$

Sort requests in increasing order of start times $(s_1, f_1), \ldots, (s_n, f_n)$

$d = 1$
schedule request 1 in room 1
$last_1 = f_1$ $O(1)$
insert 1 into priority queue Q with key = $last_1$
for $i = 2$ to n { $O(n \log d)$
  j = findmin(Q)
  if $s_i \geq last_j$ then { $O(\log d)$
    schedule request i in room j
    $last_j = f_i$
    increasekey(j,Q) to $last_j$ }
  else {
    $d = d + 1$ $O(\log d)$
    schedule request i in room d
    $last_d = f_i$
    insert d into priority queue Q with key = $last_d$ }
}

$\Theta(n \log n)$ total

Greedy Analysis Strategies

Greedy algorithm stays ahead: Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's

Structural: Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

Exchange argument: Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

Scheduling to Minimize Lateness

Scheduling to minimize lateness:

- Single resource as in interval scheduling but, instead of start and finish times, request i has
  - Time requirement t_i which must be scheduled in a contiguous block
  - Target deadline d_i by which time the request would like to be finished
- Overall start time s for all jobs

Requests are scheduled by the algorithm into time intervals $[s_i, f_i]$ s.t. $t_i = f_i - s_i$

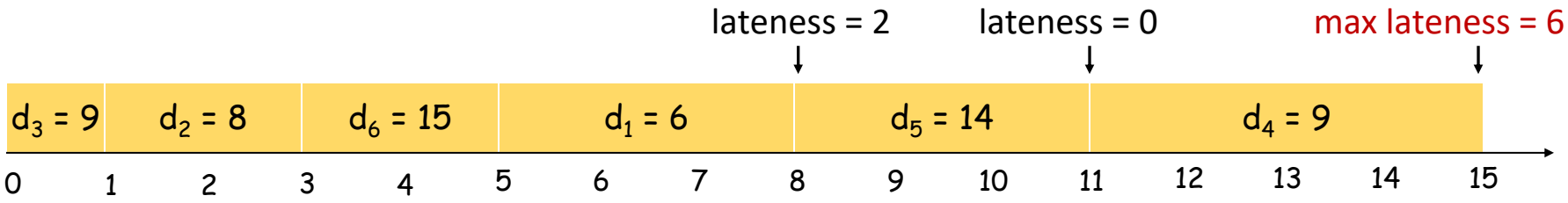
- Lateness of schedule for request i is
  - If $f_i > d_i$ then request i is late by $L_i = f_i - d_i$; otherwise its lateness $L_i = 0$
- Maximum lateness $L = \max_i Li$

Goal: Find a schedule for **all** requests (values of s_i and f_i for each request i) to minimize the maximum lateness, L.

Scheduling to Minimizing Lateness

- Example:

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Minimizing Lateness: Greedy Algorithms

Greedy template: Consider jobs in some order.

[Shortest processing time first] Consider jobs in ascending order of processing time t_j.

[Earliest deadline first] Consider jobs in ascending order of deadline d_j.

[Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

Minimizing Lateness: Greedy Algorithms

Greedy template: Consider jobs in some order.

[Shortest processing time first] Consider jobs in ascending order of processing time t_j.

	1	2
t_j	1	10
d_j	100	10

counterexample

Will schedule 1 (length 1) before 2 (length 10).
2 can only be scheduled at time 1
1 will finish at time 11 >10. Lateness 1.
Lateness 0 possible If 1 goes last.

[Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

	1	2
t_j	1	10
d_j	2	10

counterexample

Will schedule 2 (slack 0) before 1 (slack 1).
1 can only be scheduled at time 10
1 will finish at time 11 >10. Lateness 9.
Lateness 1 possible if 1 goes first.

Minimizing Lateness: Greedy Algorithms

Greedy template: Consider jobs in some order.

[Earliest deadline first] Consider jobs in ascending order of deadline d_j.

