CSE 421 Winter 2025
Lecture 4: Asymptotics and Graph Search

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Complexity analysis

- Problem size n
  - **Worst-case complexity**:
    maximum # steps algorithm takes on any input of size n
  - **Best-case complexity:**
    minimum # steps algorithm takes on any input of size n
  - **Average-case complexity**:
    Expected # steps algorithm takes on inputs of size n

Complexity

- The complexity of an algorithm associates a number $\boldsymbol{T(n)}$, the worst/average-case/best time the algorithm takes, with each problem size **n**.

- Mathematically,
  - $\boldsymbol{T}$ is a function that maps positive integers giving problem size to positive real numbers giving number of steps.

- Sometimes we have more than one size parameter
  - e.g. $\boldsymbol{n}$=# of vertices, $\boldsymbol{m}$=# of edges in a graph.

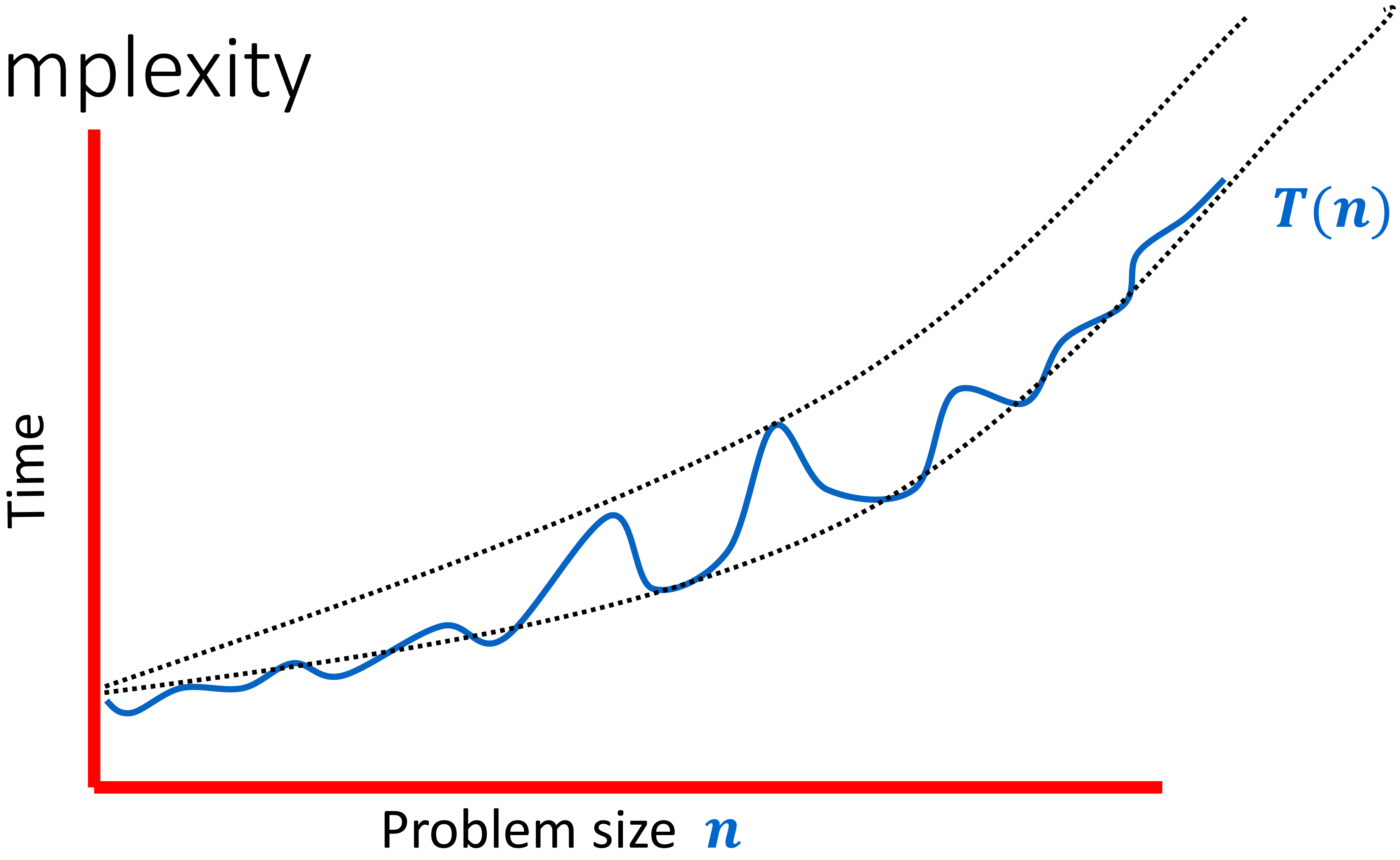
Efficient = Polynomial Time

- Polynomial time
  - Running time $T(n) \le cn^k + d$ for some $c, d, k \geq 0$

- Why polynomial time?
  - If problem size grows by at most a constant factor then so does the running time
    - e.g. $T(2n) \le c\,(2n)^k + d = 2^k cn^k + d \le 2^k\left(cn^k + d\right) = 2^k\,T(n)$
    - polynomial-time is exactly the set of running times that have this property

  - Typical running times are small degree polynomials, mostly less than n^3, at worst n^6, not n^{100}

Complexity



O-notation etc

- Given two positive functions $\boldsymbol{f}$ and $\boldsymbol{g}$
  - $\boldsymbol{f(n)}$ is $\boldsymbol{O(g(n))}$ iff there is a constant $\boldsymbol{c > 0}$ so that $\boldsymbol{f(n)}$ is eventually always $\le \boldsymbol{c \cdot g(n)}$
  - $\boldsymbol{f(n)}$ is $\boldsymbol{o(g(n))}$ iff for every constant $\boldsymbol{c > 0}$ $\boldsymbol{f(n)}$ is eventually always $\le \boldsymbol{c \cdot g(n)}$
  - $\boldsymbol{f(n)}$ is $\boldsymbol{\Omega(g(n))}$ iff there is a constant $\boldsymbol{\varepsilon > 0}$ so that $\boldsymbol{f(n) \ge \varepsilon \cdot g(n)}$ for infinitely many values of $\boldsymbol{n}$
  - $\boldsymbol{f(n)}$ is $\boldsymbol{\omega(g(n))}$ iff for every constant $\boldsymbol{c > 0}$ $\boldsymbol{f(n)}$ is eventually always $\ge \boldsymbol{c \cdot g(n)}$
  - $\boldsymbol{f(n)}$ is $\boldsymbol{\Theta(g(n))}$ iff $\boldsymbol{f(n)}$ is $\boldsymbol{O(g(n))}$ and $\boldsymbol{f(n)}$ is $\boldsymbol{\Omega(g(n))}$

Note: The definition of "$\boldsymbol{f(n)}$ is $\boldsymbol{\Omega(g(n))}$" is *almost* the same as "$\boldsymbol{f(n)}$ is **not** $\boldsymbol{o(g(n))}$"

The definition of "$\boldsymbol{f(n)}$ is $\boldsymbol{\Omega(g(n))}$" is *almost* the same as "$\boldsymbol{f(n)}$ is **not** $\boldsymbol{o(g(n))}$"

Asymptotic Notation intuition

$f(n)$ is...

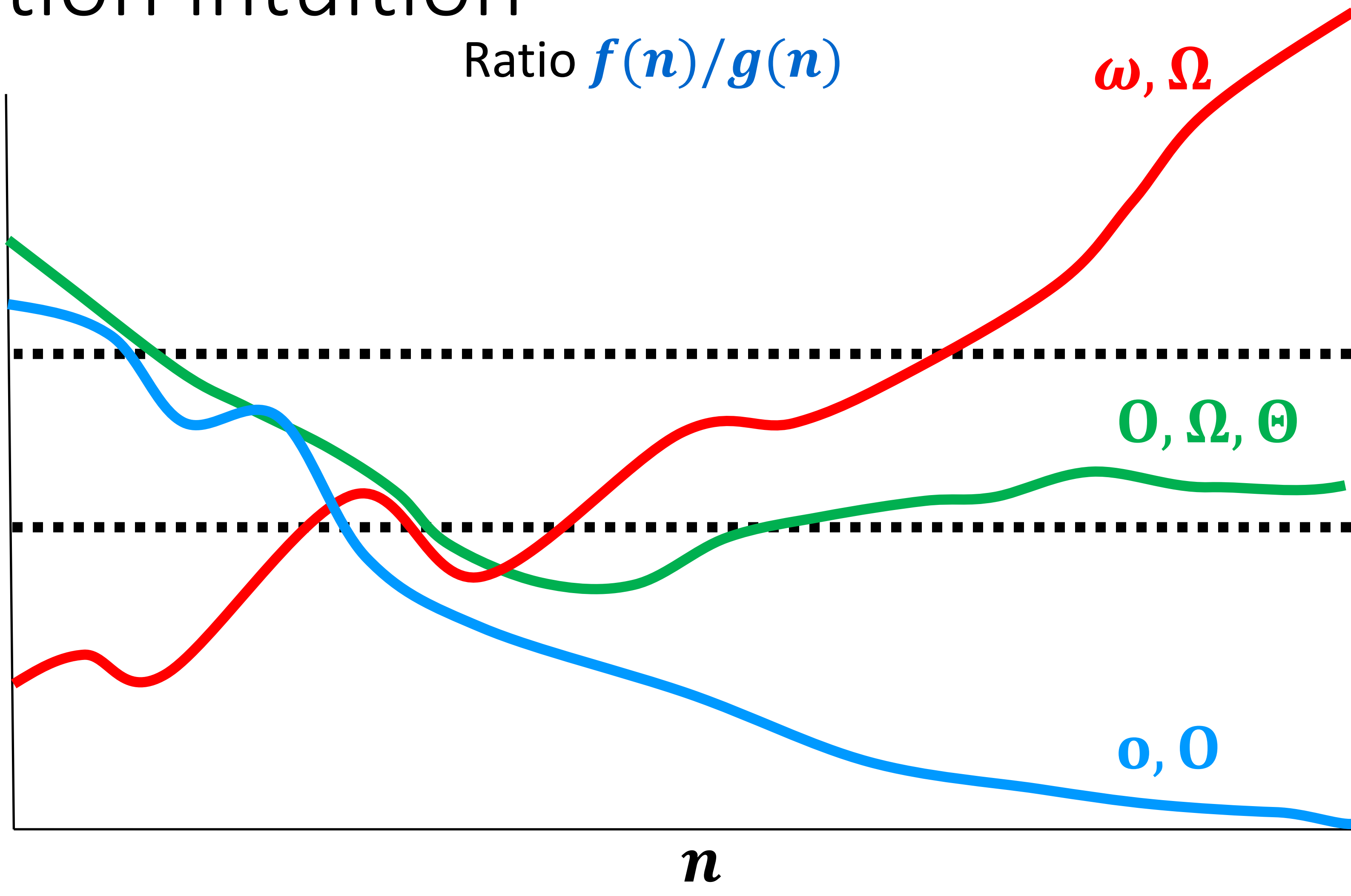
$O(g(n))$: ratio eventually below some line forever

