CSE 421 Winter 2025
Lecture 5: Graph Search and Greedy

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Graph Traversal

Learn the basic structure of a graph

Walk from a fixed starting vertex $\boldsymbol{s}$ to find all vertices reachable from $\boldsymbol{s}$

Three states of vertices

- **unvisited**
- **visited/discovered** (in $\boldsymbol{R}$, i.e. reachable)
- **fully-explored** (in $\boldsymbol{R}$ and all neighbors have been visited)

BFS(s)

Global initialization: mark all vertices "unvisited"

BFS(s)

    Mark s "visited"
    Add s to Q
    i = 0
    Mark s as "layer i"
    while Q not empty
        u = next item removed from Q
        i = "layer of u"
        for each edge (u, v)
            if (v is "unvisited")
                add v to Q
                mark v "visited"
                mark v as "layer $i + 1$"

        mark u "fully-explored"

Properties of BFS

BFS(s) visits x iff there is a path in G from s to x.

Edges followed to undiscovered vertices define a
breadth first spanning tree of G

Layer i in this tree:

L_i = set of vertices u with shortest path in G from root s of length i.

Properties of BFS

Claim: For undirected graphs:

All edges join vertices on the same or adjacent layers of BFS tree

Proof: Suppose not...

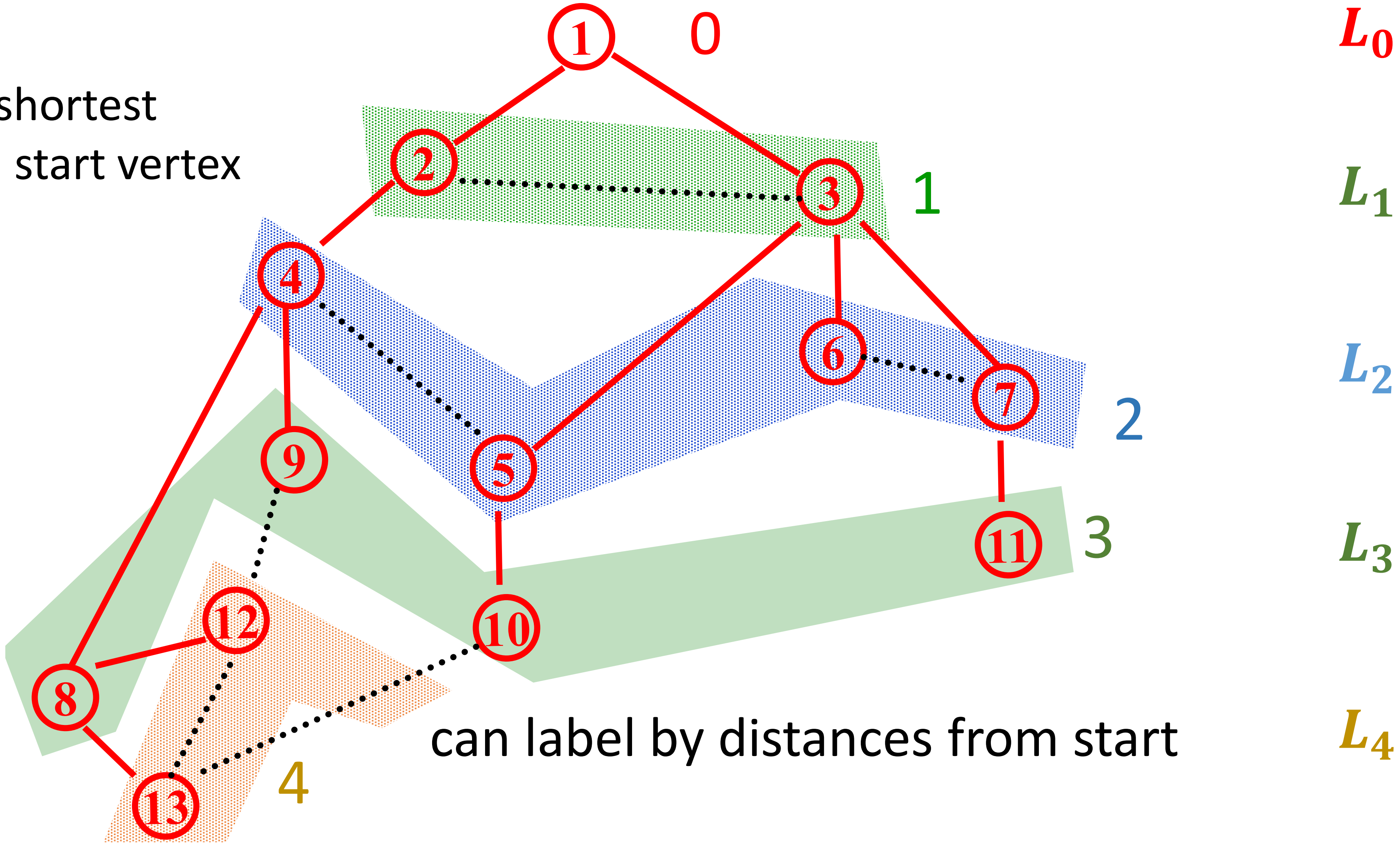
Then there would be vertices $(\boldsymbol{x}, \boldsymbol{y})$ s.t. $\boldsymbol{x} \in \boldsymbol{L_i}$ and $\boldsymbol{y} \in \boldsymbol{L_j}$ and $\boldsymbol{j} > \boldsymbol{i} + \mathbf{1}$.

Then, when vertices adjacent to $\boldsymbol{x}$ are considered in BFS,
$\boldsymbol{y}$ would be added with layer $\boldsymbol{i} + \mathbf{1}$ and not layer $\boldsymbol{j}$.

Contradiction.

BFS Application: Shortest Paths

Tree gives shortest paths from start vertex



can label by distances from start

Applications of Graph Traversal: Bipartiteness Testing

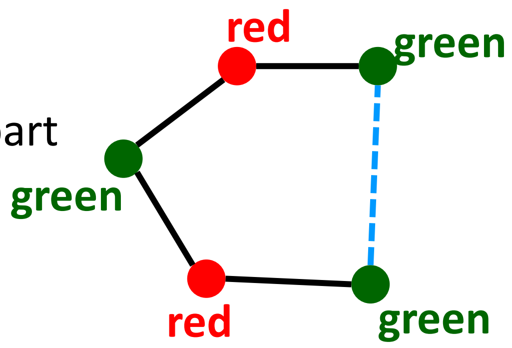
Definition: An undirected graph G is bipartite iff we can color its vertices **red** and **green** so each edge has different color endpoints

Input: Undirected graph G

Goal: If G is bipartite, output a coloring;
otherwise, output "NOT Bipartite".

Fact: Graph G contains an odd-length cycle $\Rightarrow$ it is not bipartite

Just coloring the cycle part of G is impossible



On a cycle the two colors must alternate, so

- **green** every 2nd vertex
- **red** every 2nd vertex

Can't have either if length is not divisible by 2.

Applications of Graph Traversal: Bipartiteness Testing

WLOG ("without loss of generality"): Can assume that G is connected

- Otherwise run on each component

Simple idea: start coloring nodes starting at a given node s

- Color s **red**
- Color all neighbors of s **green**
- Color all their neighbors **red**, etc.
- If you ever hit a node that was already colored
  - the **same** color as you want to color it, ignore it
  - the **opposite** color, output "NOT Bipartite" and halt

BFS gives Bipartiteness

Run BFS assigning all vertices from layer $\boldsymbol{L_i}$ the color $\boldsymbol{i} \bmod \mathbf{2}$

- i.e., **red** if they are in an even layer, **green** if in an odd layer

- if no edge joining two vertices of the same color
  - then it is a good coloring
- otherwise
  - there is a bad edge; output "Not Bipartite"

Why is that "Not Bipartite" output correct?

Why does BFS work for Bipartiteness?

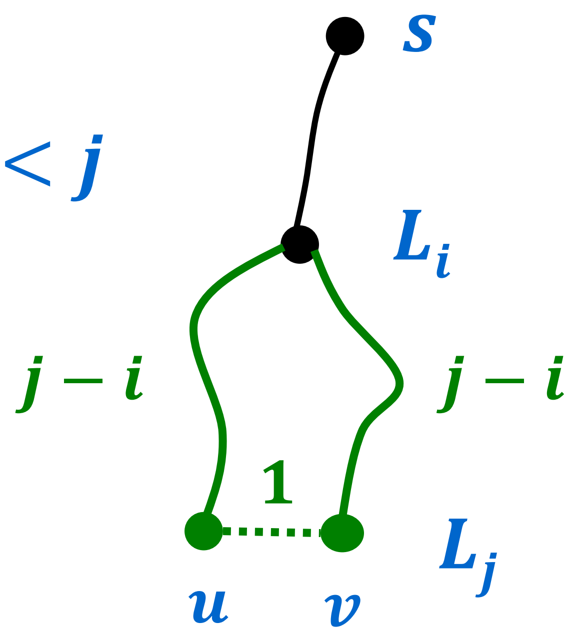
Recall: All edges join vertices on the same or adjacent BFS layers

⇒ Any "bad" edge must join two vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ in the same layer

Say the layer with $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{L_j}$

$\boldsymbol{u}$ and $\boldsymbol{v}$ have common ancestor at some level $\boldsymbol{L_i}$ for $\boldsymbol{i < j}$

Odd cycle of length $\mathbf{2(j-i)+1}$
⇒ Not Bipartite



Undirected Graph Search Application: Connected Components

Want to answer questions of the form:

Given: vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{G}$

Is there a path from $\boldsymbol{u}$ to $\boldsymbol{v}$?

Idea: create array $\boldsymbol{A}$ s.t

$\boldsymbol{A}[\boldsymbol{u}]$ = smallest numbered vertex connected to $\boldsymbol{u}$

Answer is yes iff $\boldsymbol{A}[\boldsymbol{u}] = \boldsymbol{A}[\boldsymbol{v}]$

Q: Why is this better than an array **Path**$[\boldsymbol{u}, \boldsymbol{v}]$?

Undirected Graph Search Application: Connected Components

Initial state: all $\boldsymbol{v}$ unvisited
for $\boldsymbol{s}$ from 1 to $\boldsymbol{n}$ do:
  if state($\boldsymbol{s}$) $\neq$ fully-explored then
    BFS($\boldsymbol{s}$): setting $\boldsymbol{A}[\boldsymbol{u}] = \boldsymbol{s}$ for each $\boldsymbol{u}$ found
      (and marking $\boldsymbol{u}$ visited/fully-explored)

Total cost: $O(\boldsymbol{n} + \boldsymbol{m})$

- Each vertex is touched once in outer procedure and edges examined in different BFS runs are disjoint
- Works also with Depth First Search ...

DFS($\boldsymbol{u}$) – Recursive Procedure

Global Initialization: mark all vertices "unvisited"

DFS($\boldsymbol{u}$)

mark $\boldsymbol{u}$ “visited” and add $\boldsymbol{u}$ to $\boldsymbol{R}$

for each edge $(\boldsymbol{u}, \boldsymbol{v})$

if ($\boldsymbol{v}$ is “unvisited”)

DFS($\boldsymbol{v}$)

mark $\boldsymbol{u}$ “fully-explored”

Properties of DFS($\boldsymbol{s}$)

Like BFS($\boldsymbol{s}$):

- DFS($\boldsymbol{s}$) visits $\boldsymbol{x}$ iff there is a path in $\boldsymbol{G}$ from $\boldsymbol{s}$ to $\boldsymbol{x}$
- Edges into undiscovered vertices define depth-first spanning tree of $\boldsymbol{G}$

Unlike the BFS tree:

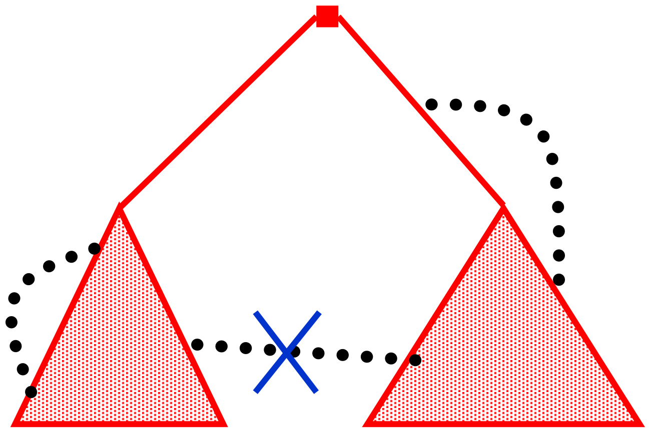
- the DFS spanning tree *isn't* minimum depth
- its levels *don't* reflect min distance from the root
- non-tree edges *never* join vertices on the same or adjacent levels

BUT…

Non-tree edges in DFS tree of **undirected** graphs

Claim: All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree

- In other words ... No "cross edges".