Greedy Algorithm: Earliest Deadline First

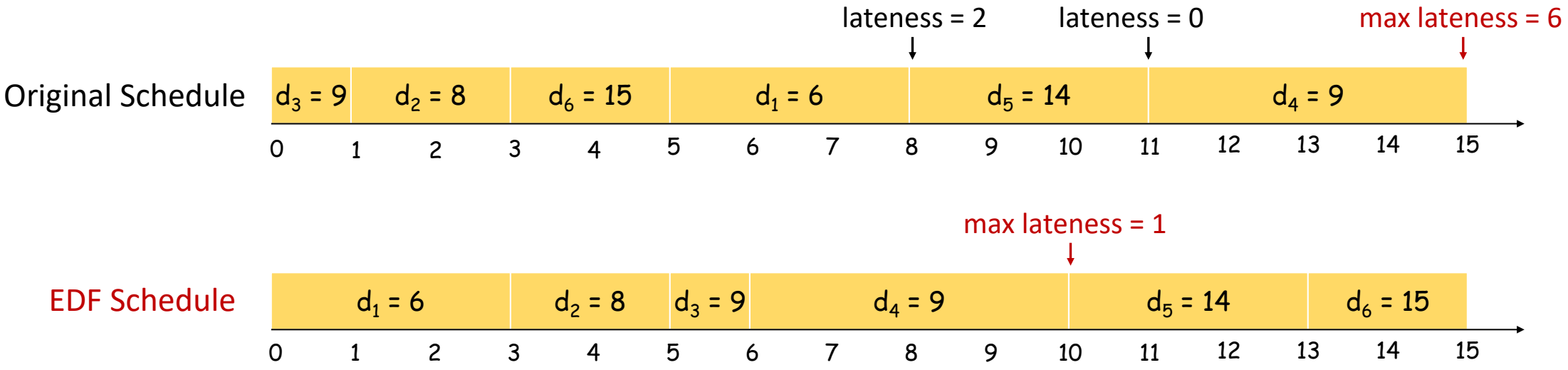
Consider requests in increasing order of deadlines

Schedule the request with the earliest deadline as soon as the resource is available

Scheduling to Minimizing Lateness

- Example:

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Proof for Greedy EDF Algorithm: Exchange Argument

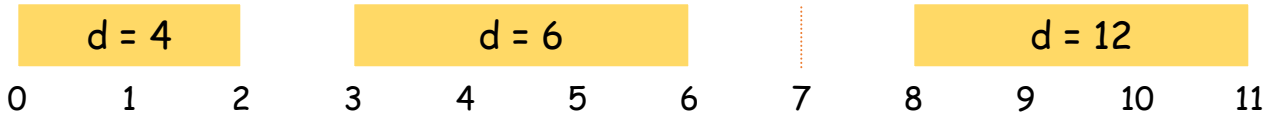
Show that if there is another schedule **O** (think optimal schedule) then we can gradually change **O** so that…

- at each step the maximum lateness in **O** never gets worse
- it eventually becomes the same cost as **A**

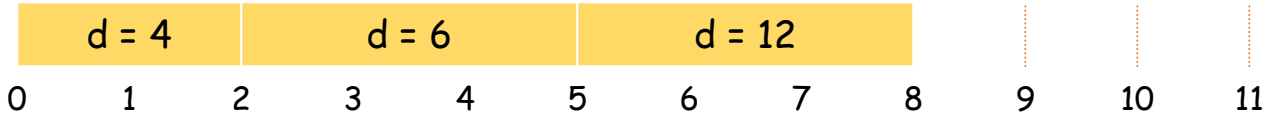
This means that **A** is at least as good as **O**, so **A** is also optimal!