$o(g(n))$: ratio eventually below every line forever

$\Omega(g(n))$: ratio eventually above some line forever

$\omega(g(n))$: ratio eventually above every line forever

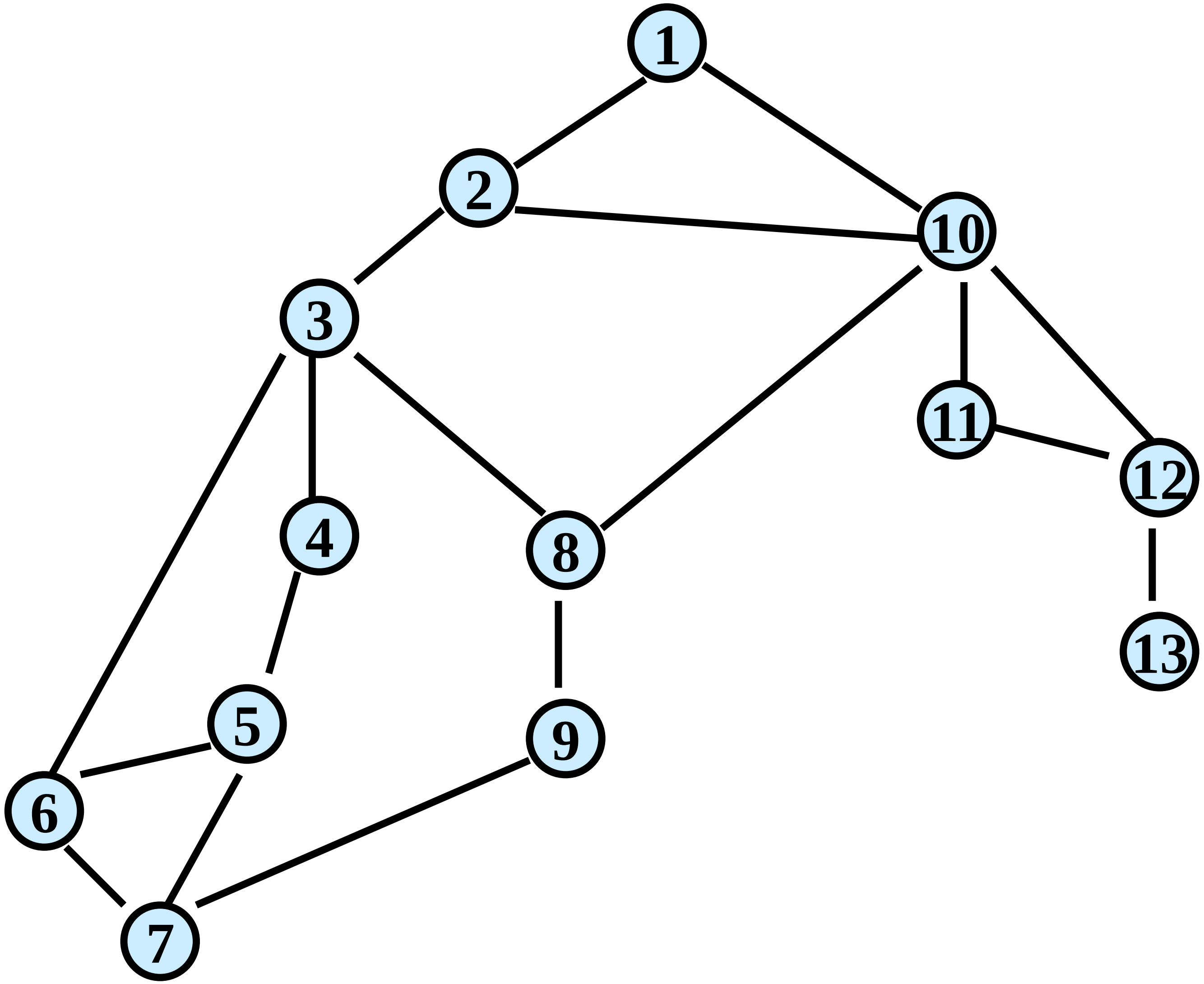
$\Theta(g(n))$: both O and Ω



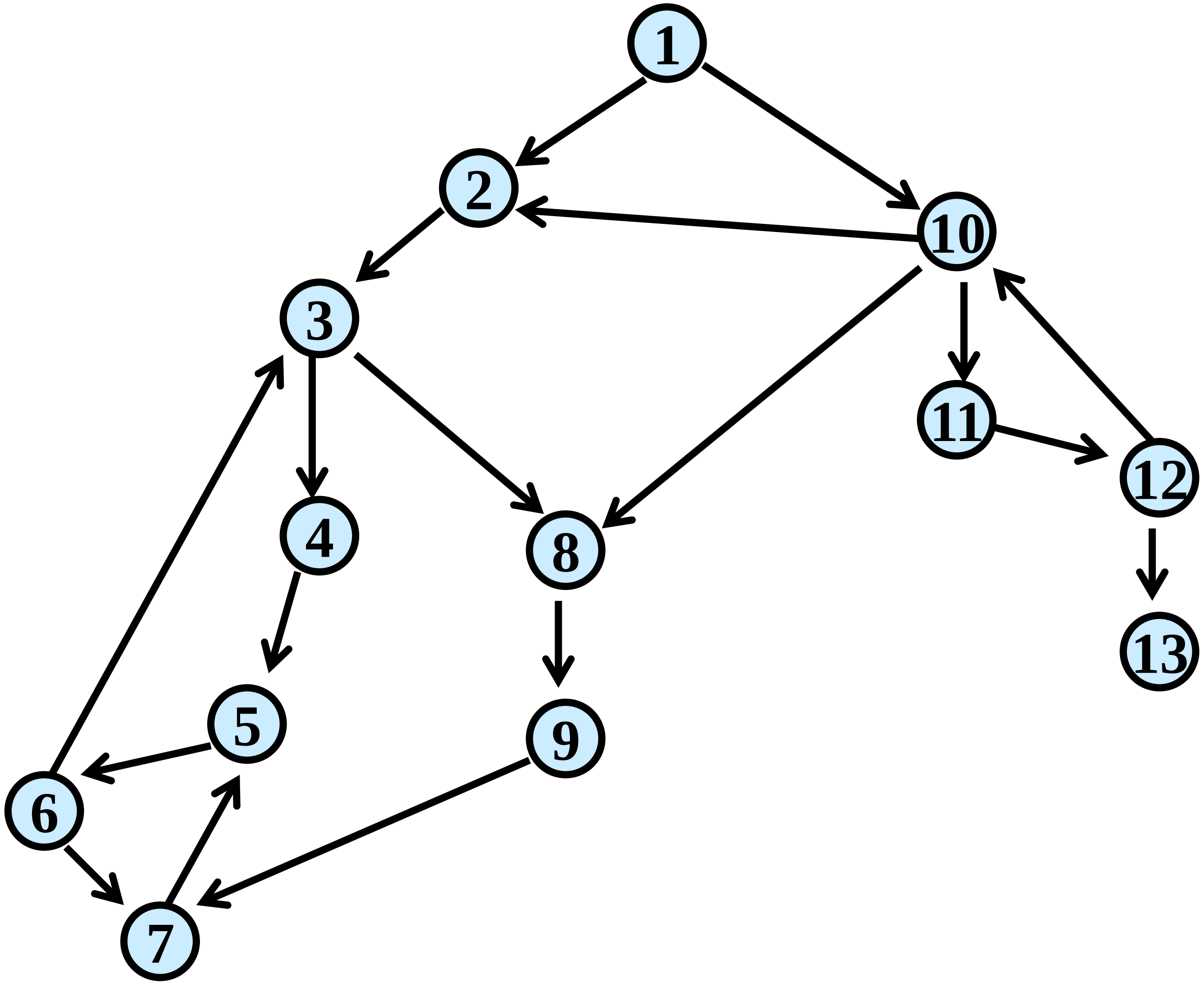
Introduction to Algorithms

- **Graph Search/Traversal**

Undirected Graph G = (V,E)



Directed Graph G = (V,E)



Graph Traversal

Learn the basic structure of a graph

Walk from a fixed starting vertex $\boldsymbol{s}$ to find all vertices reachable from $\boldsymbol{s}$

Generic Graph Traversal Algorithm

Given: Graph graph $\boldsymbol{G = (V, E)}$ vertex $\boldsymbol{s \in V}$

Find: set $\boldsymbol{R}$ of vertices reachable from $\boldsymbol{s \in V}$

Reachable($\boldsymbol{s}$):

Add $\boldsymbol{s}$ to $\boldsymbol{R}$

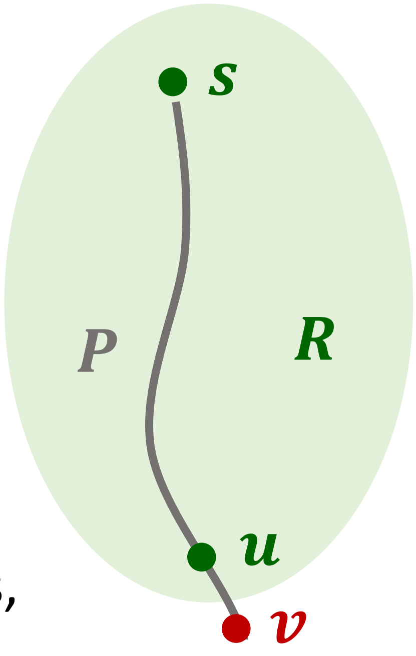
while there is a $\boldsymbol{(u, v) \in E}$ where $\boldsymbol{u \in R}$ and $\boldsymbol{v \notin R}$

Add $\boldsymbol{v}$ to $\boldsymbol{R}$

return $\boldsymbol{R}$

Generic Traversal Always Works

Claim: At termination, R is the set of nodes reachable from s



Proof

$\subseteq$: For every node $v \in R$ there is a path from s to v

- Induction based on edges found.
  - Base case: s is reachable from s
  - Inductive step: If there is a path to every member of R after i iterations, then there is a path to every member of R after $i + 1$ iterations

$\supseteq$: Suppose there is a node $w \notin R$ reachable from s via a path P

- Take first node v on P such that $v \notin R$
- Predecessor u of v in P satisfies
  - $u \in R$
  - $(u, v) \in E$
- But this contradicts the fact that the algorithm exited the while loop.

Graph Traversal

Learn the basic structure of a graph

Walk from a fixed starting vertex $\boldsymbol{s}$ to find all vertices reachable from $\boldsymbol{s}$

Three states of vertices

- **unvisited**
- **visited/discovered** (in $\boldsymbol{R}$)
- **fully-explored** (in $\boldsymbol{R}$ and all neighbors have been visited)

Breadth-First Search

Completely explore the vertices in order of their distance from s

Naturally implemented using a queue

BFS(s)

Global initialization: mark all vertices "unvisited"

BFS(s)

    Mark s "visited"
    Add s to Q
    i = 0
    Mark s as "layer i"
    while Q not empty
        u = next item removed from Q
        i = "layer of u"
        for each edge (u, v)
            if (v is "unvisited")
                mark v "visited"
                mark s as "layer $i + 1$"
        mark u "fully-explored"

Properties of BFS

BFS(s) visits x iff there is a path in G from s to x.

Edges followed to undiscovered vertices define a breadth first spanning tree of G

Layer i in this tree:

L_i = set of vertices u with shortest path in G from root s of length i.

Properties of BFS

Claim: For undirected graphs:

All edges join vertices on the same or adjacent layers of BFS tree

Proof: Suppose not...

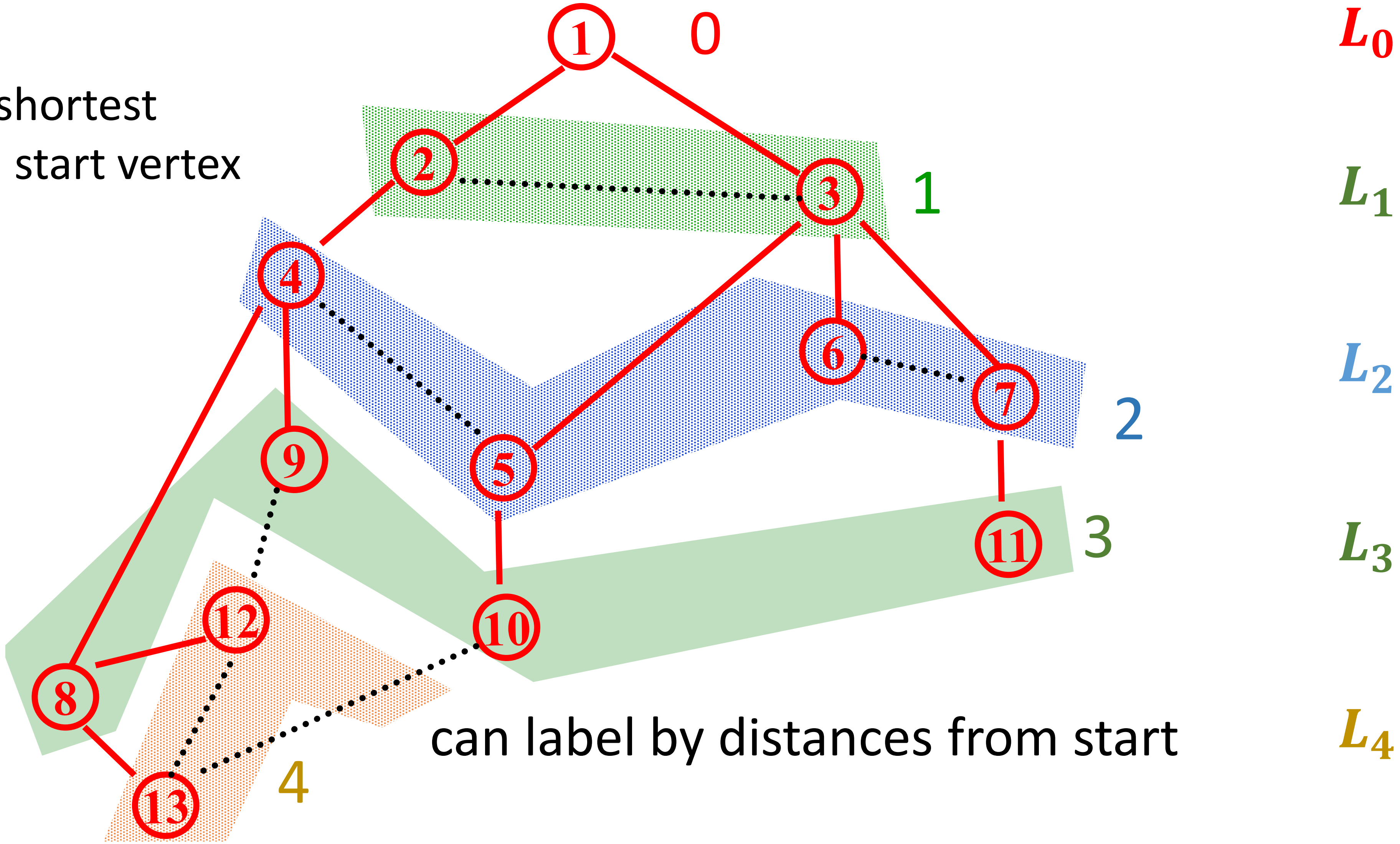
Then there would be vertices (x, y) s.t. $x \in L_i$ and $y \in L_j$ and $j > i + 1$.