No cross edges in DFS on undirected graphs

Claim: During DFS($\boldsymbol{x}$) every vertex marked "visited" is a descendant of $\boldsymbol{x}$ in the DFS tree $\boldsymbol{T}$

Claim: For every $\boldsymbol{x}, \boldsymbol{y}$ in the DFS tree $\boldsymbol{T}$, if $(\boldsymbol{x}, \boldsymbol{y})$ is an edge *not* in $\boldsymbol{T}$ then one of $\boldsymbol{x}$ or $\boldsymbol{y}$ is an ancestor of the other in $\boldsymbol{T}$

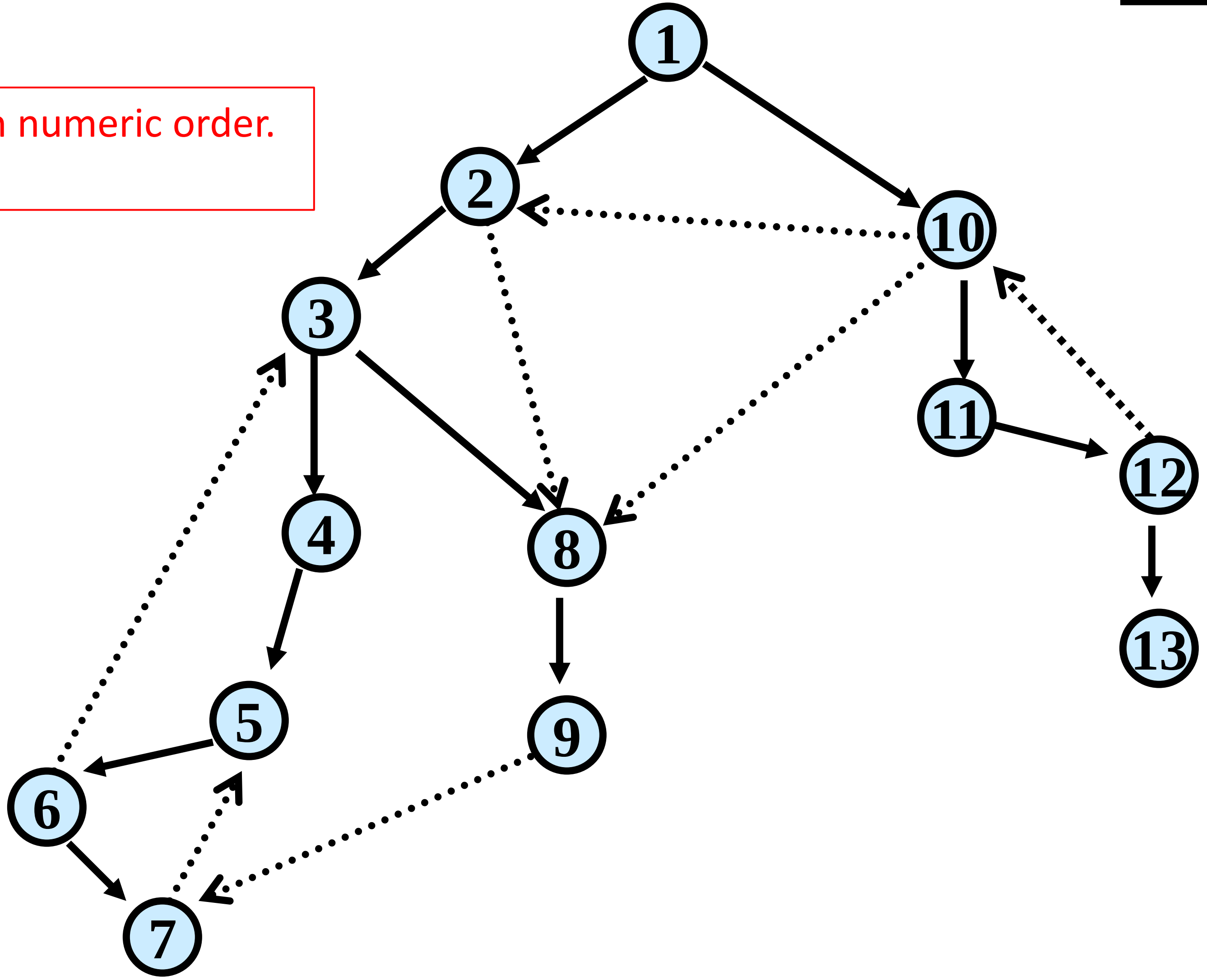
Proof:

- One of DFS($\boldsymbol{x}$) or DFS($\boldsymbol{y}$) is called first, suppose WLOG that DFS($\boldsymbol{x}$) was called before DFS($\boldsymbol{y}$)
- During DFS($\boldsymbol{x}$), the edge $(\boldsymbol{x}, \boldsymbol{y})$ is examined
- Since $(\boldsymbol{x}, \boldsymbol{y})$ is a *not* an edge of $\boldsymbol{T}$, $\boldsymbol{y}$ was already visited when edge $(\boldsymbol{x}, \boldsymbol{y})$ was examined during DFS($\boldsymbol{x}$)
- Therefore $\boldsymbol{y}$ was visited during the call to DFS($\boldsymbol{x}$) so $\boldsymbol{y}$ is a descendant of $\boldsymbol{x}$.

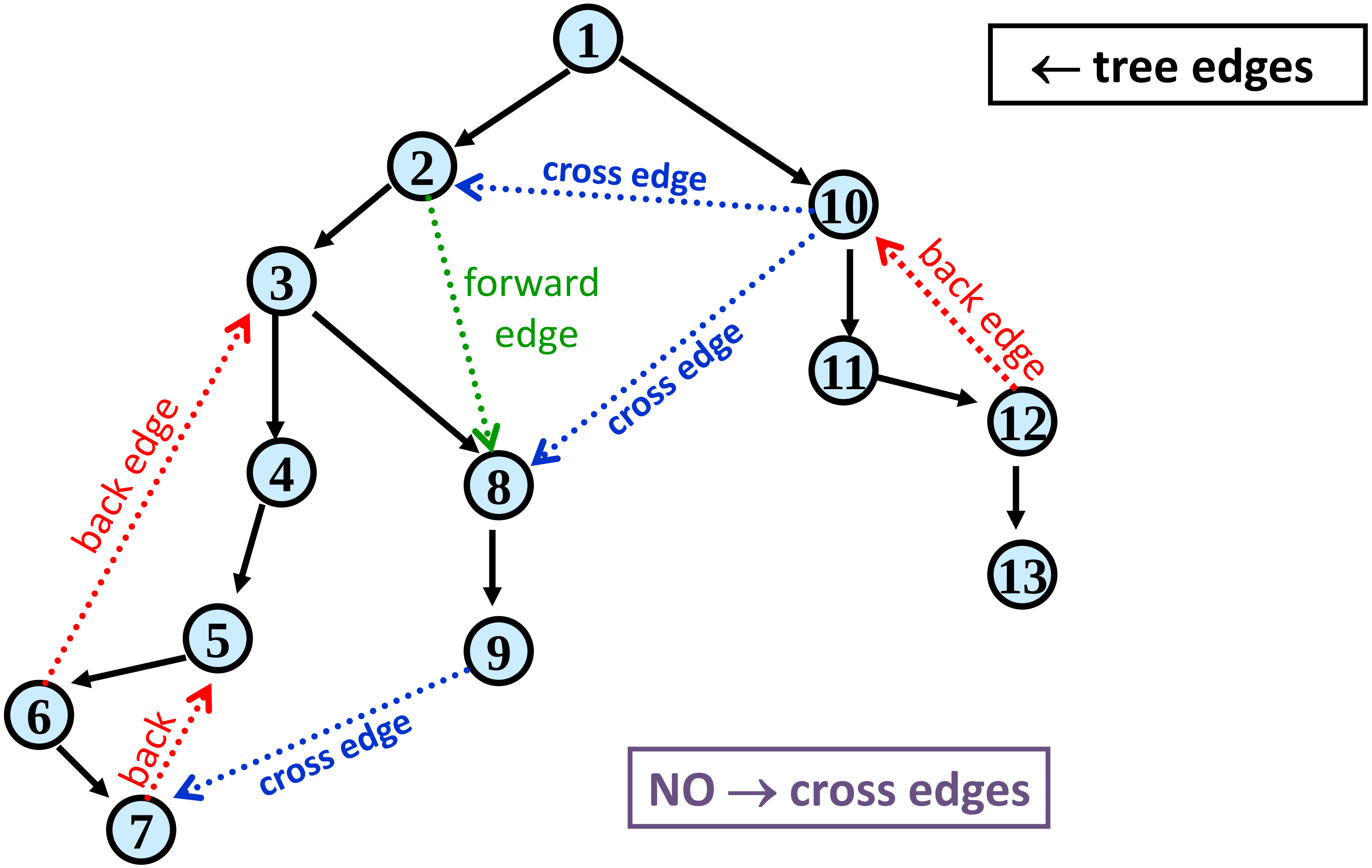
DFS($\boldsymbol{v}$) for a directed graph

← tree edges

Nodes were visited in numeric order.
How can we tell?



DFS($\boldsymbol{v}$)



Properties of Directed DFS

- Before DFS($\boldsymbol{s}$) returns, it visits all previously unvisited vertices reachable via directed paths from $\boldsymbol{s}$

- Every cycle contains a back edge in the DFS tree

Directed Acyclic Graphs

A directed graph $G = (V, E)$ is acyclic iff it has no directed cycles

Terminology: A directed acyclic graph is also called a DAG

After shrinking the strongly connected components of a directed graph to single vertices, the result is a DAG

Topological Sort

Given: a directed acyclic graph (DAG) $\boldsymbol{G = (V, E)}$

Output: numbering of the vertices of $\boldsymbol{G}$ with distinct numbers from $\mathbf{1}$ to $\boldsymbol{n}$ so that edges only go from lower numbered to higher numbered vertices

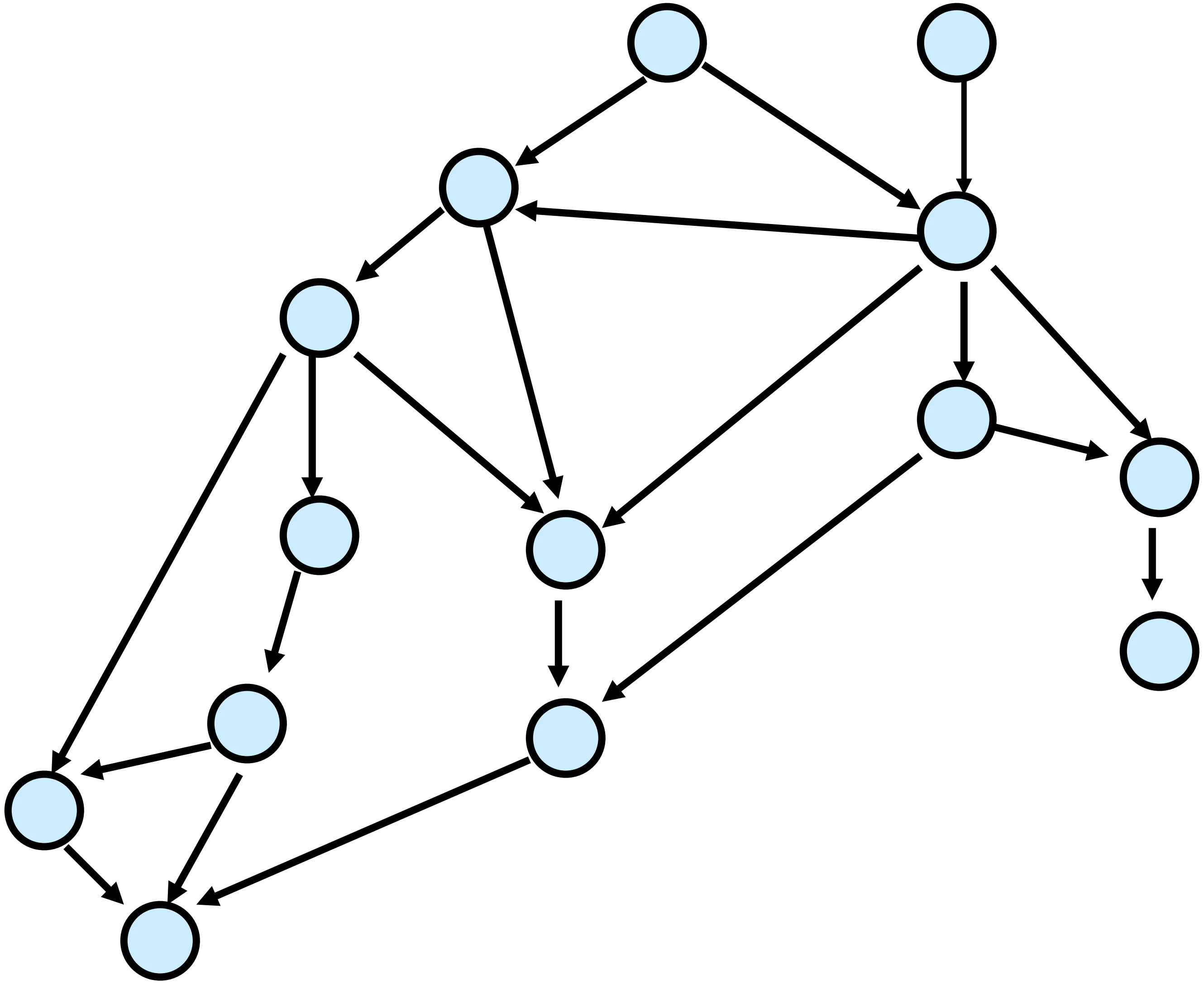
Applications:

- nodes represent tasks
- edges represent precedence between tasks
- topological sort gives a sequential schedule for solving them

Nice algorithmic paradigm for general directed graphs:

- Process strongly connected components one-by-one in the order given by topological sort of the DAG you get from shrinking them.