Minimizing Lateness: No Idle Time

Observation: There exists an optimal schedule with no idle time



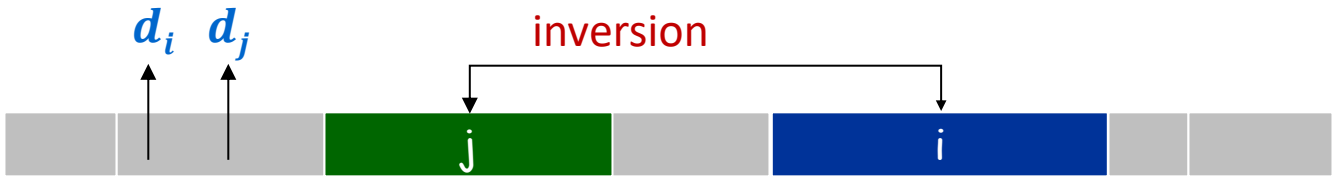
At least as good



Observation: The greedy EDF schedule has no idle time.

Minimizing Lateness: Inversions

Defn: An inversion in schedule S is a pair of jobs i and j such that $d_i < d_j$ but j is scheduled before i.



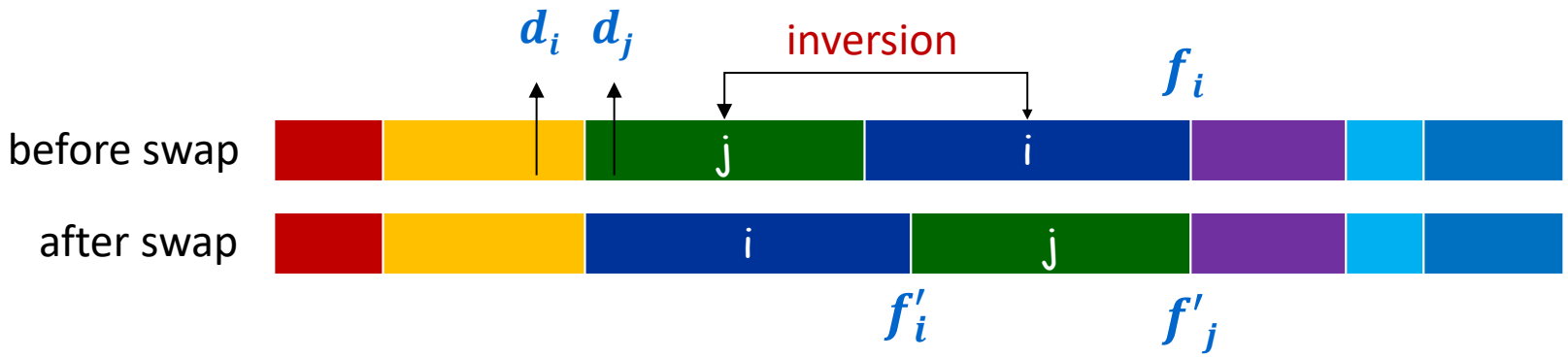
Observation: Greedy EDF schedule has no inversions.

Observation: If schedule S (with no idle time) has an inversion it has two adjacent jobs that are inverted

- Any job in between would be inverted w.r.t. one of the two ends

Minimizing Lateness: Inversions

Defn: An inversion in schedule S is a pair of jobs i and j such that $d_i < d_j$ but j is scheduled before i.