Then, when vertices adjacent to x are considered in BFS,
y would be added with layer $i + 1$ and not layer j.

Contradiction.

BFS Application: Shortest Paths

Tree gives shortest paths from start vertex



can label by distances from start

Undirected Graph Search Application: Connected Components

Want to answer questions of the form:

Given: vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{G}$

Is there a path from $\boldsymbol{u}$ to $\boldsymbol{v}$?

Idea: create array $\boldsymbol{A}$ s.t

$\boldsymbol{A}[\boldsymbol{u}]$ = smallest numbered vertex connected to $\boldsymbol{u}$

Answer is yes iff $\boldsymbol{A}[\boldsymbol{u}] = \boldsymbol{A}[\boldsymbol{v}]$

Q: Why is this better than an array **Path**$[\boldsymbol{u}, \boldsymbol{v}]$?

Undirected Graph Search Application: Connected Components

Initial state: all v unvisited
for s from 1 to n do:
    if state(s) $\neq$ fully-explored then
        BFS(s): setting $A[u] = s$ for each u found
            (and marking u visited/fully-explored)

Total cost: $O(n+m)$

- Each vertex is touched once in outer procedure and edges examined in different BFS runs are disjoint
- Works also with Depth First Search ...

DFS($\boldsymbol{u}$) – Recursive Procedure

Global Initialization: mark all vertices "unvisited"

DFS($\boldsymbol{u}$)

mark $\boldsymbol{u}$ “visited” and add $\boldsymbol{u}$ to $\boldsymbol{R}$

for each edge $(\boldsymbol{u}, \boldsymbol{v})$

if ($\boldsymbol{v}$ is “unvisited”)

DFS($\boldsymbol{v}$)

mark $\boldsymbol{u}$ “fully-explored”

Properties of DFS(s)

Like BFS(s):

- DFS(s) visits x iff there is a path in G from s to x
- Edges into undiscovered vertices define depth-first spanning tree of G

Unlike the BFS tree:

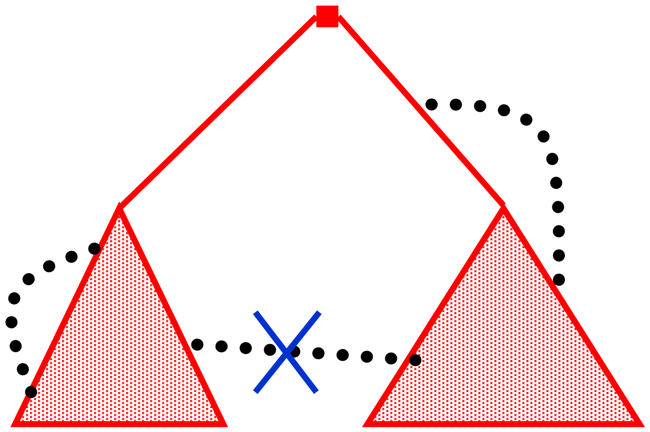
- the DFS spanning tree *isn't* minimum depth
- its levels *don't* reflect min distance from the root
- non-tree edges *never* join vertices on the same or adjacent levels

BUT…

Non-tree edges in DFS tree of undirected graphs

Claim: All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree

- In other words ... No "cross edges".



No cross edges in DFS on undirected graphs

Claim: During DFS($\boldsymbol{x}$) every vertex marked "visited" is a descendant of $\boldsymbol{x}$ in the DFS tree $\boldsymbol{T}$

Claim: For every $\boldsymbol{x}, \boldsymbol{y}$ in the DFS tree $\boldsymbol{T}$, if $(\boldsymbol{x}, \boldsymbol{y})$ is an edge *not* in $\boldsymbol{T}$ then one of $\boldsymbol{x}$ or $\boldsymbol{y}$ is an ancestor of the other in $\boldsymbol{T}$

Proof:

- One of DFS($\boldsymbol{x}$) or DFS($\boldsymbol{y}$) is called first, suppose WLOG that DFS($\boldsymbol{x}$) was called before DFS($\boldsymbol{y}$)
- During DFS($\boldsymbol{x}$), the edge $(\boldsymbol{x}, \boldsymbol{y})$ is examined
- Since $(\boldsymbol{x}, \boldsymbol{y})$ is a *not* an edge of $\boldsymbol{T}$, $\boldsymbol{y}$ was already visited when edge $(\boldsymbol{x}, \boldsymbol{y})$ was examined during DFS($\boldsymbol{x}$)
- Therefore $\boldsymbol{y}$ was visited during the call to DFS($\boldsymbol{x}$) so $\boldsymbol{y}$ is a descendant of $\boldsymbol{x}$.

Applications of Graph Traversal: Bipartiteness Testing

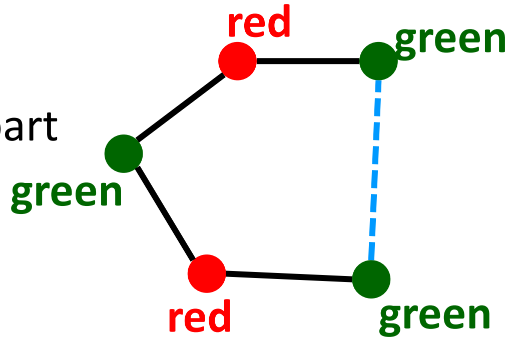
Definition: An undirected graph G is bipartite iff we can color its vertices **red** and **green** so each edge has different color endpoints

Input: Undirected graph G

Goal: If G is bipartite, output a coloring;
otherwise, output "NOT Bipartite".

Fact: Graph G contains an odd-length cycle $\Rightarrow$ it is not bipartite

Just coloring the cycle part of G is impossible



On a cycle the two colors must alternate, so

- **green** every 2nd vertex
- **red** every 2nd vertex

Can't have either if length is not divisible by 2.

Applications of Graph Traversal: Bipartiteness Testing

WLOG ("without loss of generality"): Can assume that G is connected

- Otherwise run on each component

Simple idea: start coloring nodes starting at a given node s

- Color s **red**
- Color all neighbors of s **green**
- Color all their neighbors **red**, etc.
- If you ever hit a node that was already colored
  - the **same** color as you want to color it, ignore it
  - the **opposite** color, output "NOT Bipartite" and halt

BFS gives Bipartiteness

Run BFS assigning all vertices from layer L_i the color $i \bmod 2$

- i.e., **red** if they are in an even layer, **green** if in an odd layer

- if no edge joining two vertices of the same color
  - then it is a good coloring
- otherwise
  - there is a bad edge; output “Not Bipartite”

Why is that “Not Bipartite” output correct?

Why does BFS work for Bipartiteness?

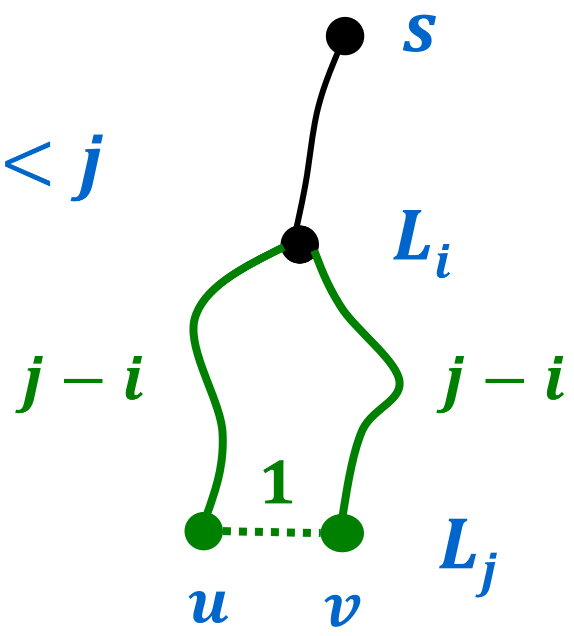
Recall: All edges join vertices on the same or adjacent BFS layers

⇒ Any "bad" edge must join two vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ in the same layer

Say the layer with $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{L_j}$

$\boldsymbol{u}$ and $\boldsymbol{v}$ have common ancestor at some level $\boldsymbol{L_i}$ for $\boldsymbol{i < j}$

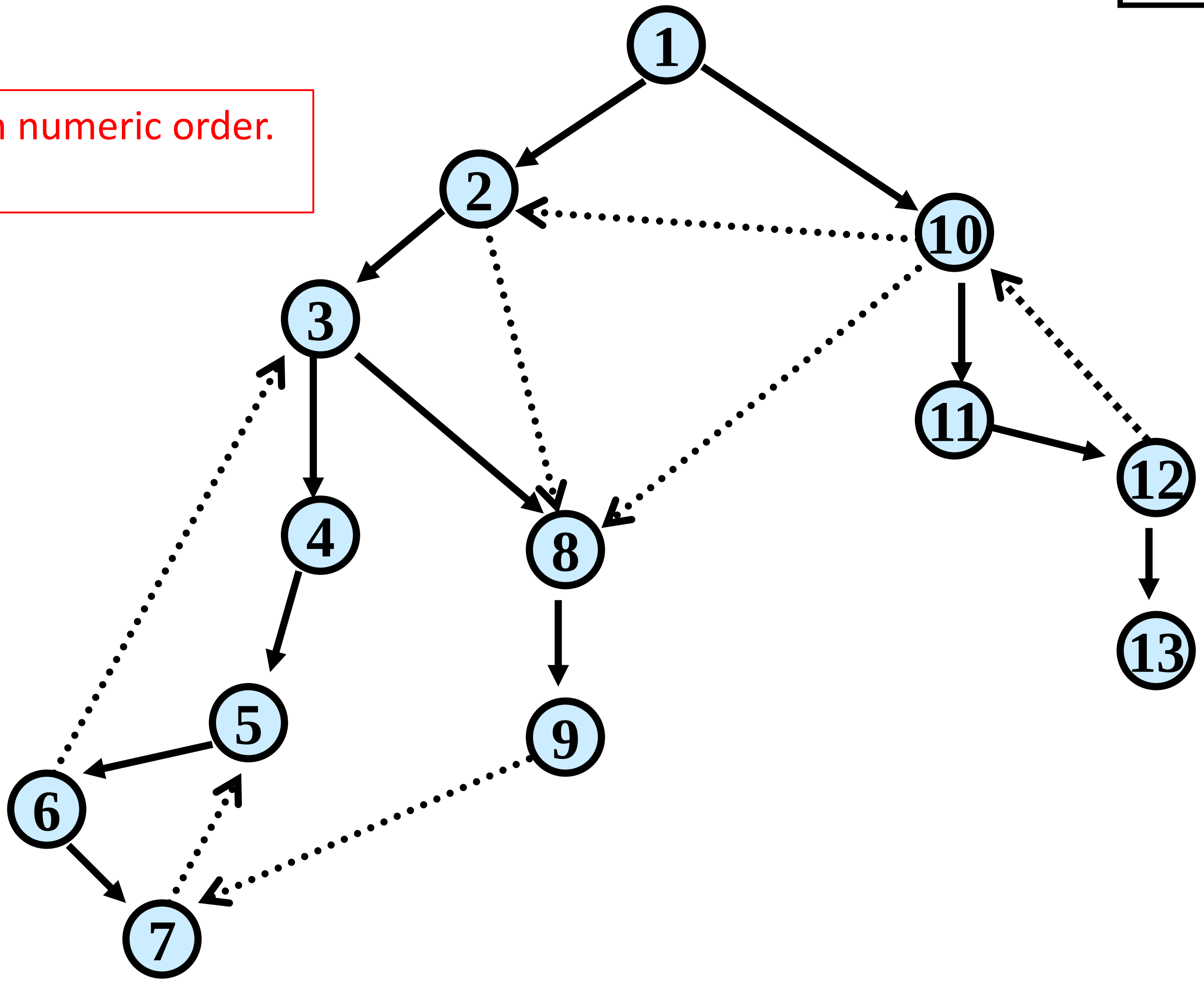
Odd cycle of length $\mathbf{2}(\boldsymbol{j - i}) + \mathbf{1}$
⇒ Not Bipartite