Directed Acyclic Graph



In-degree 0 vertices

Claim: Every DAG has a vertex of in-degree 0

Proof: By contradiction

Suppose every vertex has some incoming edge

Consider following procedure:

while (true) do

$\boldsymbol{v}$ = some predecessor of $\boldsymbol{v}$

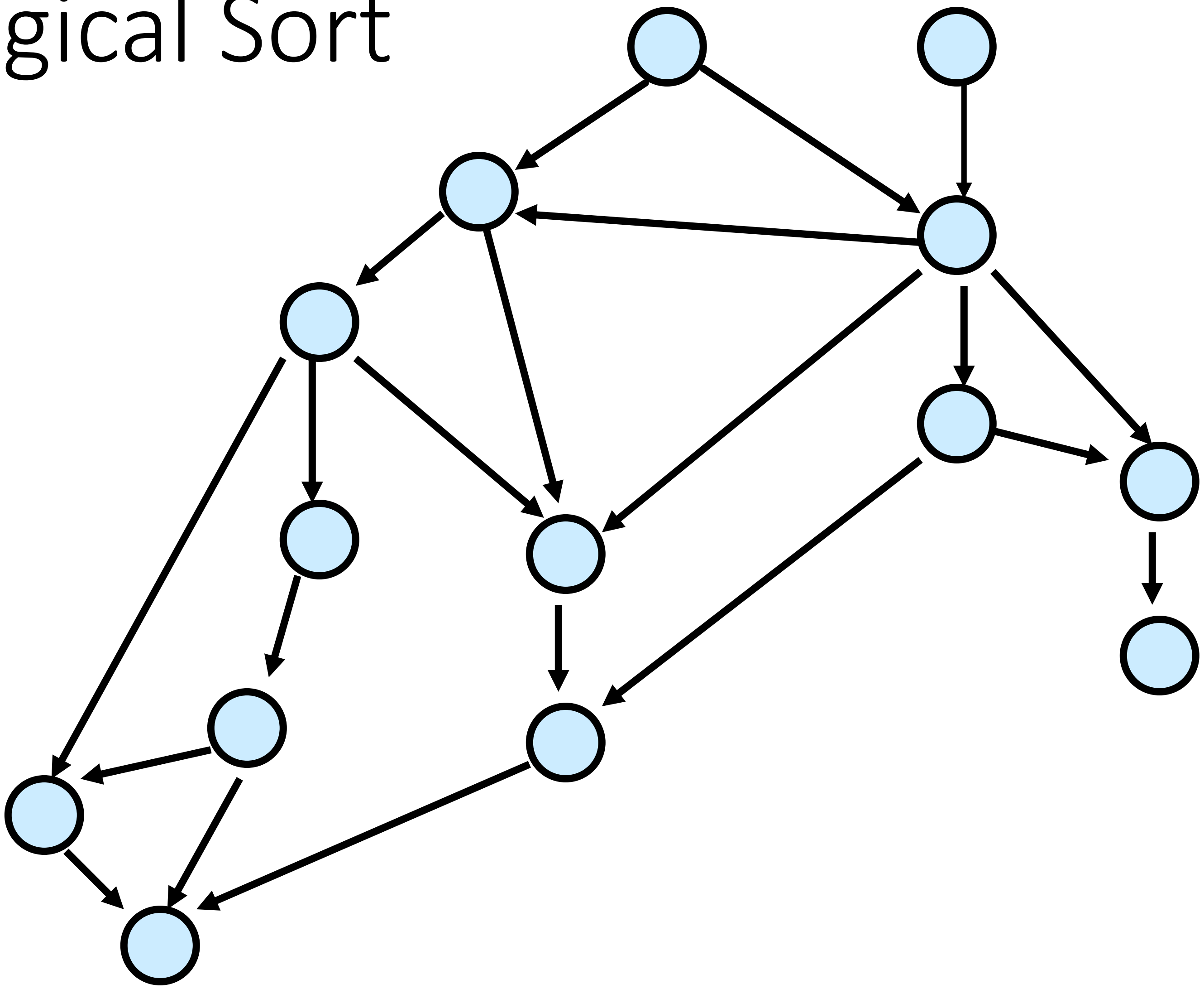
- After $\boldsymbol{n+1}$ steps where $\boldsymbol{n = |V|}$ there will be a repeated vertex
  - This yields a cycle, contradicting that it is a DAG.

Topological Sort

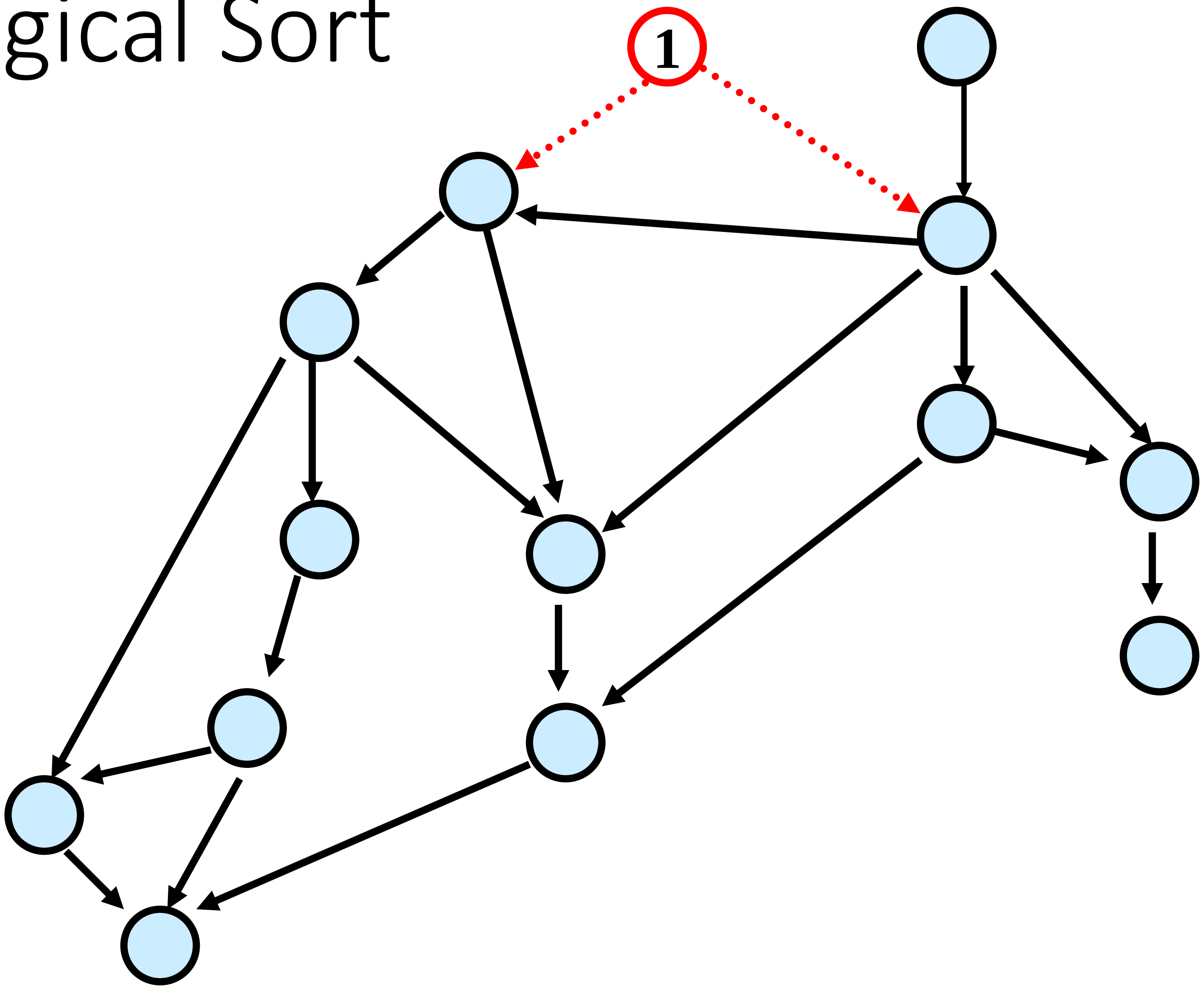
- Can do using DFS

- Alternative simpler idea:
  - Any vertex of in-degree 0 can be given number 1 to start
  - Remove it from the graph
  - Then give a vertex of in-degree 0 number 2
  - Etc.