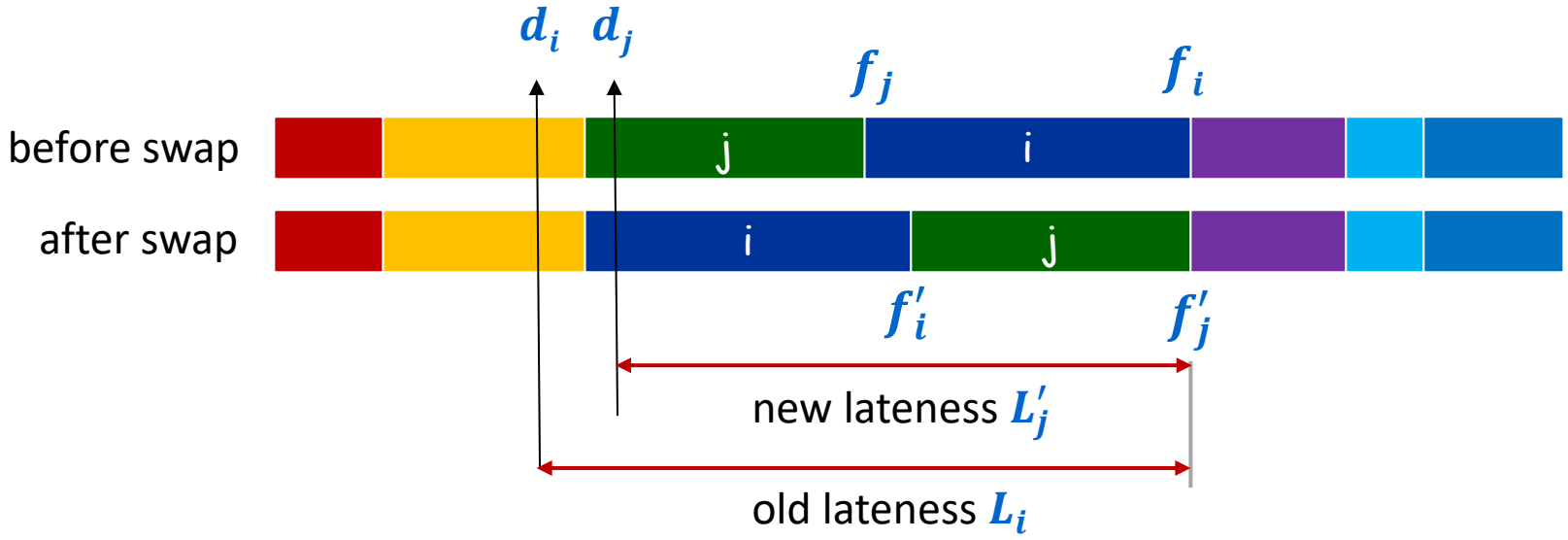


Claim: Swapping two adjacent, inverted jobs

- reduces the # of inversions by **1**
- does not increase the max lateness.

Minimizing Lateness: Inversions

Defn: An inversion in schedule S is a pair of jobs i and j such that $d_i < d_j$ but j is scheduled before i.



Claim: Maximum lateness does not increase

Optimal schedules and inversions

Claim: There is an optimal schedule with no idle time and no inversions

Proof:

By previous argument there is an optimal schedule **O** with no idle time

If **O** has an inversion then it has an **adjacent** pair of requests in its schedule that are inverted and can be swapped without increasing lateness

… we just need to show one more claim that eventually this swapping stops

Optimal schedules and inversions

Claim: Eventually these swaps will produce an optimal schedule with no inversions.

Proof:

Each swap decreases the # of inversions by $\mathbf{1}$

There are a bounded # of inversions possible in the worst case

- at most $\boldsymbol{n(n-1)/2}$ but we only care that this is finite.

The # of inversions can't be negative so this must stop.

Idleness and Inversions are the only issue

Claim: All schedules with no inversions and no idle time have the same maximum lateness.

Proof:

Schedules can differ only in how they order requests with equal deadlines

Consider all requests having some common deadline $\boldsymbol{d}$.

- Maximum lateness of these jobs is based only on finish time of the last one …

and the set of these requests occupies the same time segment in both schedules.

⇒ The last of these requests finishes at the same time in any such schedule.

Earliest Deadline First is optimal

We know that

- There is an optimal schedule with no idle time or inversions
- All schedules with no idle time or inversions have the same maximum lateness
- EDF produces a schedule with no idle time or inversions

So …

- EDF produces an optimal schedule

Greedy Analysis Strategies

Greedy algorithm stays ahead: Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's

Structural: Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

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