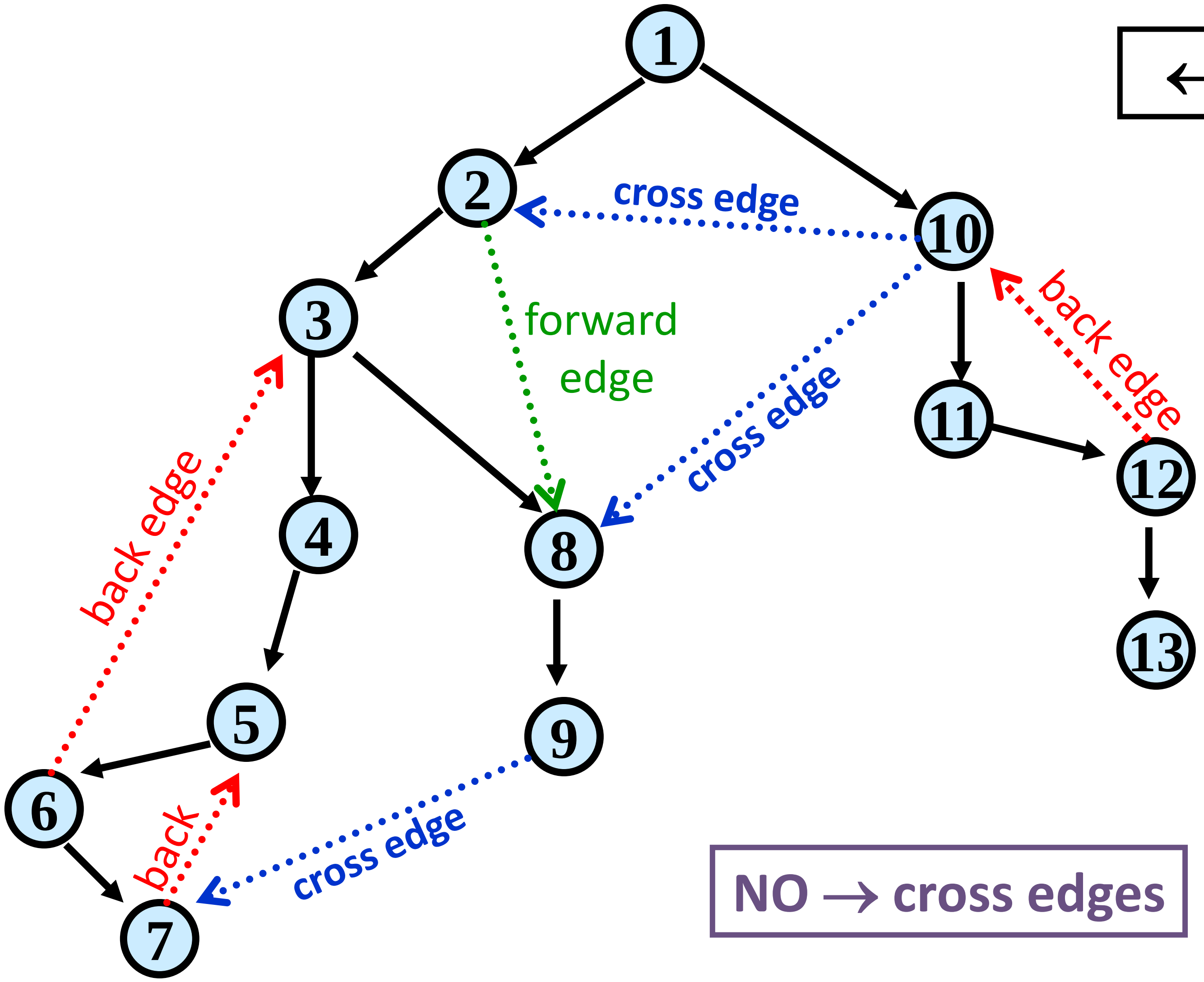
DFS($\boldsymbol{v}$) for a directed graph

← **tree edges**

Nodes were visited in numeric order.
How can we tell?



DFS($\boldsymbol{v}$)



← tree edges

NO → cross edges

Properties of Directed DFS

- Before DFS(s) returns, it visits all previously unvisited vertices reachable via directed paths from s

- Every cycle contains a back edge in the DFS tree

Directed Acyclic Graphs

A directed graph $G = (V, E)$ is acyclic iff it has no directed cycles

Terminology: A directed acyclic graph is also called a DAG

After shrinking the strongly connected components of a directed graph to single vertices, the result is a DAG

Topological Sort

Given: a directed acyclic graph (DAG) $G = (V, E)$

Output: numbering of the vertices of G with distinct numbers from 1 to n so that edges only go from lower numbered to higher numbered vertices

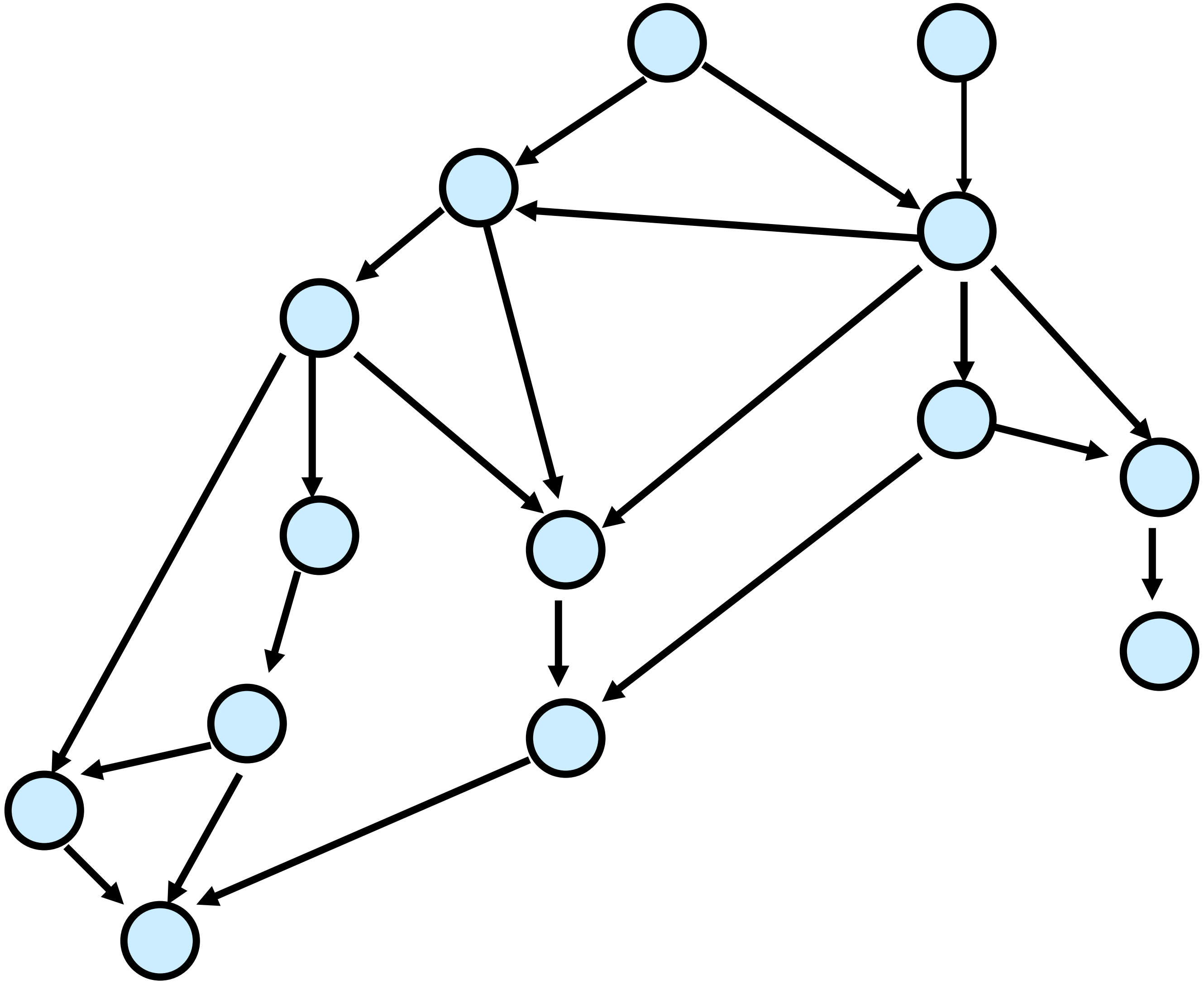
Applications:

- nodes represent tasks
- edges represent precedence between tasks
- topological sort gives a sequential schedule for solving them

Nice algorithmic paradigm for general directed graphs:

- Process strongly connected components one-by-one in the order given by topological sort of the DAG you get from shrinking them.

Directed Acyclic Graph



In-degree 0 vertices

Claim: Every DAG has a vertex of in-degree 0

Proof: By contradiction

Suppose every vertex has some incoming edge

Consider following procedure:

while (true) do

$\boldsymbol{v}$ = some predecessor of $\boldsymbol{v}$

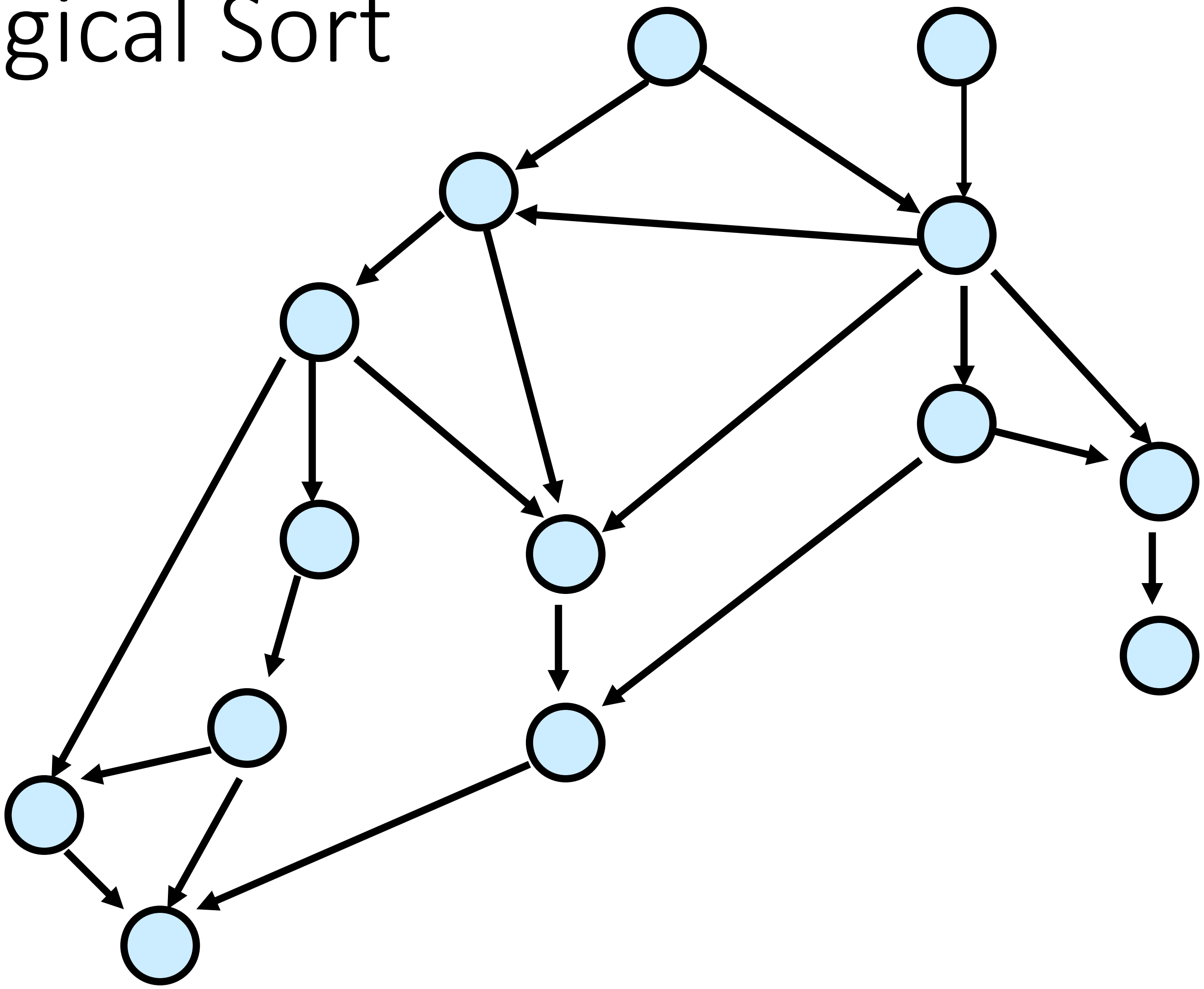
- After $\boldsymbol{n+1}$ steps where $\boldsymbol{n = |V|}$ there will be a repeated vertex
  - This yields a cycle, contradicting that it is a DAG.