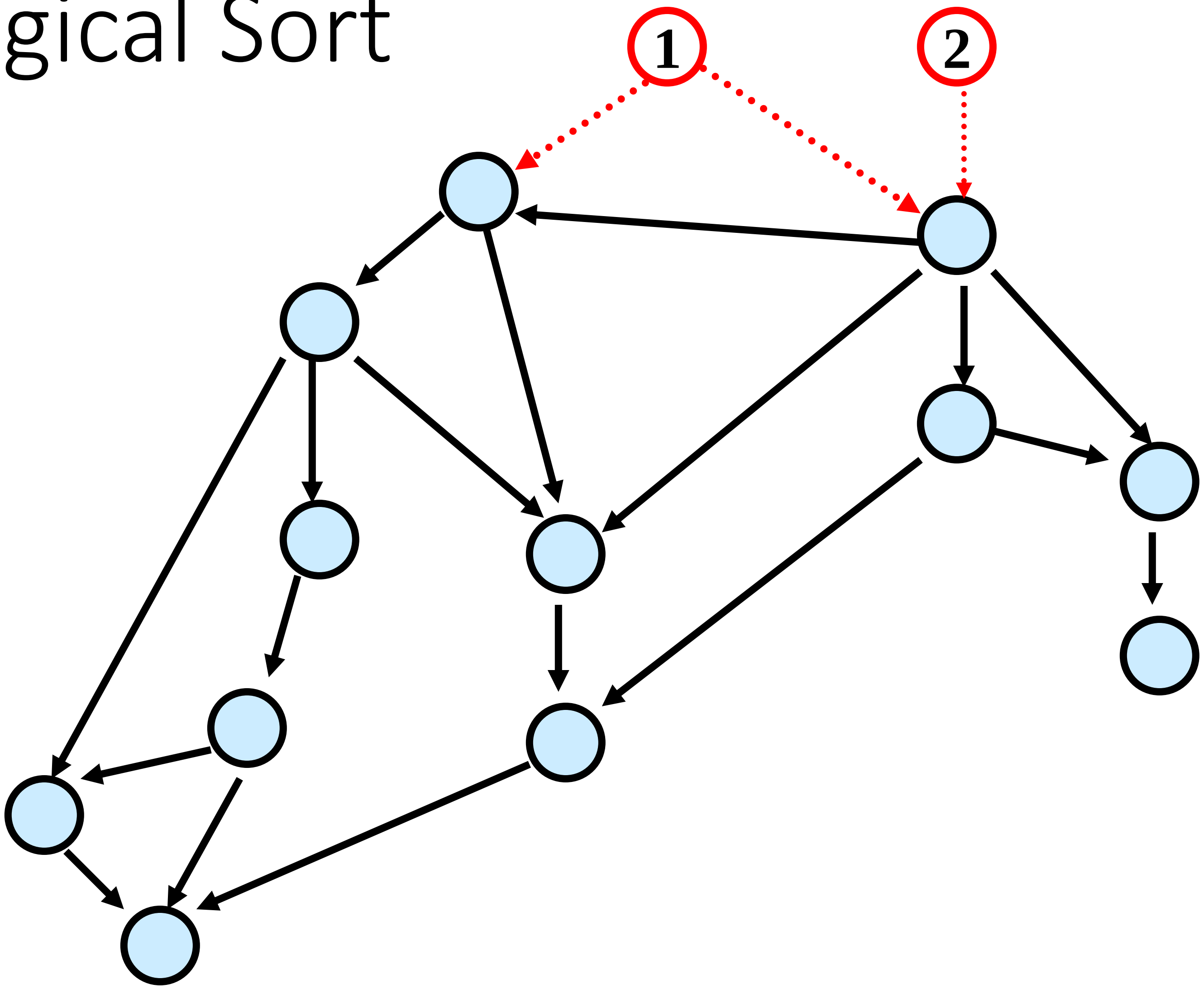
Topological Sort



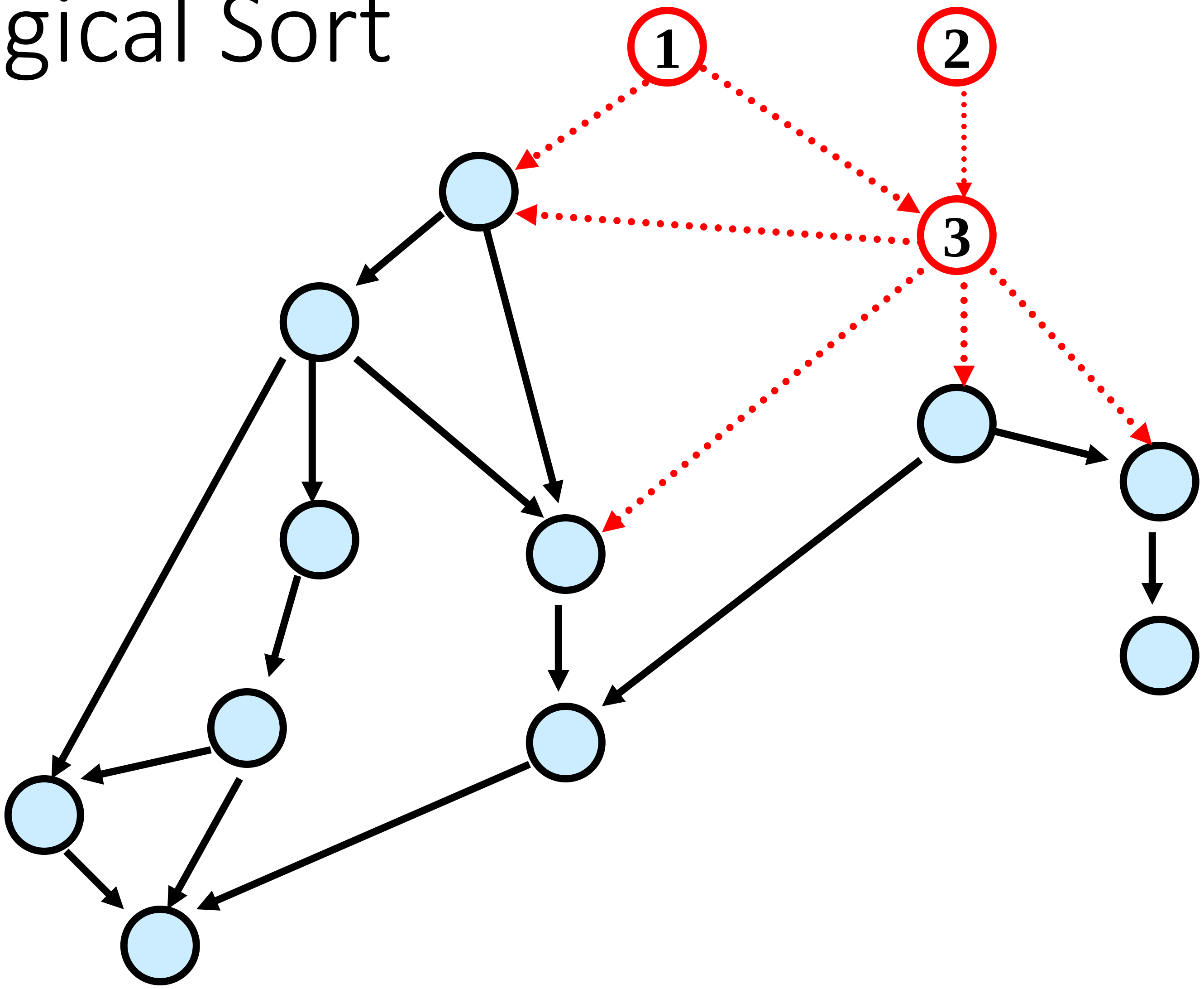
Topological Sort



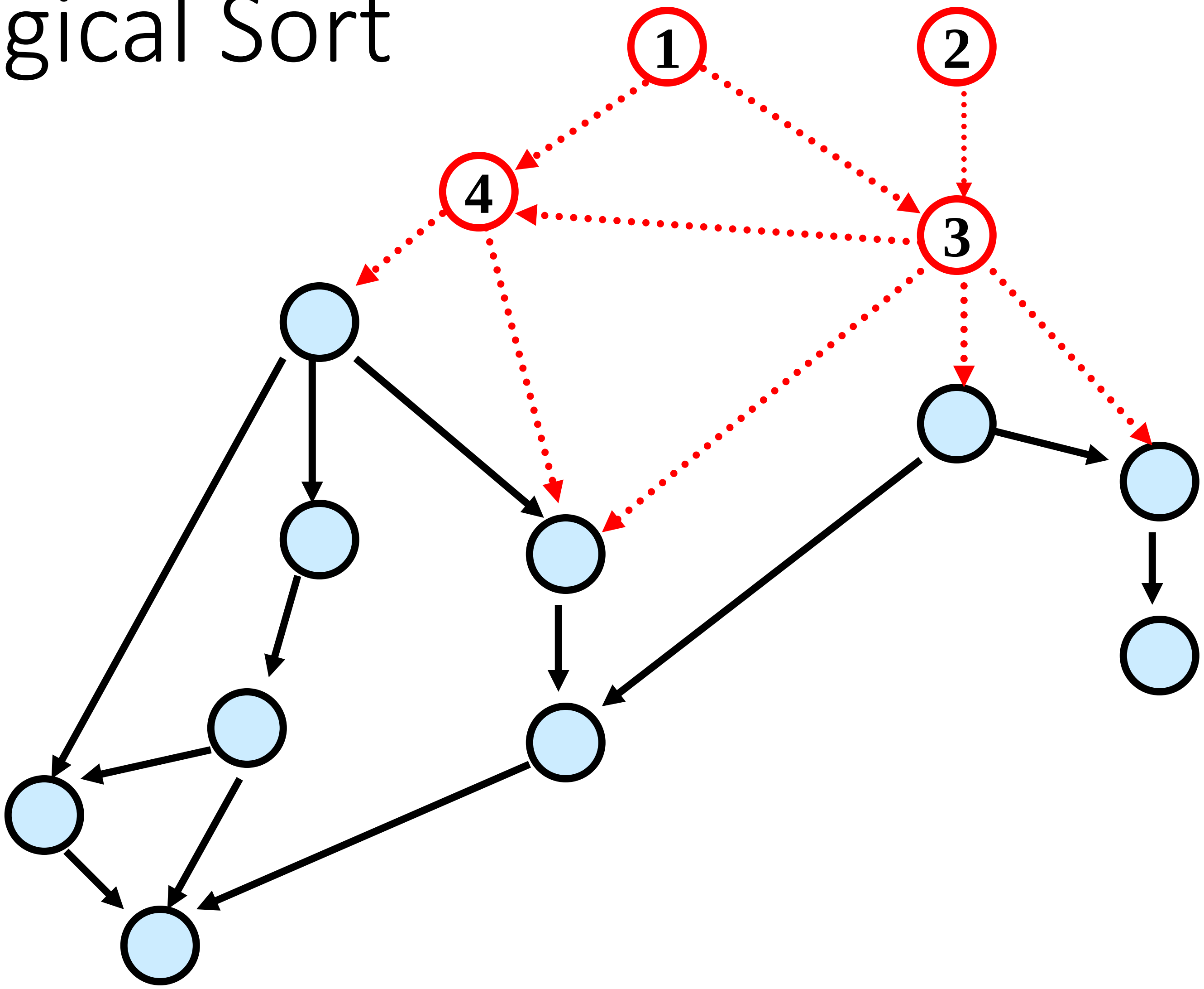
Topological Sort



Topological Sort



Topological Sort



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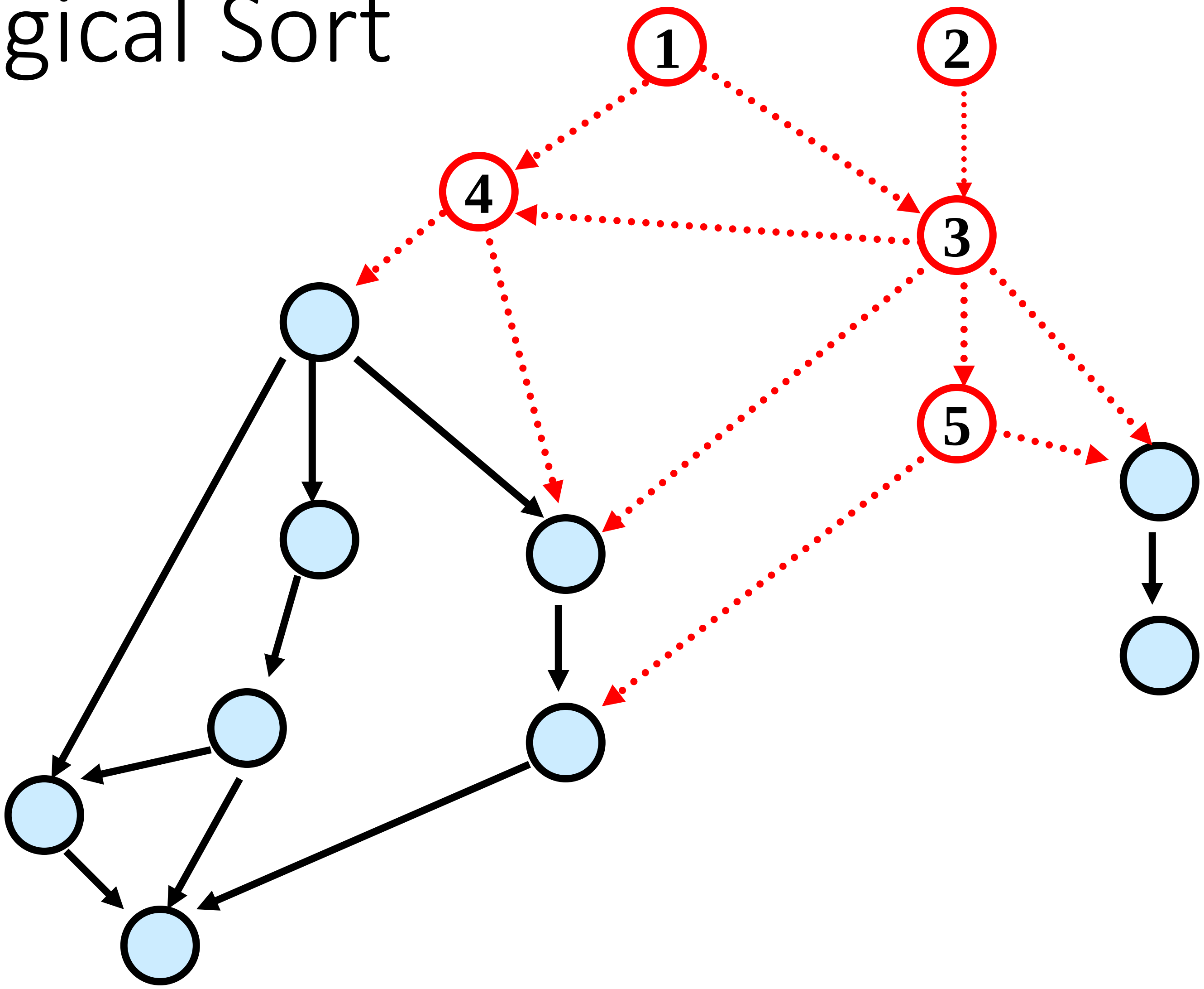

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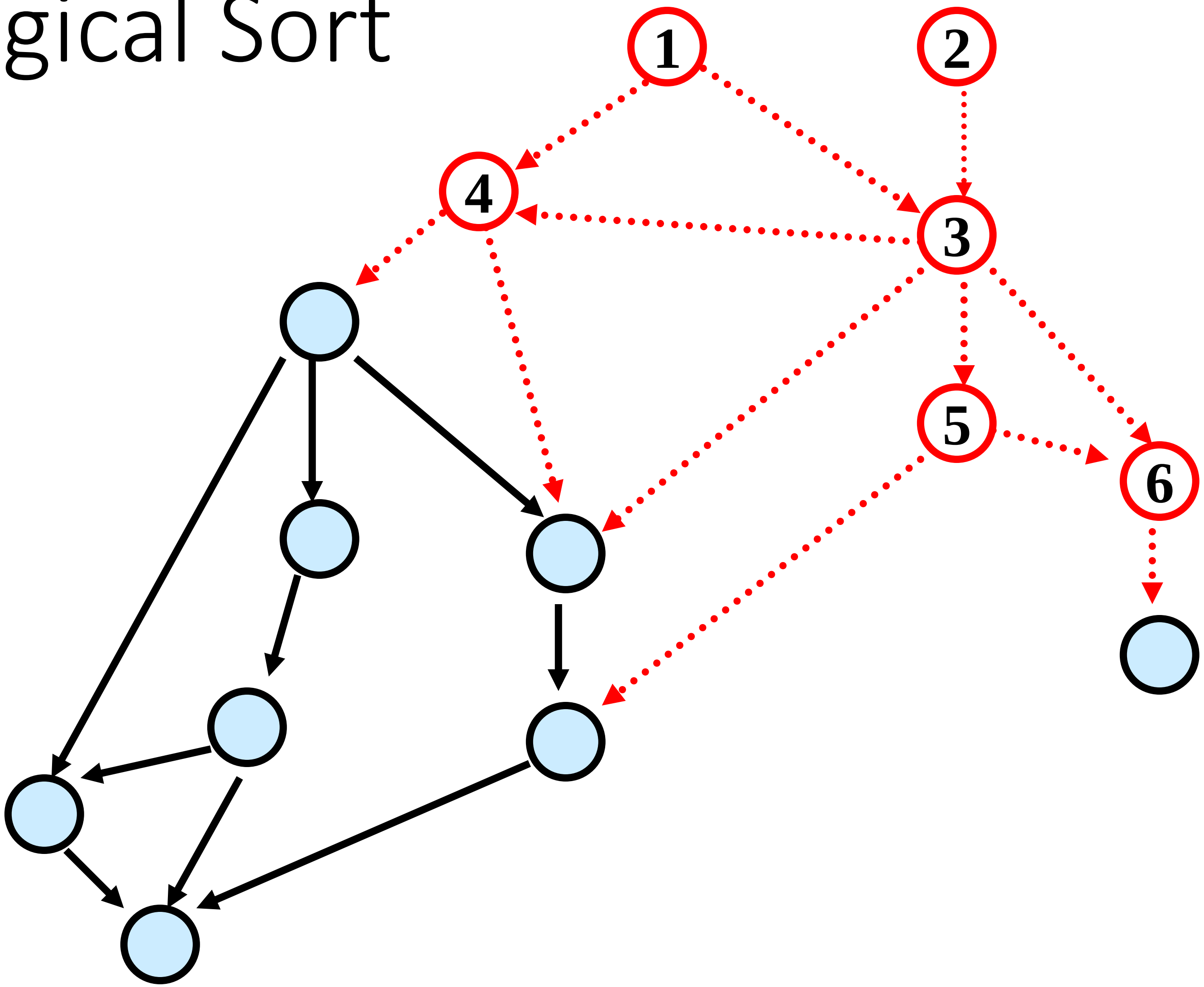

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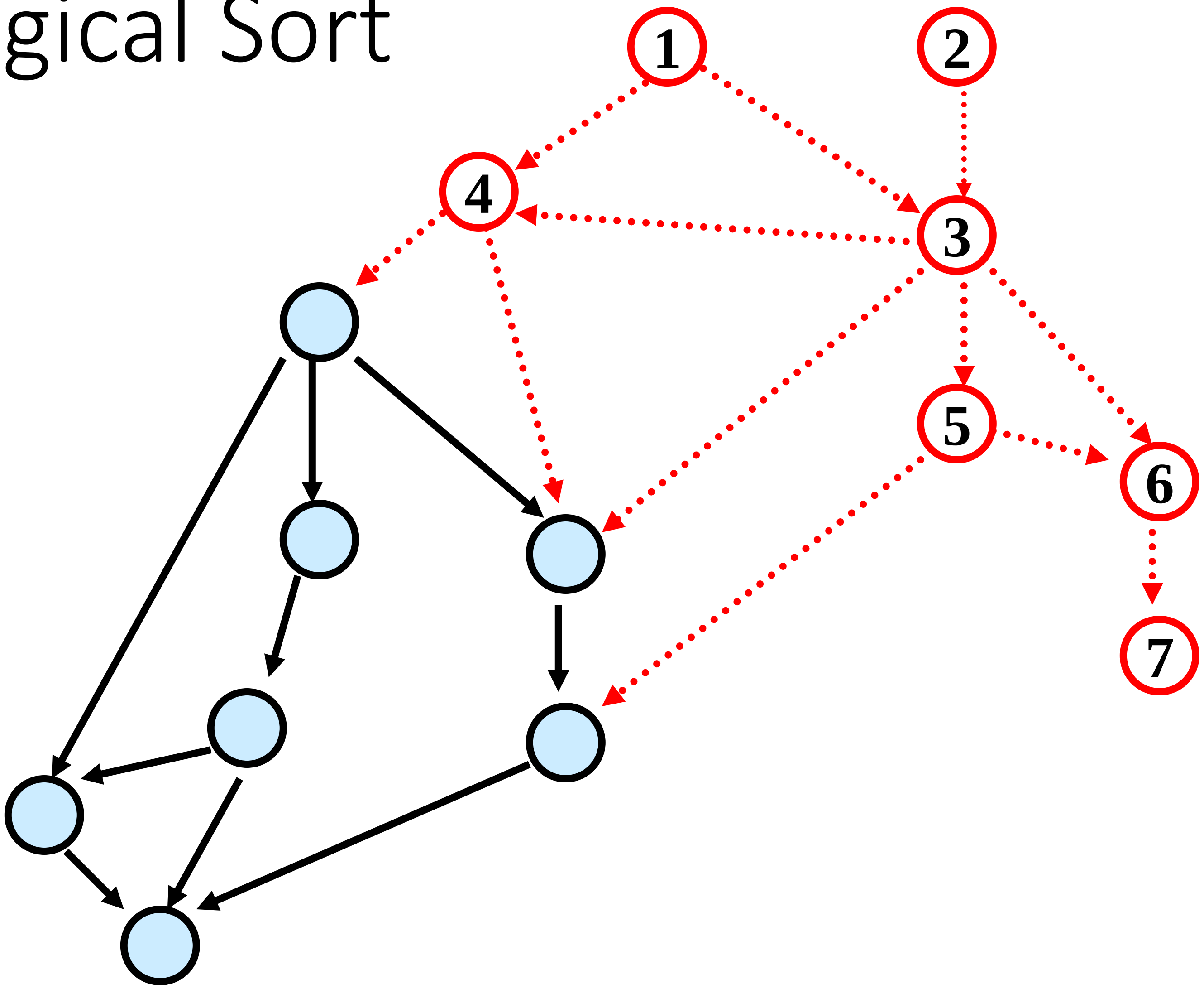

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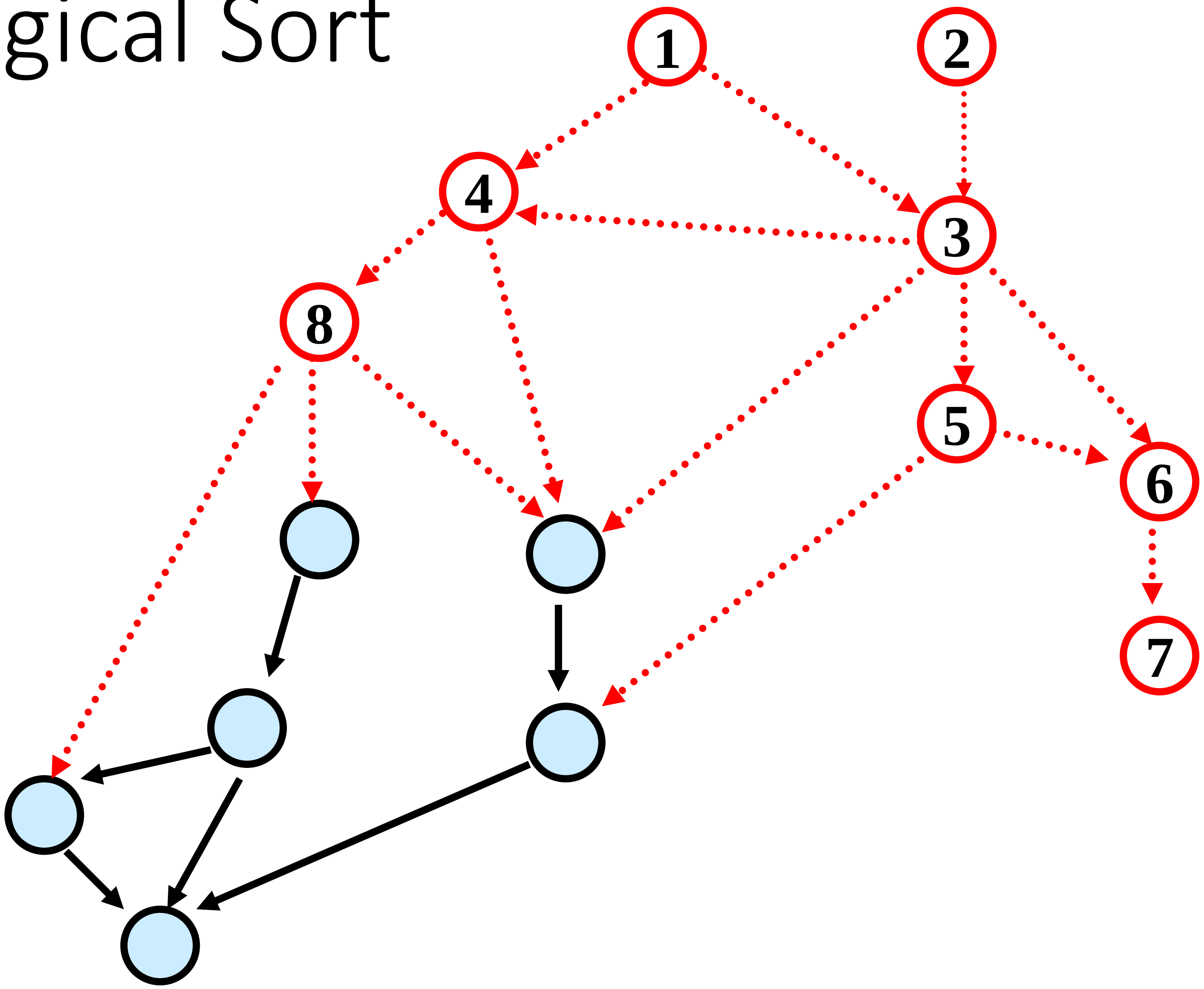

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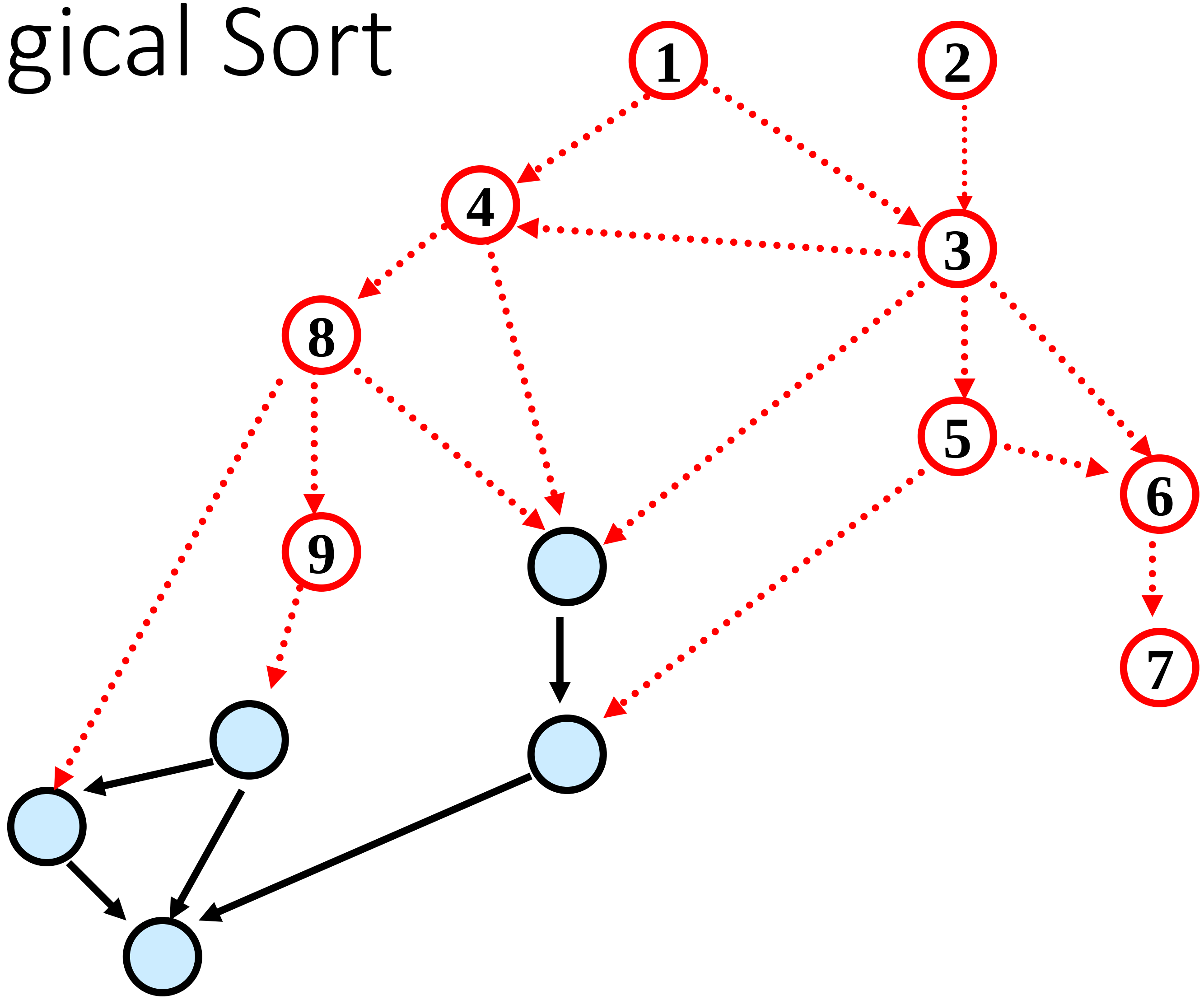