Topological Sort

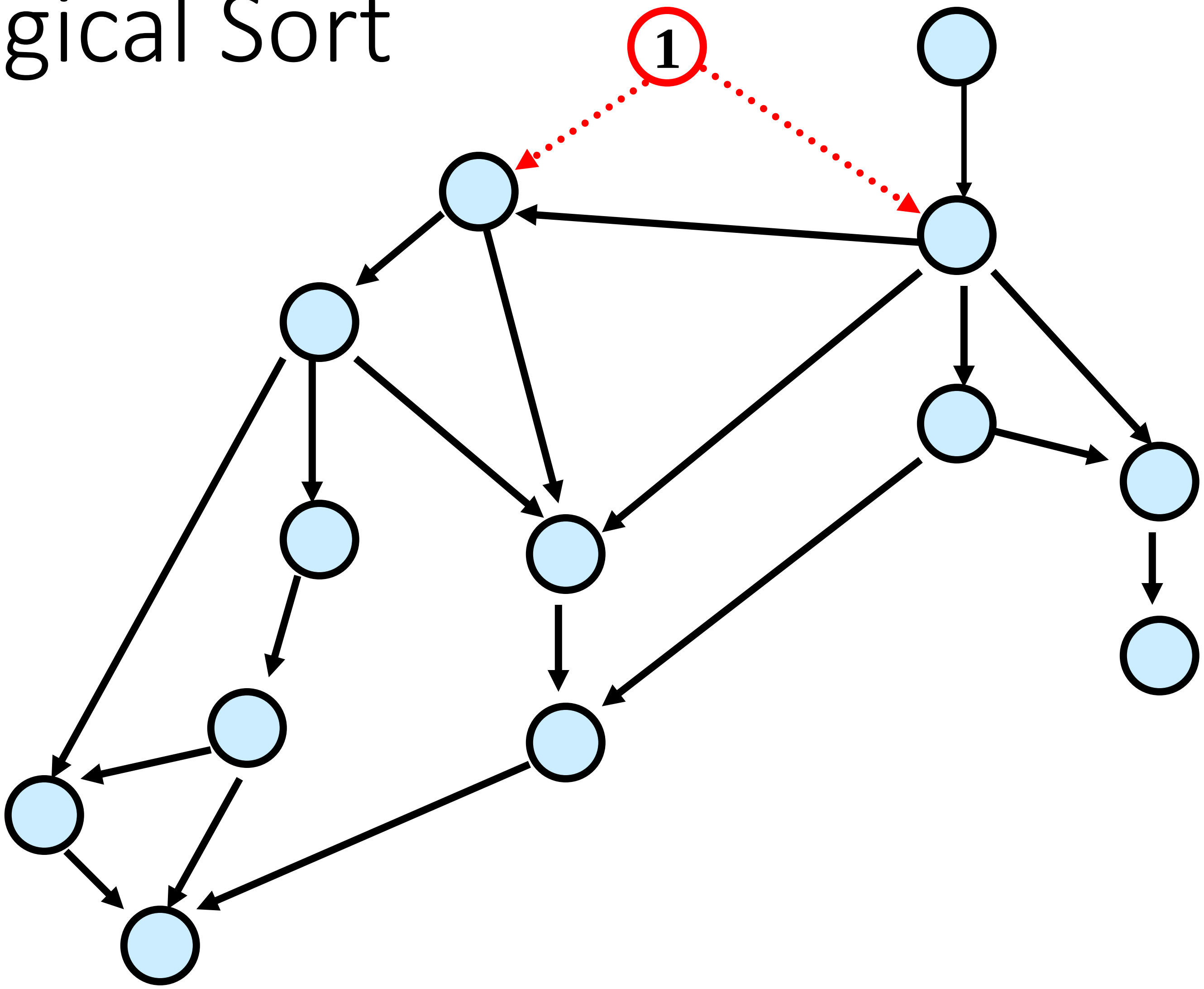
- Can do using DFS

- Alternative simpler idea:
  - Any vertex of in-degree 0 can be given number 1 to start
  - Remove it from the graph
  - Then give a vertex of in-degree 0 number 2
  - Etc.