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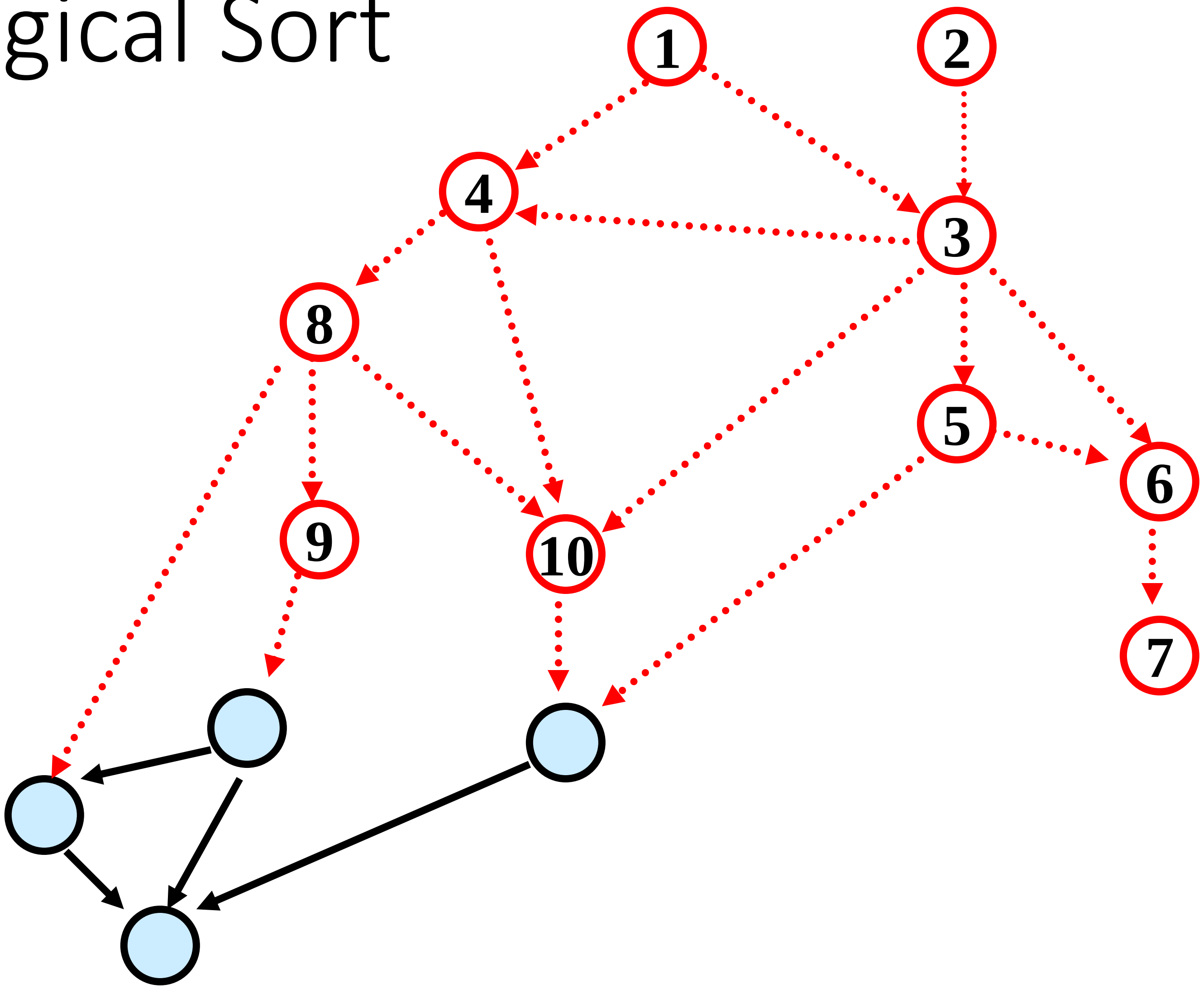

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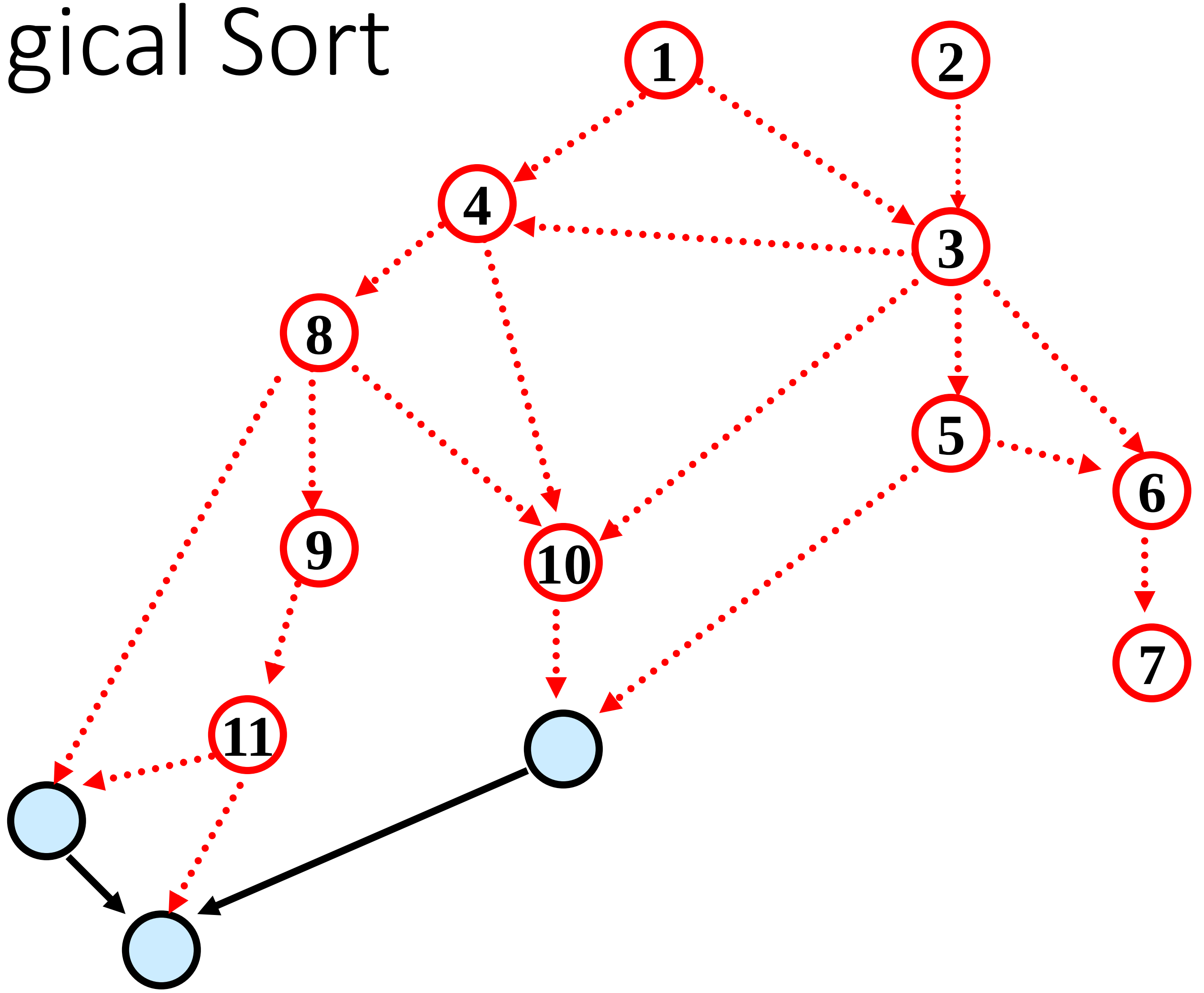

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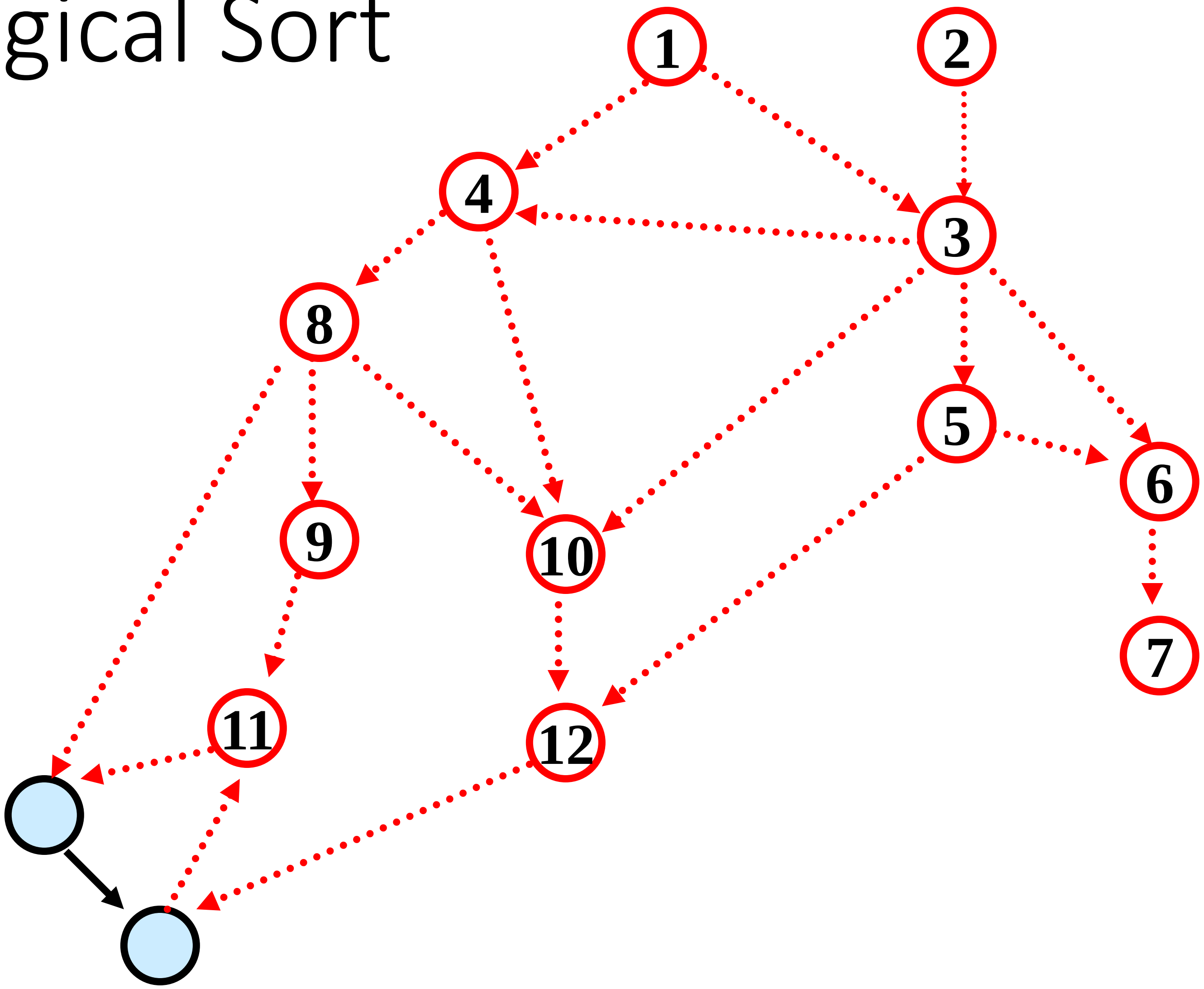

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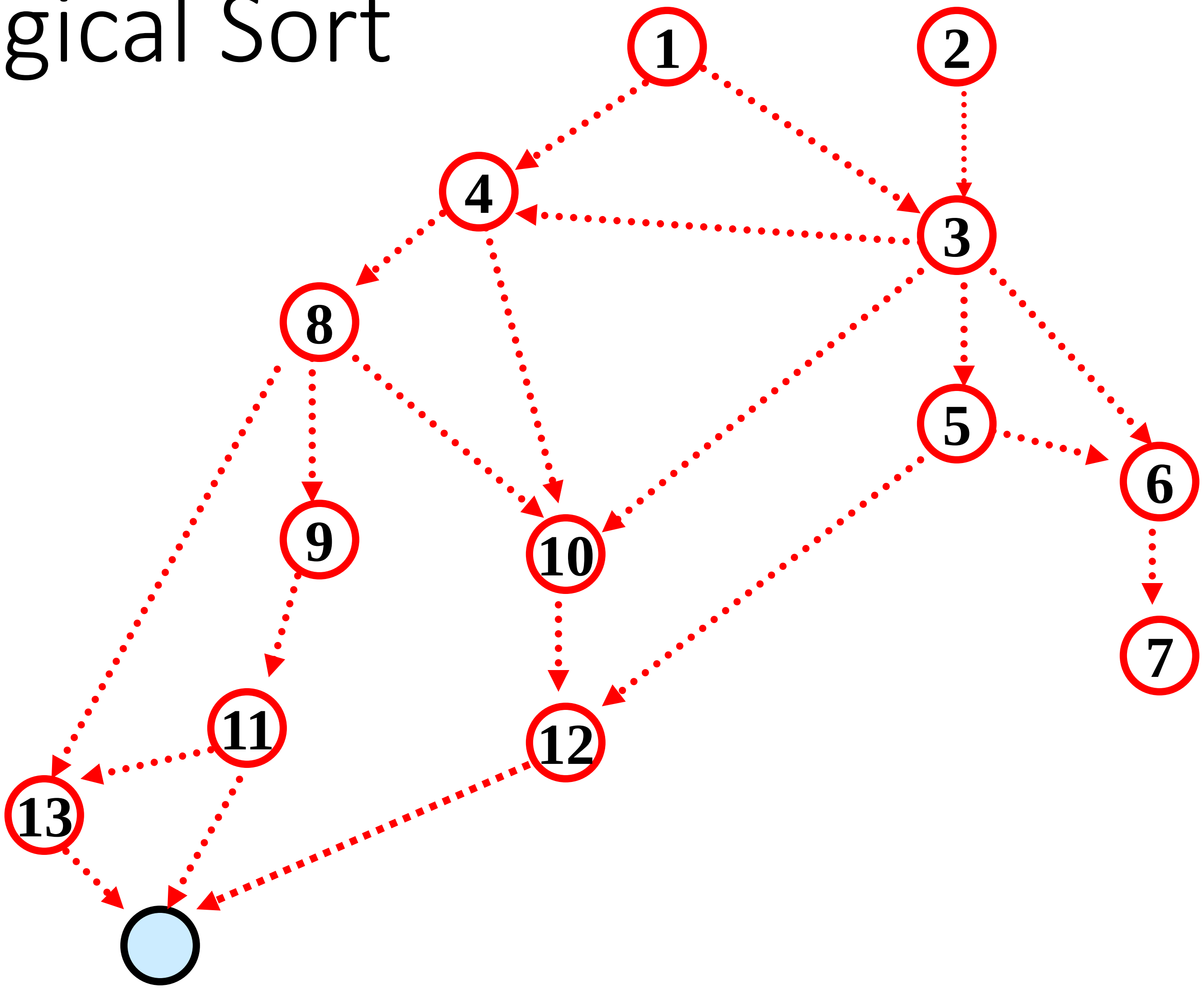

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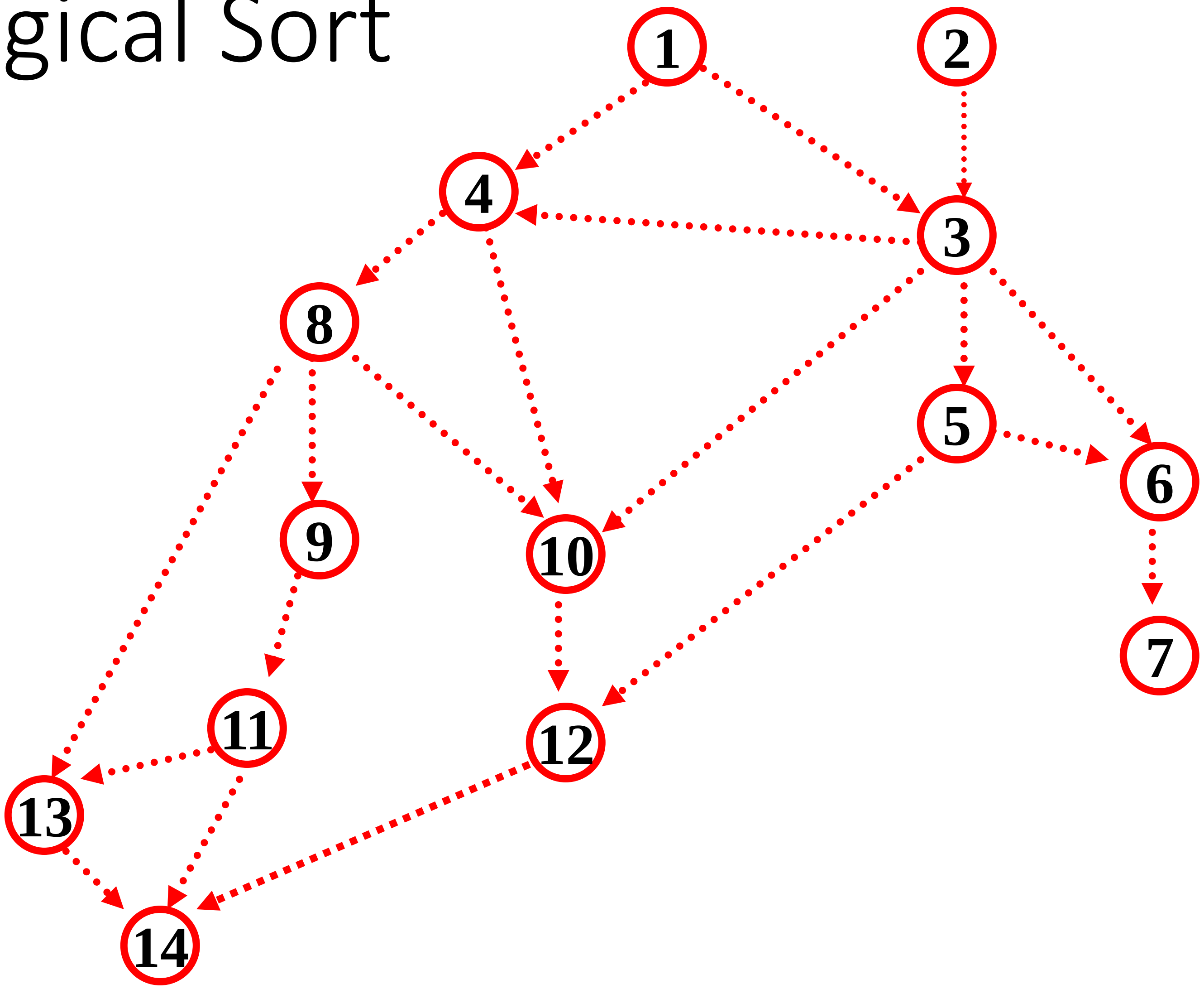


Implementing Topological Sort

- Go through all edges, computing array with in-degree for each vertex
  $\boldsymbol{O(m+n)}$

- Maintain a list of vertices of in-degree $\mathbf{0}$

- Remove any vertex in list and number it

- When a vertex is removed, decrease in-degree of each neighbor by $\mathbf{1}$ and add them to the list if their degree drops to $\mathbf{0}$

Total cost: $\boldsymbol{O(m+n)}$

Strongly Connected Components of Directed Graphs

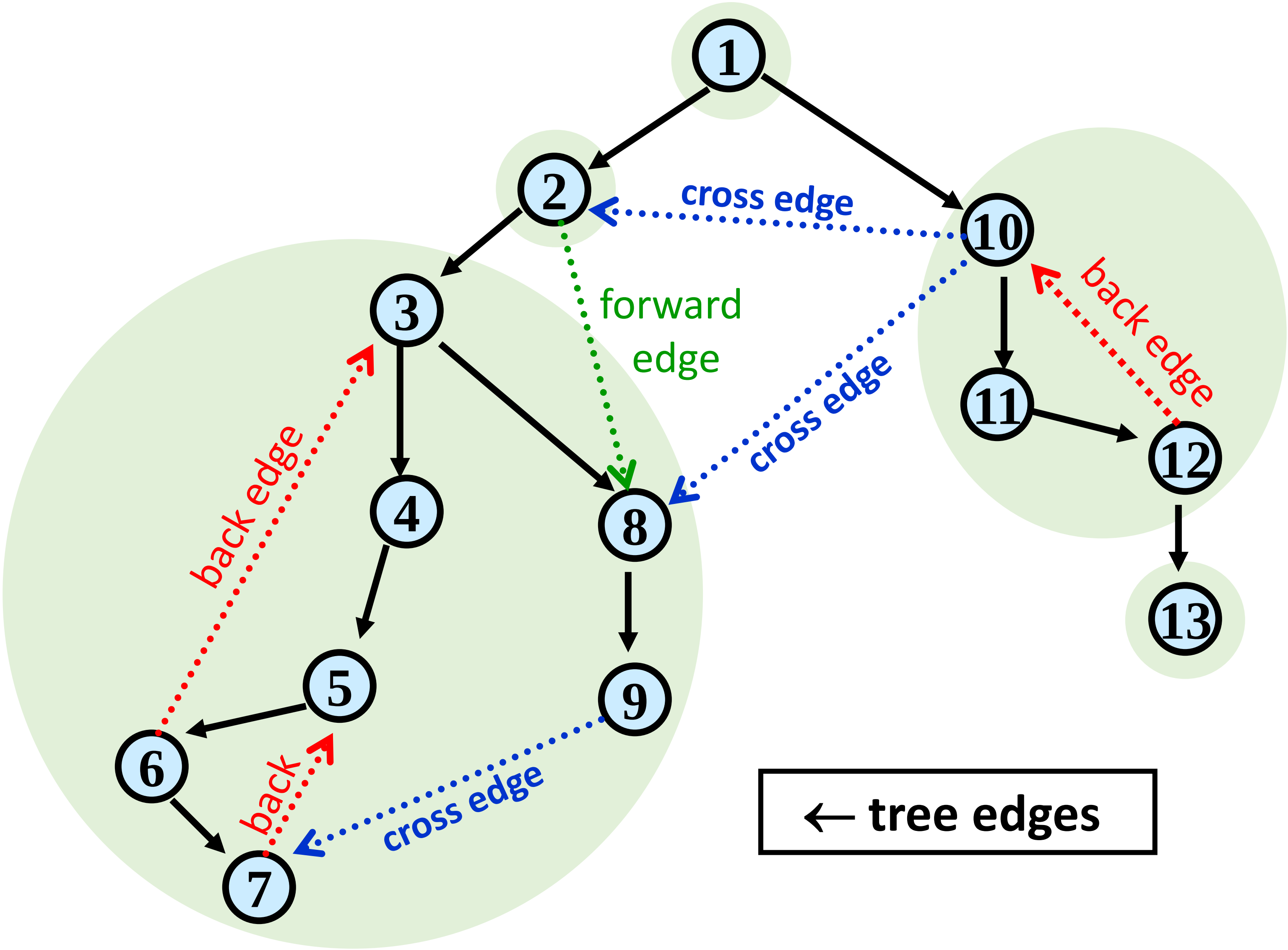
Defn: Vertices u and v are strongly connected iff they are on a directed cycle (there are paths from u to v and from v to u).

Defn: Can partition vertices of any directed graph into strongly connected components:

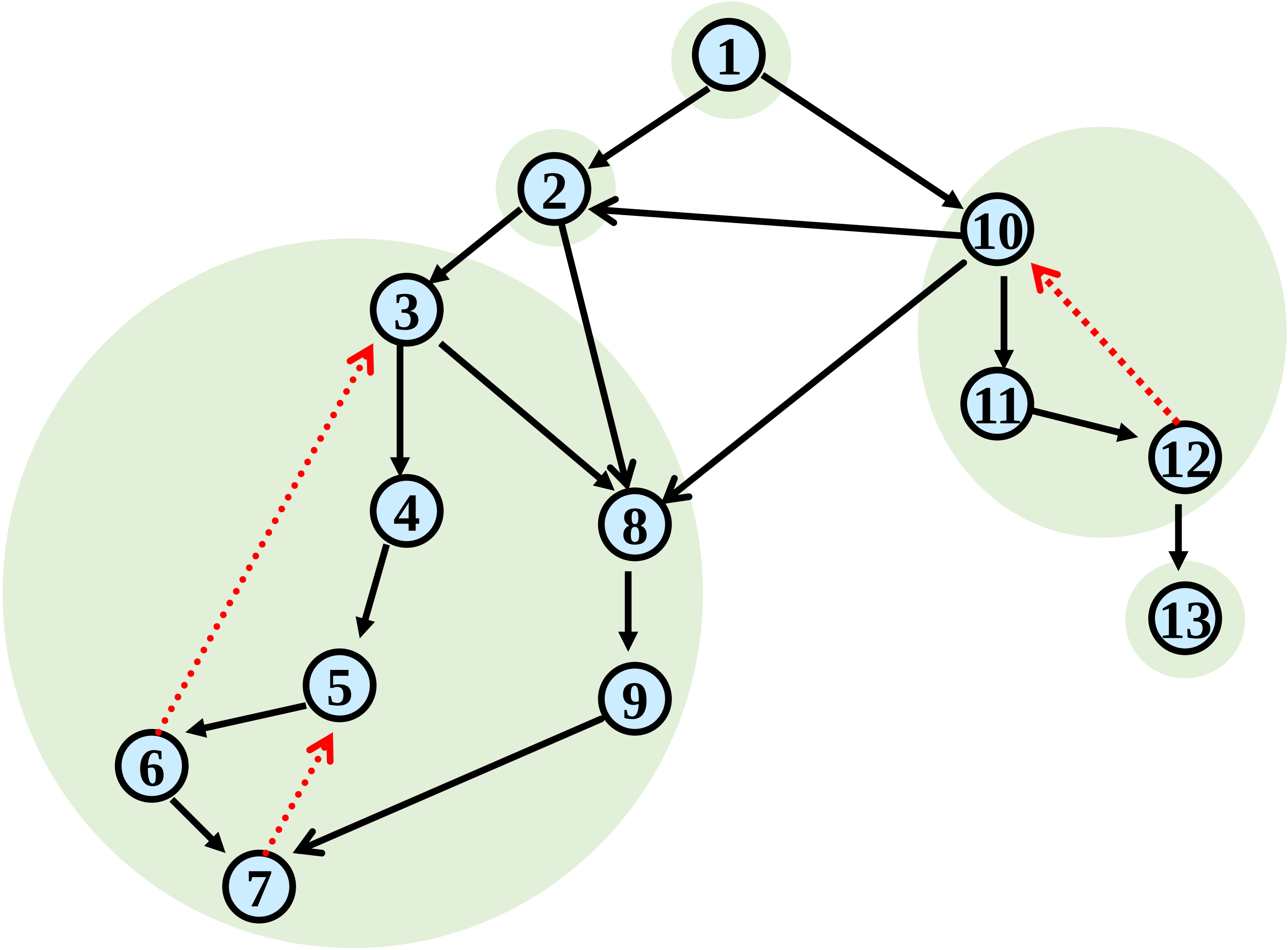
1. all pairs of vertices in the same component are strongly connected
2. can’t merge components and keep property 1

- Strongly connected components can be stored efficiently just like connected components
- Can be found in $O(n + m)$ time using a DFS then a BFS
  - Do a depth-first sort, keeping track of the order nodes are marked “fully-explored”
  - Going in order from least recent to most recent, run connected components