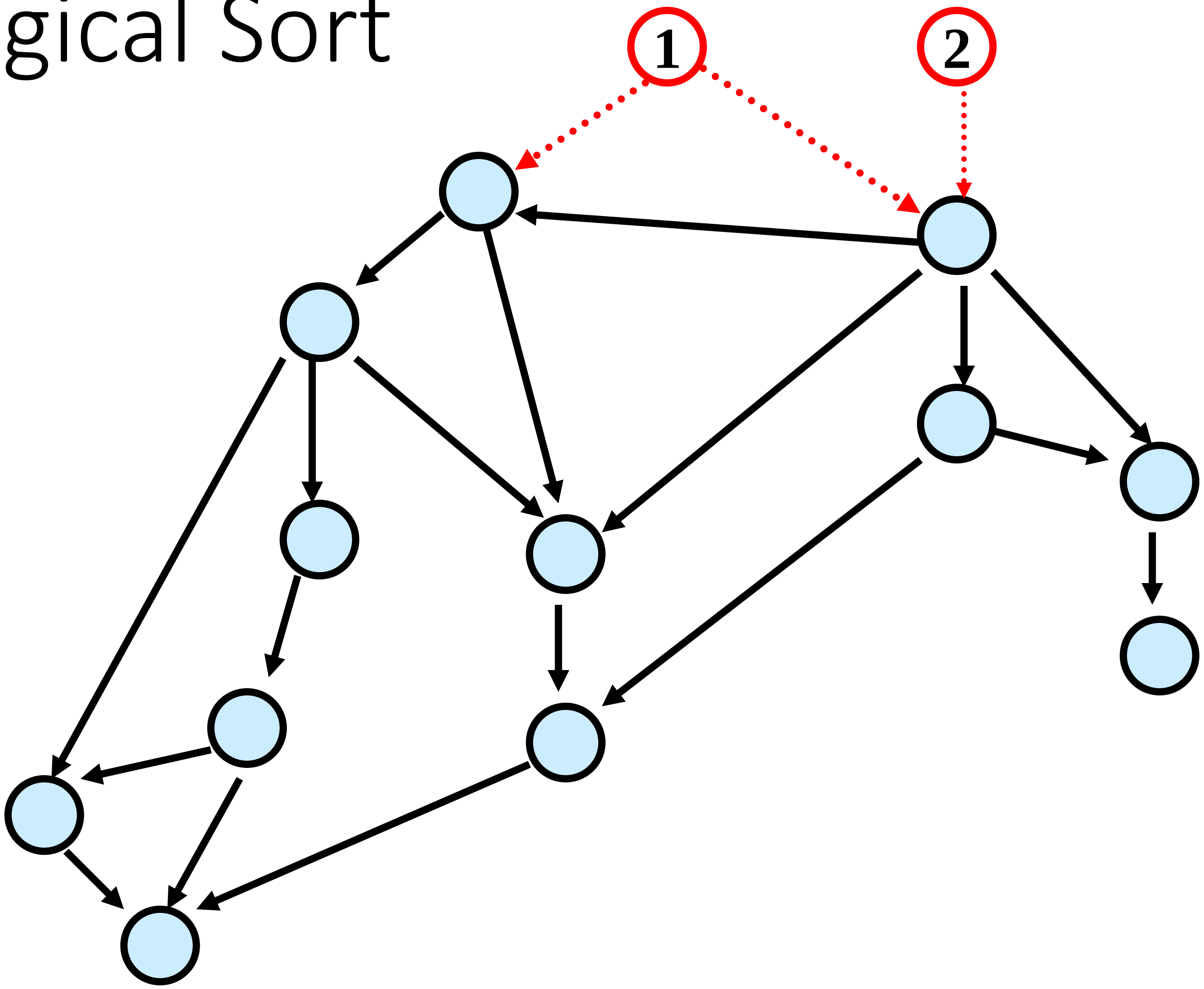
Topological Sort



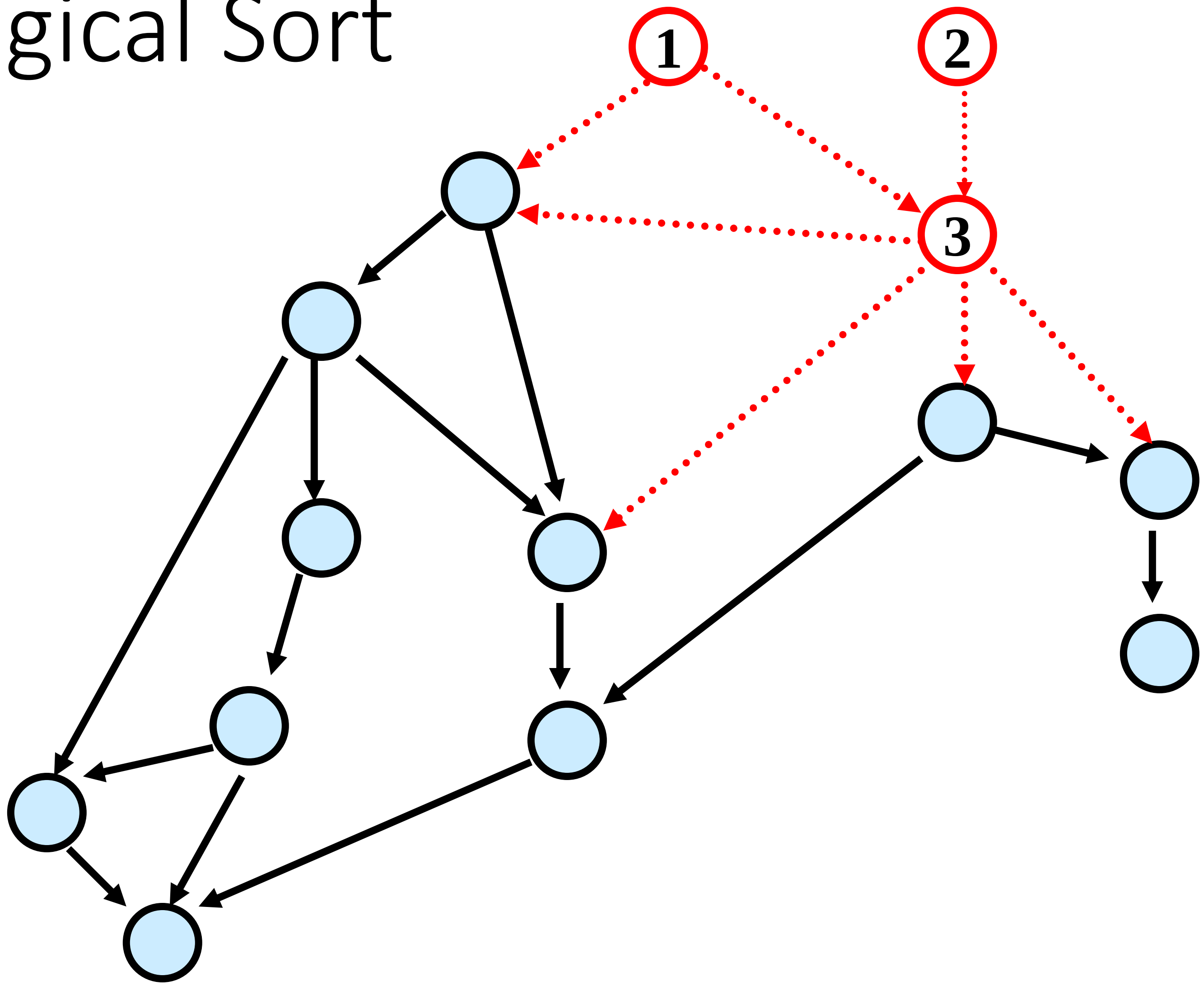
Topological Sort



Topological Sort



Topological Sort



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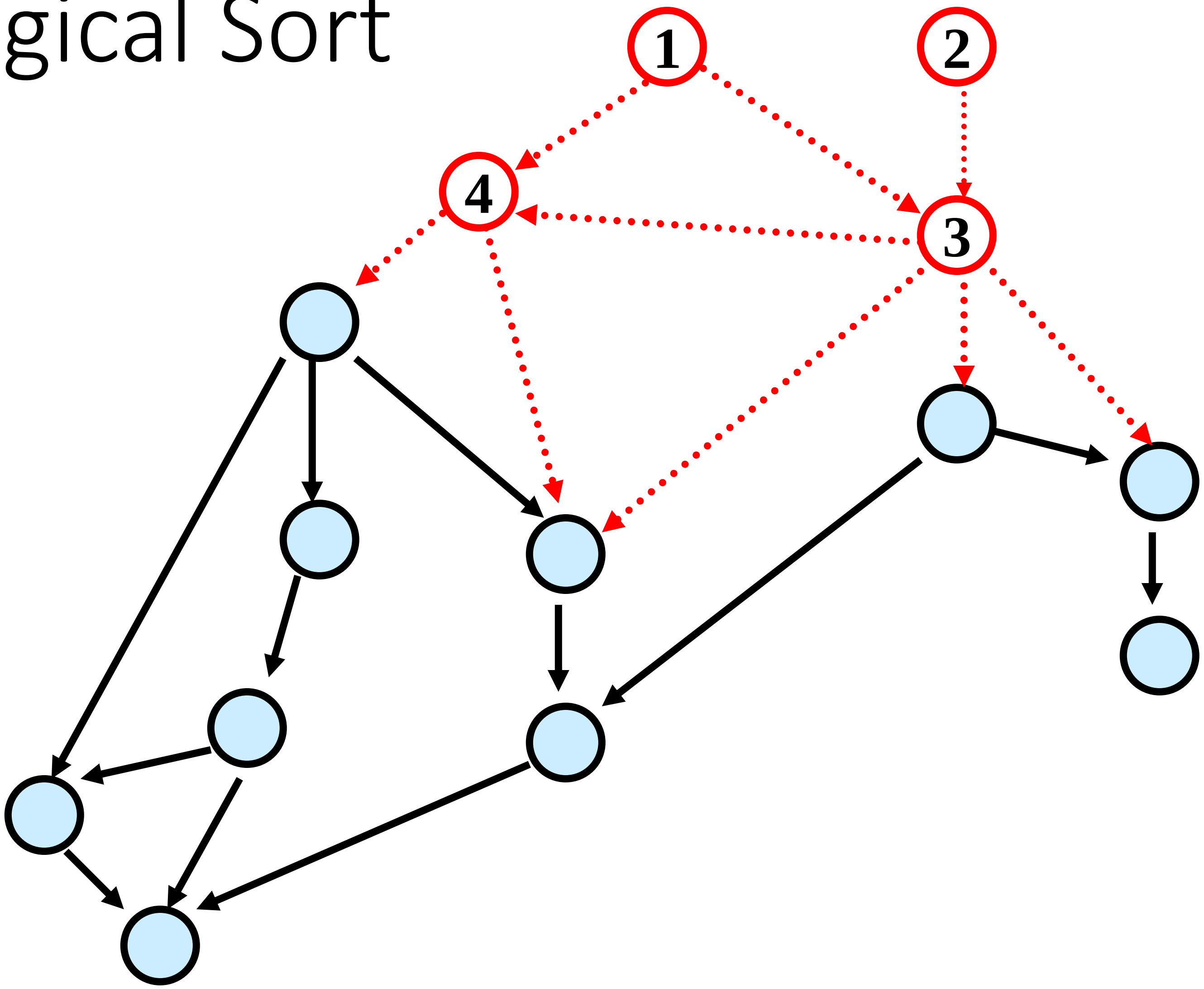
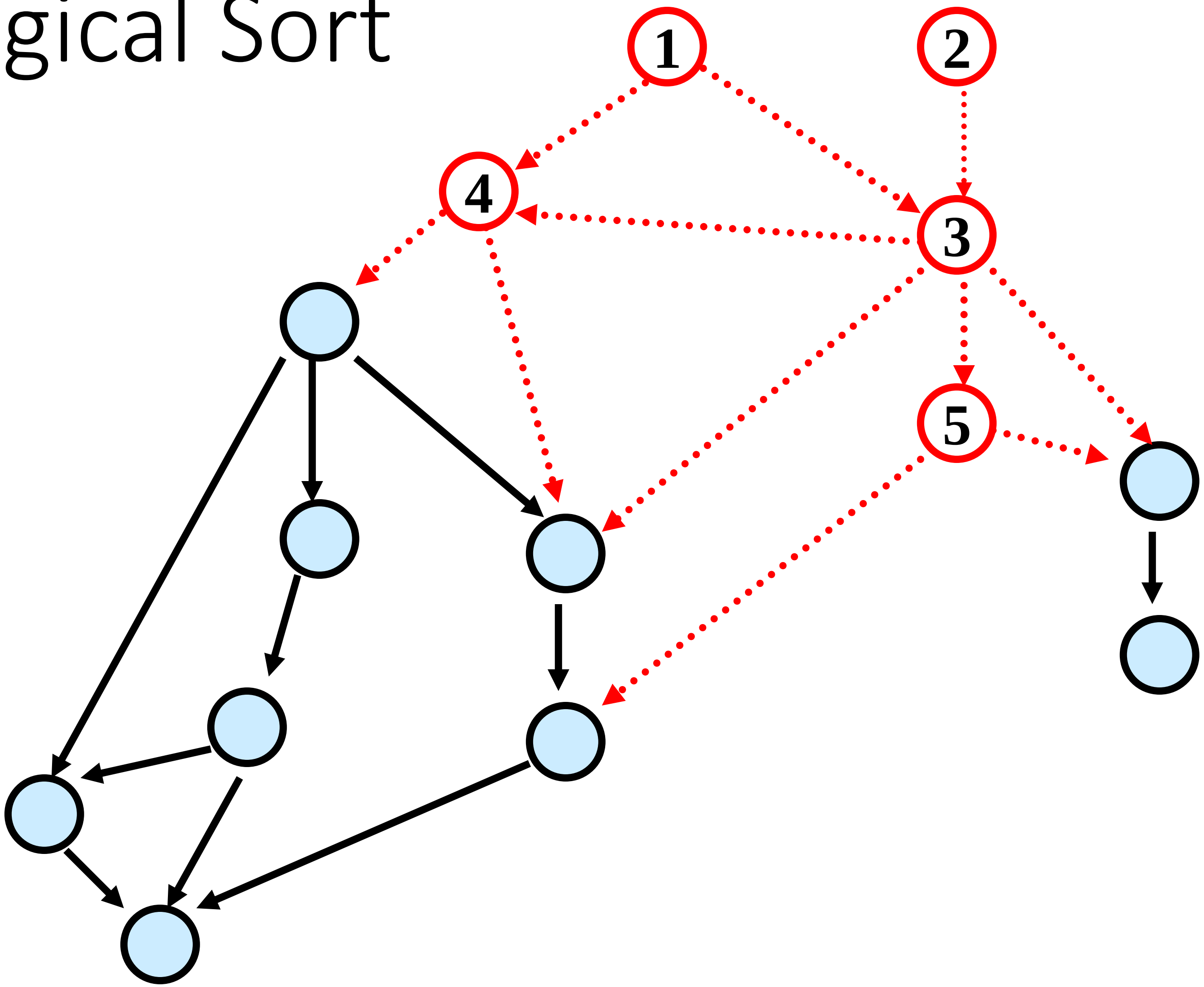


Topological Sort



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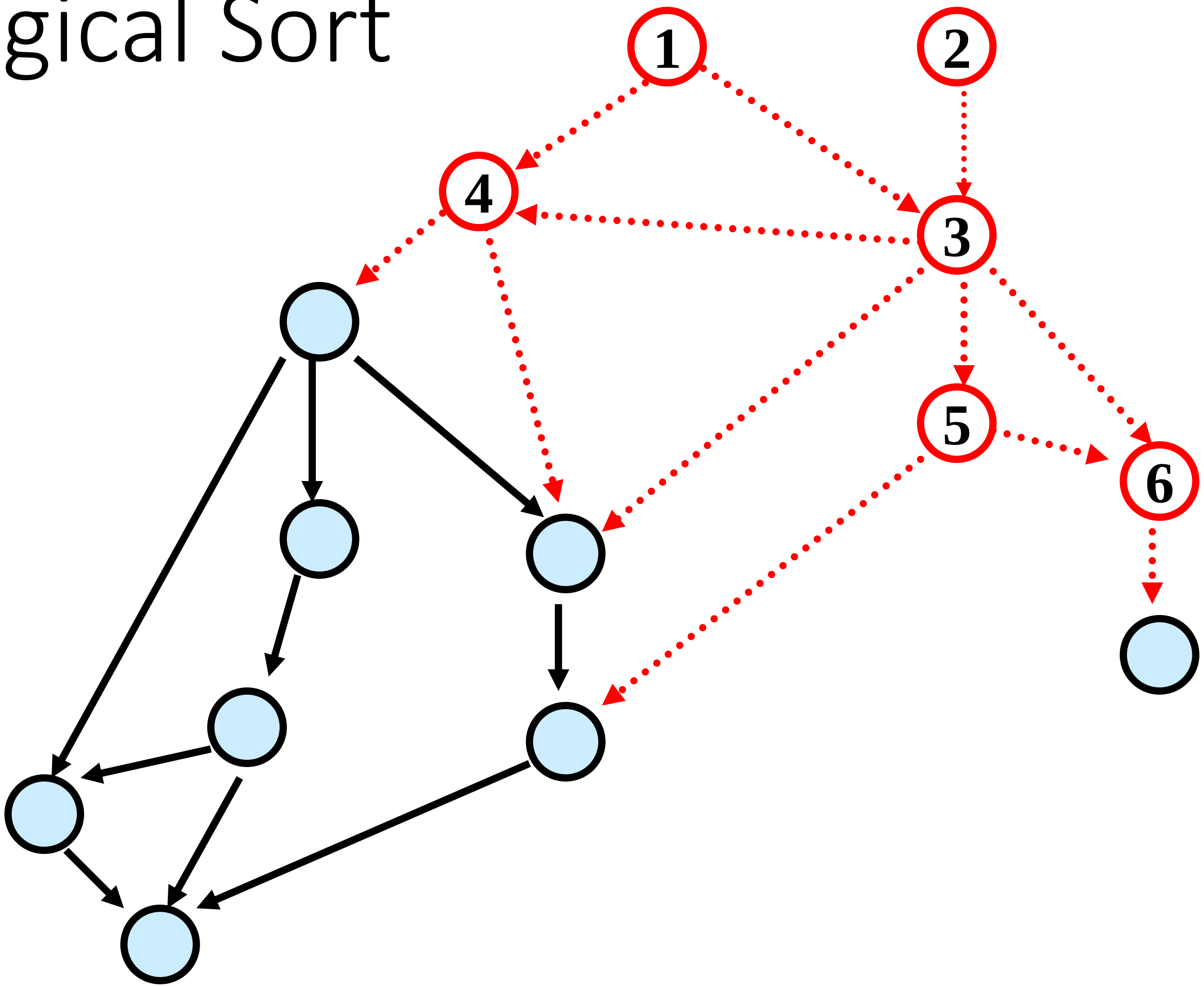

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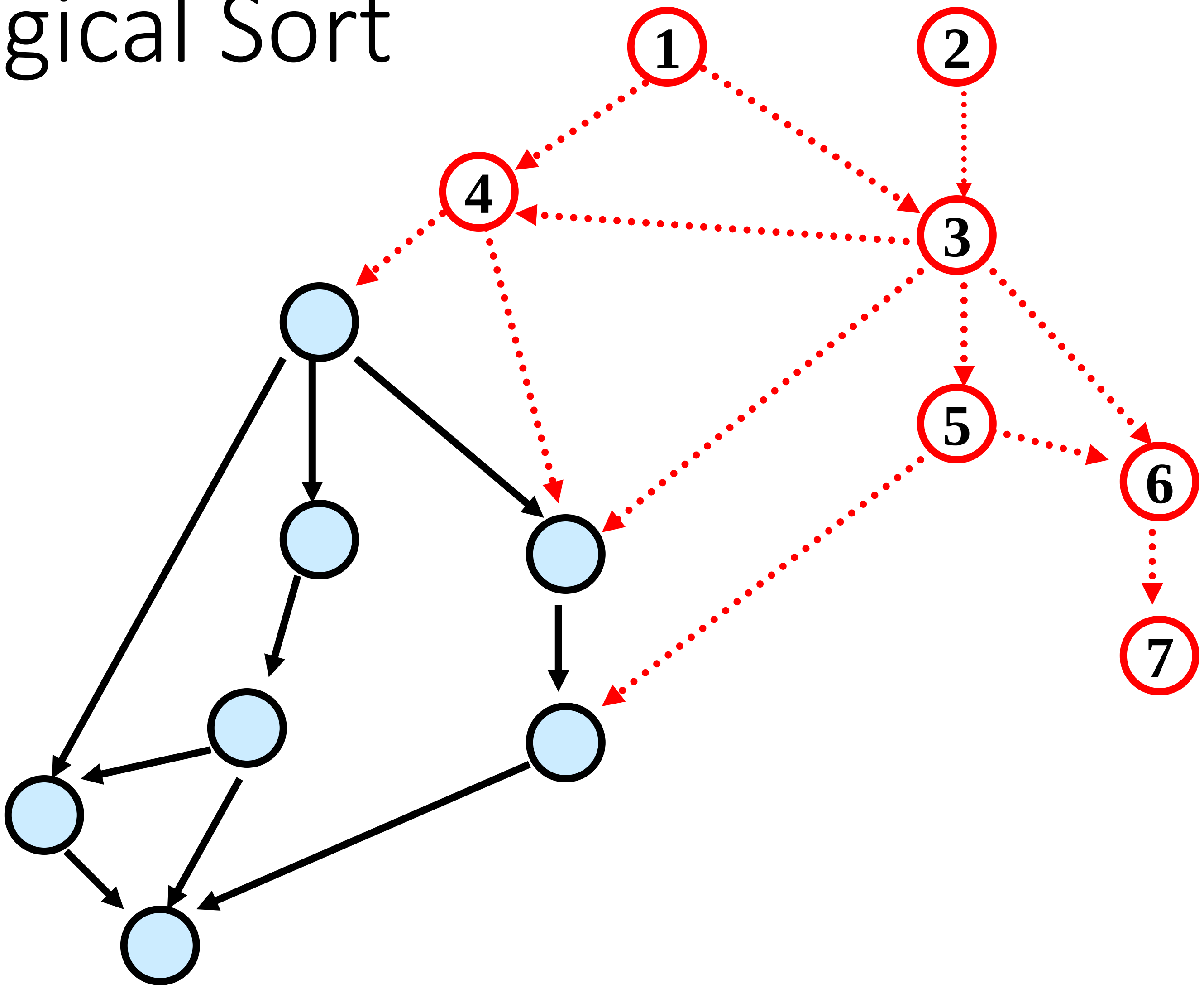

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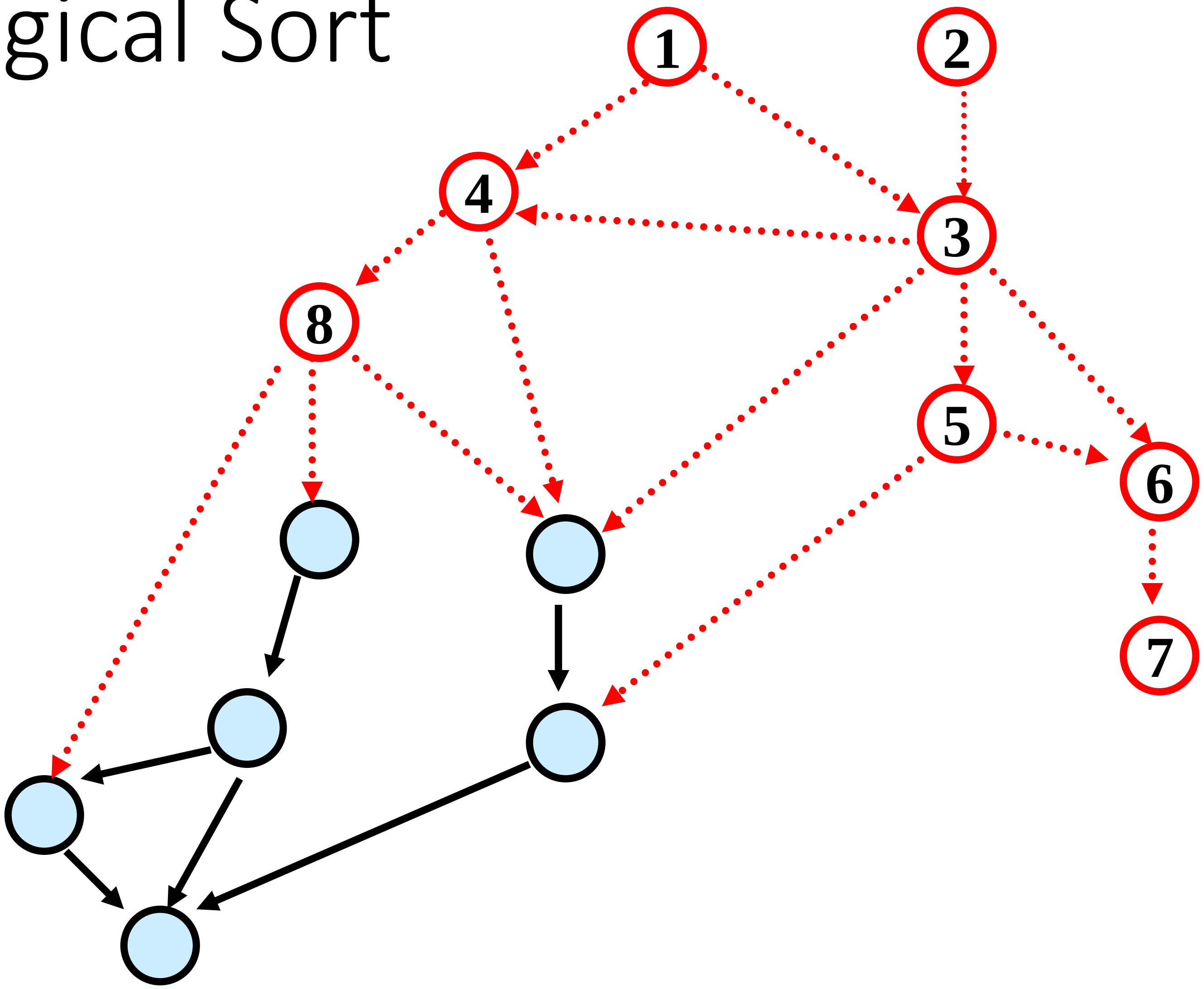

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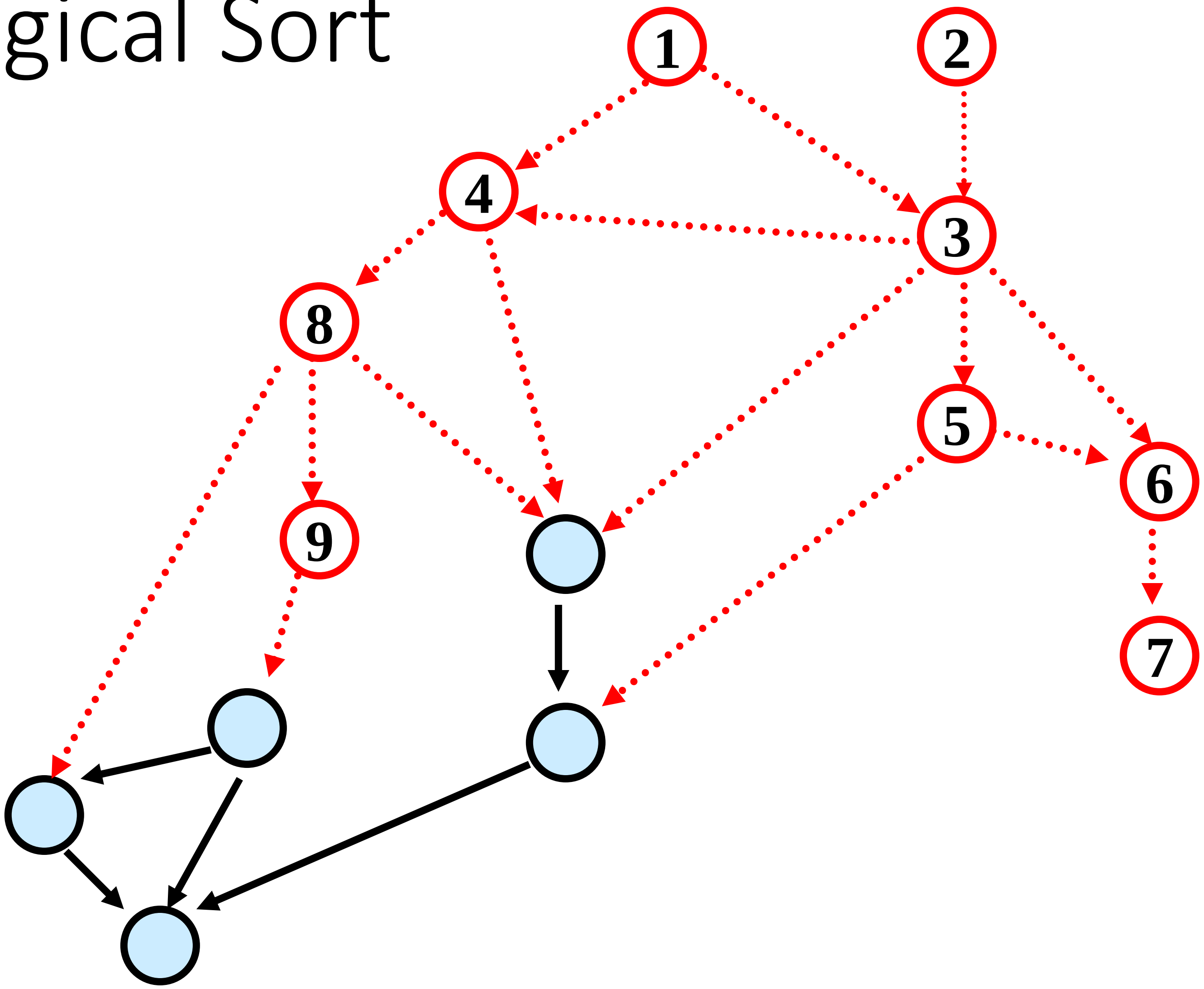

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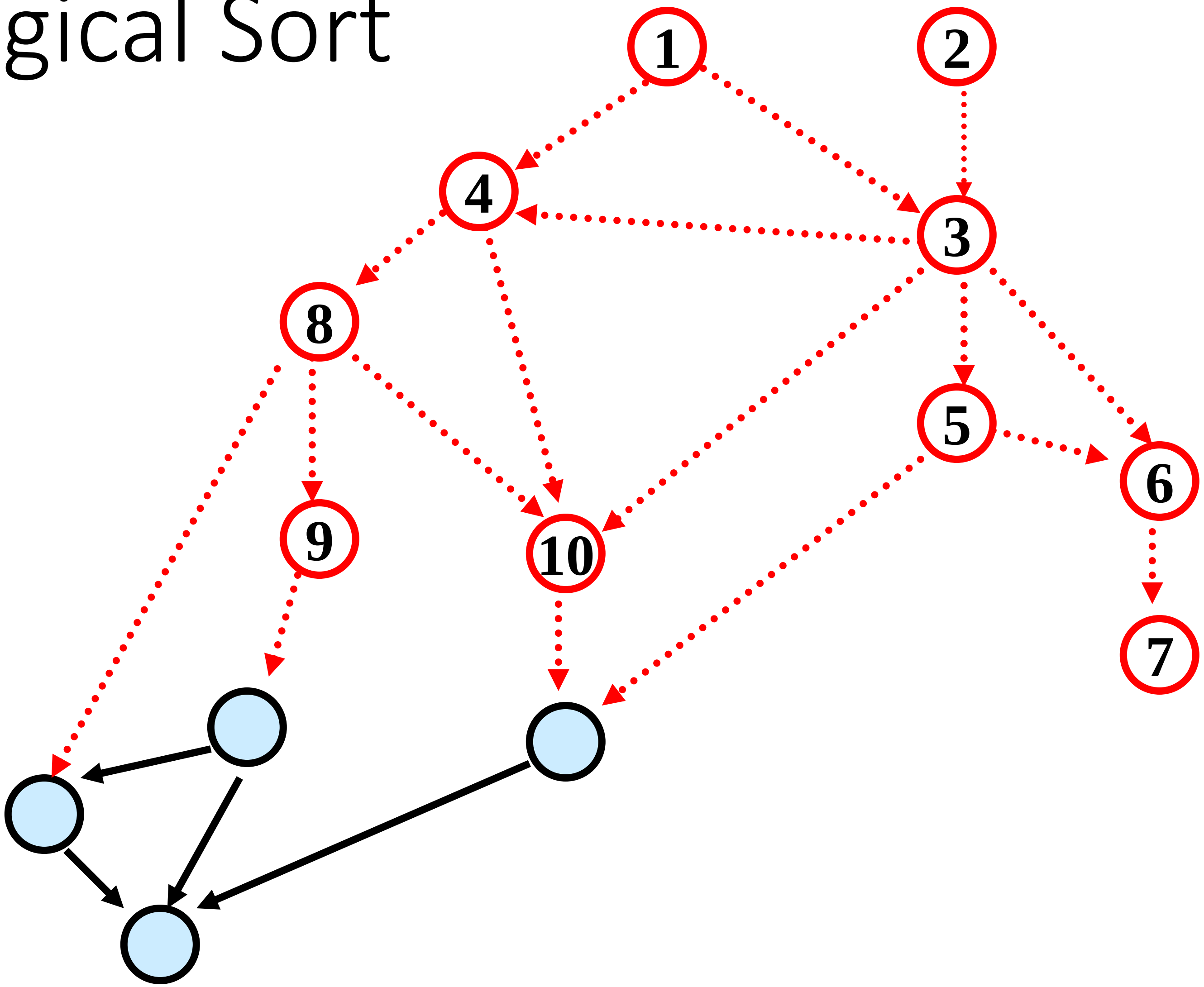