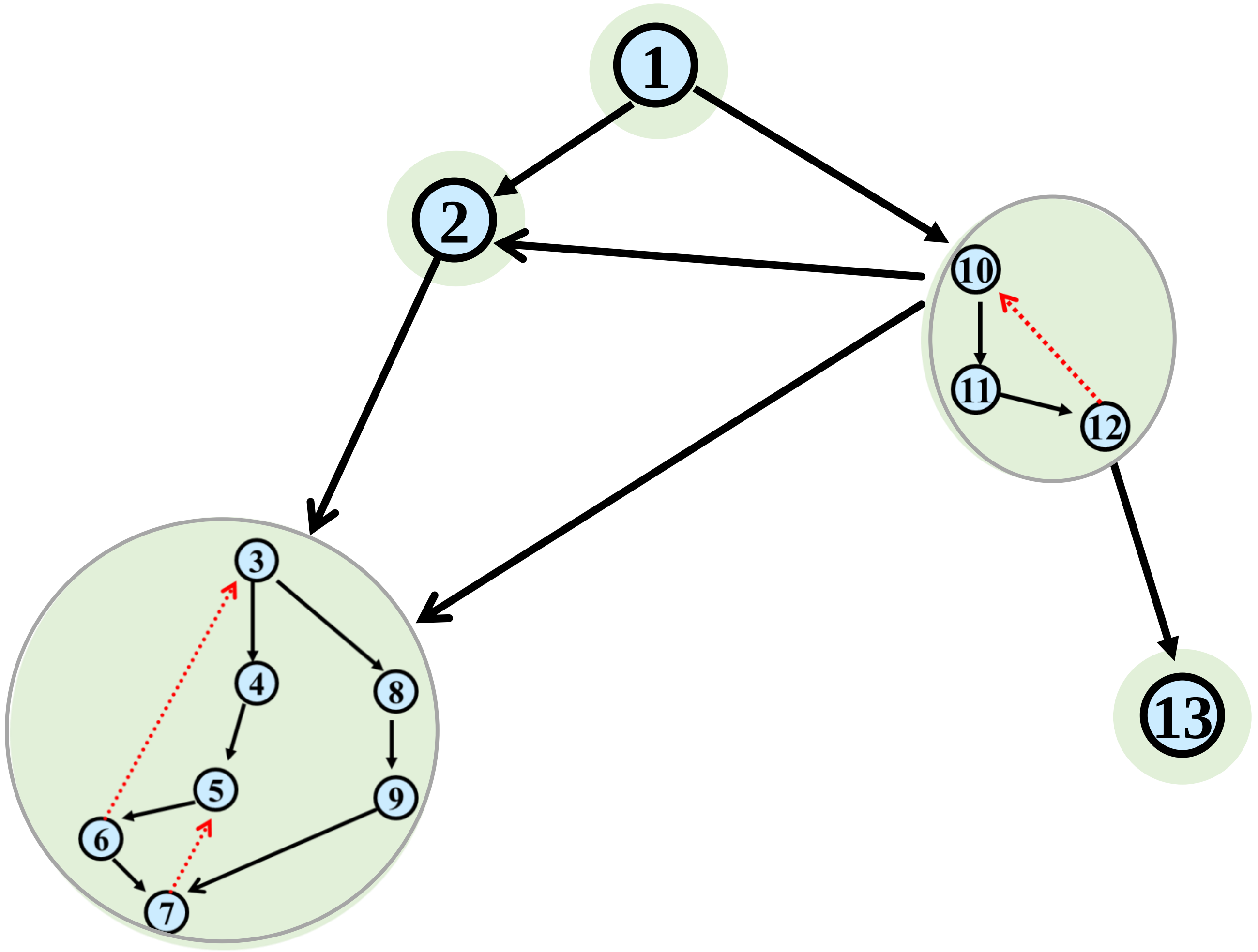
Strongly Connected Components



Strongly Connected Components



Strongly Connected Components



Strongly-Connected Components Usage

Common algorithmic paradigm for general directed graphs:

- Process strongly connected components one-by-one in the order given by topological sort of the DAG you get from shrinking them.

Greedy Algorithms

Hard to define exactly but can give general properties

- Solution is built in small steps
- Decisions on how to build the solution are made to maximize some criterion without looking to the future
  - Want the ‘best’ current partial solution as if the current step were the last step

May be more than one greedy algorithm using different criteria to solve a given problem

- Not obvious which criteria will actually work

Greedy Algorithms

- Greedy algorithms
  - Easy to describe
  - Fast running times
  - Work only on certain classes of problems
    - Hard part is showing that they are correct

- Focus on methods for proving that greedy algorithms do work

Interval Scheduling

Interval Scheduling:

- Single resource
- Reservation requests of form:

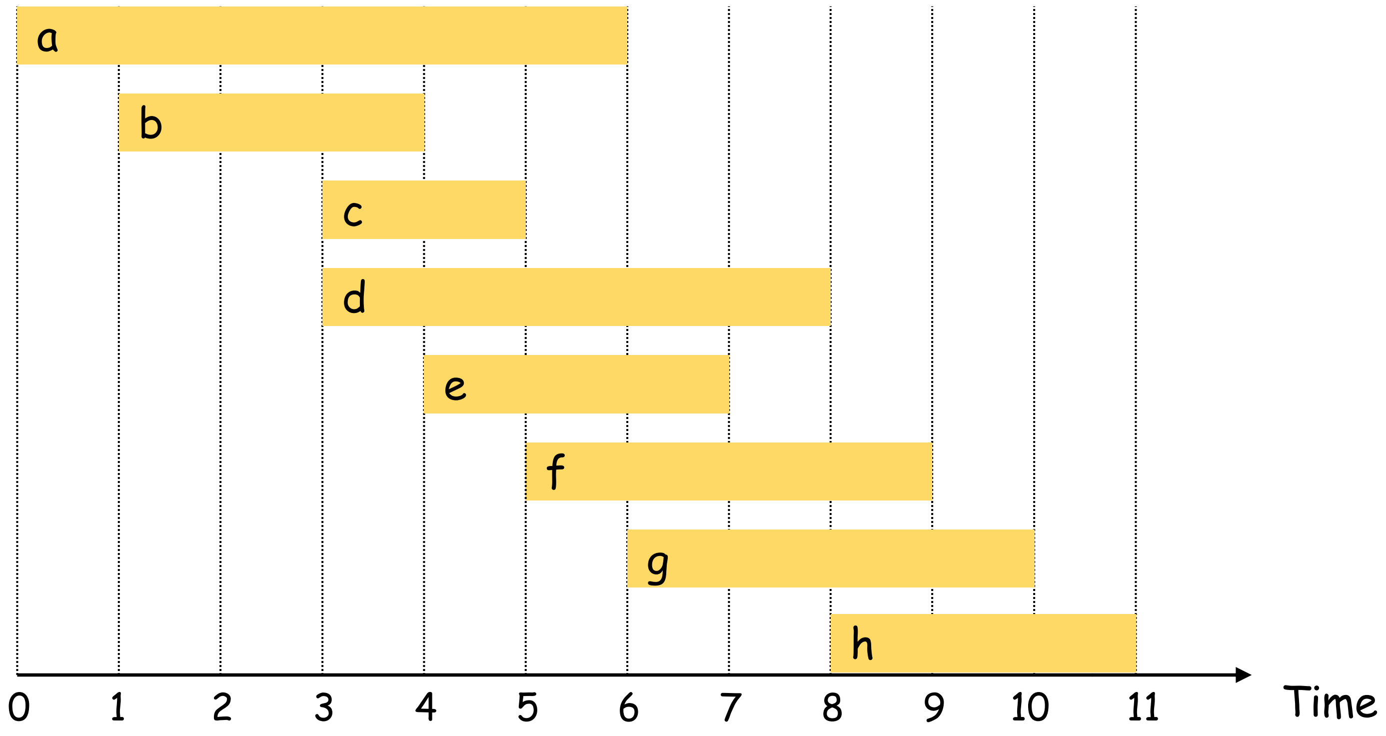
  "Can I reserve it from start time s to finish time f?"

$$s < f$$

Interval Scheduling

Interval scheduling:

- Job $\boldsymbol{j}$ starts at $\boldsymbol{s_j}$ and finishes at $\boldsymbol{f_j > s_j}$.
- Two jobs $\boldsymbol{i}$ and $\boldsymbol{j}$ are compatible if they don't overlap: $\boldsymbol{f_i \le s_j}$ or $\boldsymbol{f_j \le s_i}$
- **Goal:** find maximum size subset of mutually compatible jobs.



Greedy Algorithms for Interval Scheduling

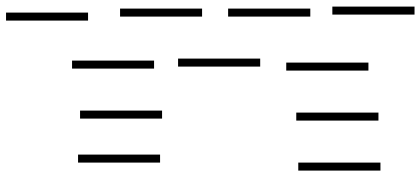
- What criterion should we try?

Greedy Algorithms for Interval Scheduling

- What criterion should we try?
  - Earliest start time $\boldsymbol{s_i}$
  - Shortest request time $\boldsymbol{f_i - s_i}$
  - Fewest conflicts

Greedy Algorithms for Interval Scheduling

- What criterion should we try?
  - Earliest start time s_i
    - Doesn't work
  - Shortest request time $f_i - s_i$
    - Doesn't work
  - Fewest conflicts
    - Doesn't work
  - Earliest finish time f_i
    - Works!



Greedy (by finish time) Algorithm for Interval Scheduling

$\boldsymbol{R}$ = set of all requests

$\boldsymbol{A} = \varnothing$

while $\boldsymbol{R} \neq \varnothing$ do:

    Choose request $\boldsymbol{i} \in \boldsymbol{R}$ with smallest finish time $\boldsymbol{f_i}$

    Add request $\boldsymbol{i}$ to $\boldsymbol{A}$

    Delete all requests in $\boldsymbol{R}$ not compatible with request $\boldsymbol{i}$

return $\boldsymbol{A}$

Greedy Analysis Strategies

Greedy algorithm stays ahead: Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's

For interval scheduling: Show that after the greedy algorithm selects each interval, any alternative schedule's selection would have also been non-conflicting.

Conclusion: Each choice from the alternative selections can be swapped with a greedy choice, making greedy no worse off.

Interval Scheduling: Analysis

Claim: A is a compatible set of requests and requests are added to A in order of finish time

- When we add a request to A we delete all incompatible ones from R

Name the finish times of requests in A as $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_t$ in order.

Claim: Let $O \subseteq R$ be a set of compatible requests whose finish times in order are $\mathbf{o}_1, \mathbf{o}_2, \ldots, \mathbf{o}_s$. Then for every integer $k \geq 1$ we have:

a) if O contains a k^{th} request then A does too, and

b) $\mathbf{a}_k \leq \mathbf{o}_k$ "A is ahead of O"

Note that a) alone implies that $t \geq s$ which means that A is optimal but we also need b) "stays ahead" to keep the induction going.

Inductive Proof of Claim

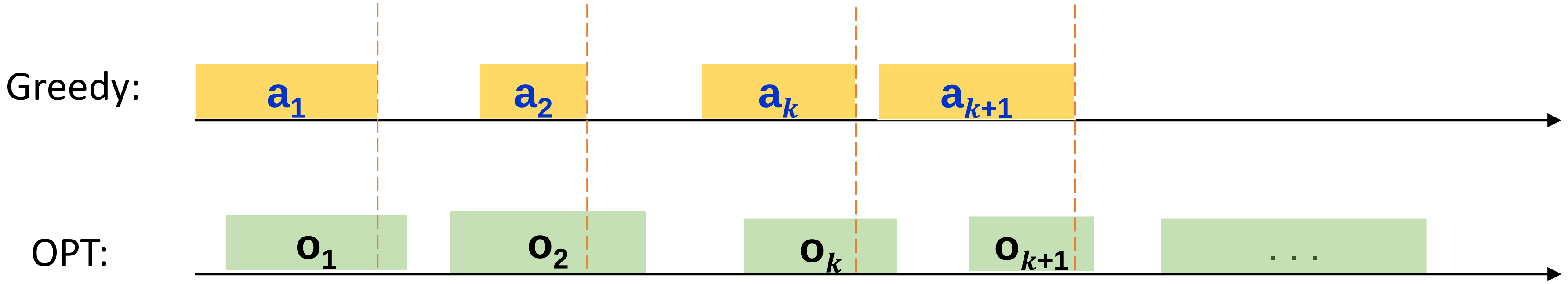
Base Case $k = 1$: A includes the request with smallest finish time, so if O is not empty then $a_1 \leq o_1$

Inductive Step: Suppose that $a_k \leq o_k$ and there is a k+1st request in O.

Then k+1st request in O is compatible with $a_1, a_2, \ldots, a_k$ since $a_k \leq o_k$ and $o_k \leq$ start time of k+1st request in O whose finish time is o_{k+1}

$\Rightarrow$ There is a k+1st request in A whose finish time is named a_{k+1}.

Also, since A would have considered both requests and chosen the one with the earlier finish time, $a_{k+1} \leq o_{k+1}$.



Interval Scheduling: Greedy Algorithm Implementation

```
Sort jobs by finish times so that 0 ≤ f₁ ≤ f₂ ≤ ... ≤ fₙ.

A = ϕ
last = 0
for j = 1 to n {
    if (last ≤ sⱼ)
      A = A ∪ {j}
      last = fⱼ
}
return A
```

$O(n \log n)$

$O(n)$