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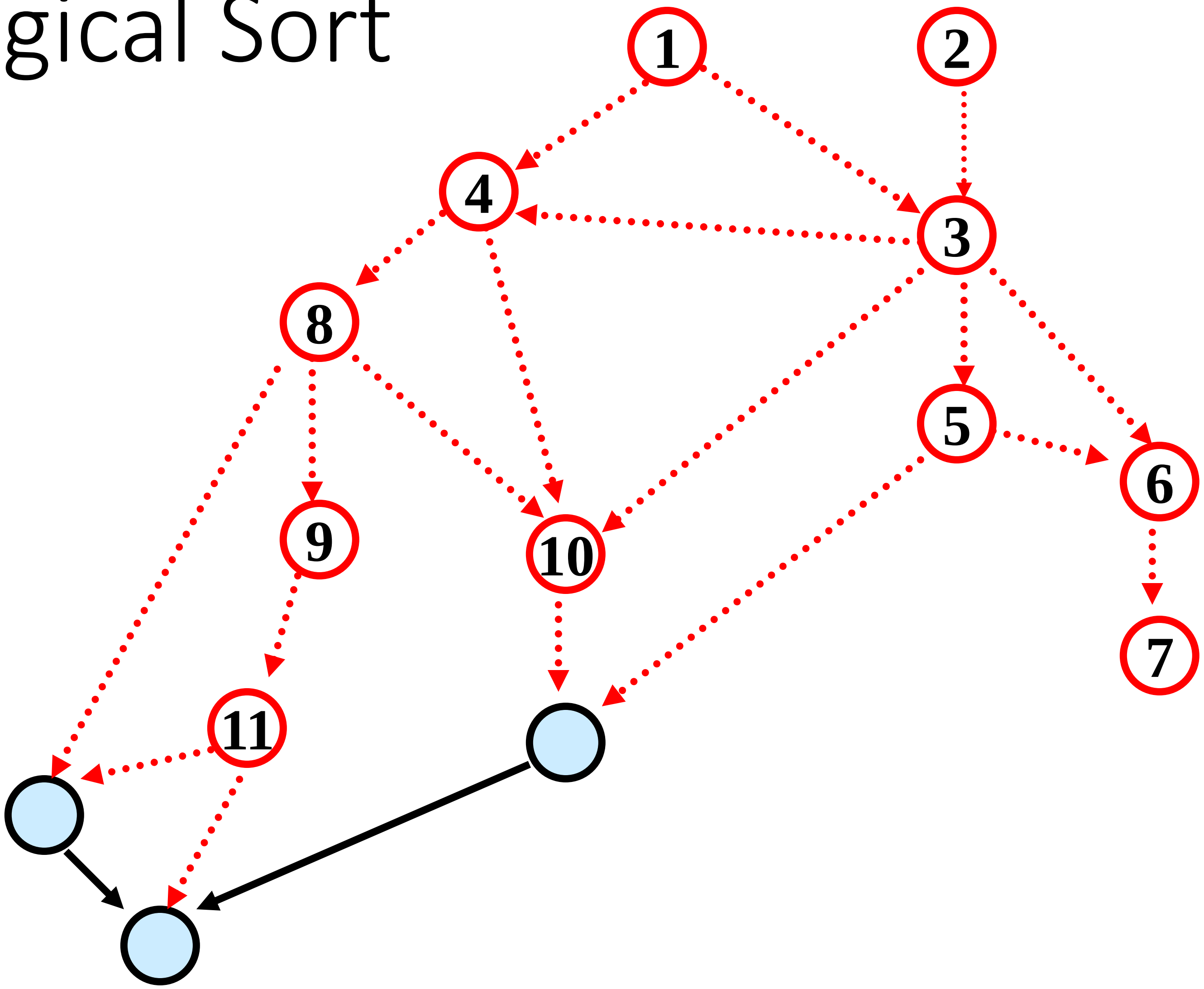

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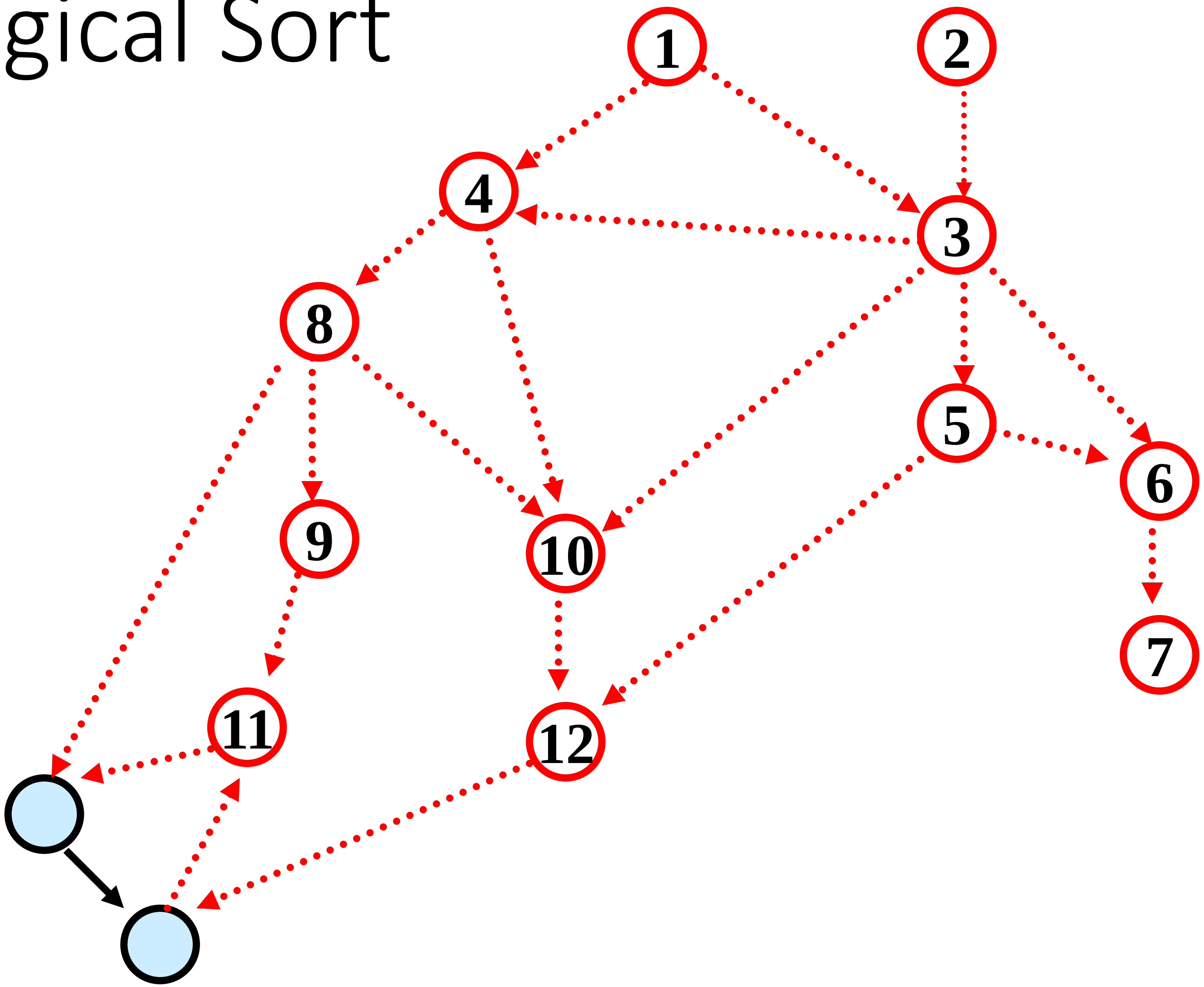

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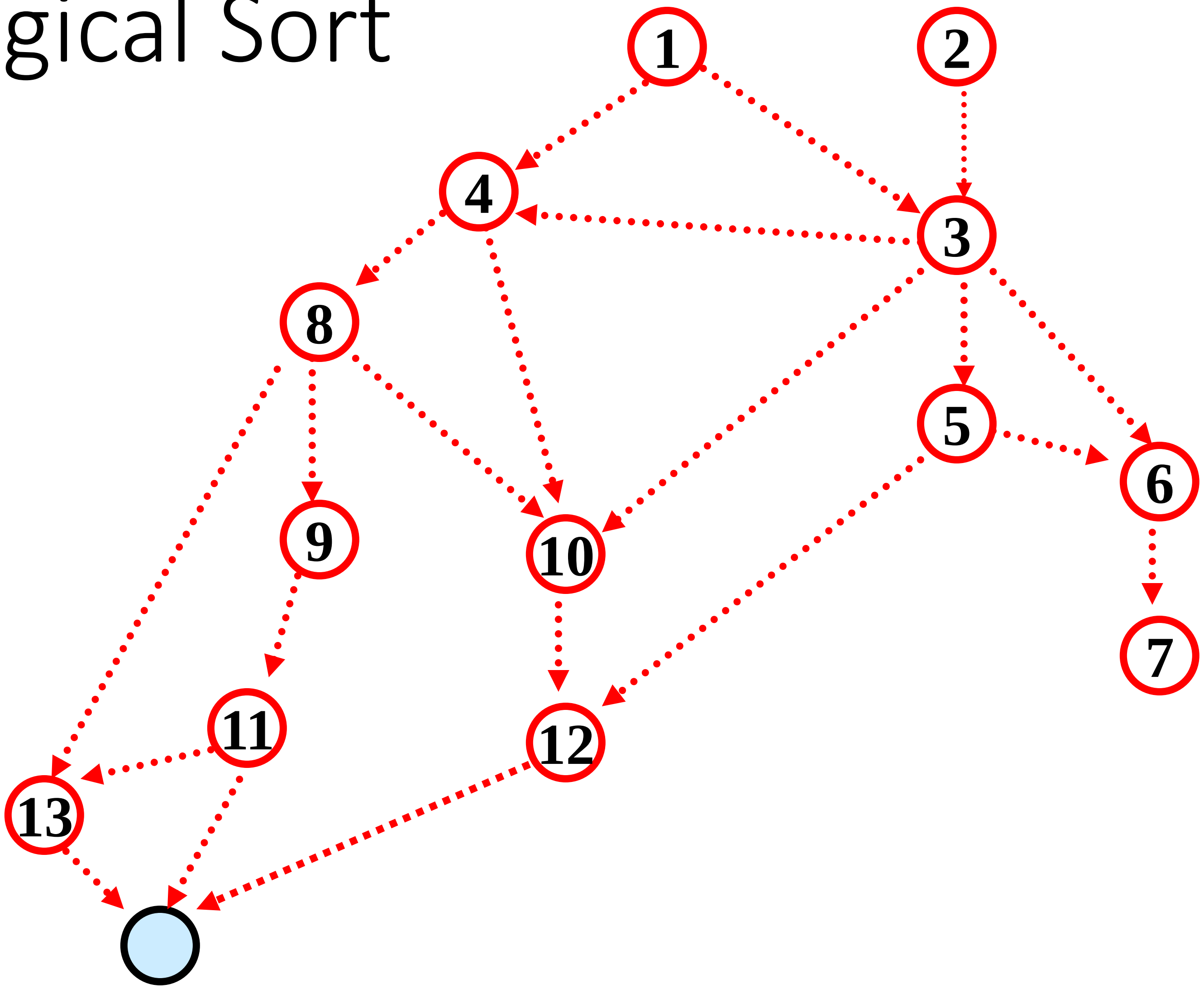

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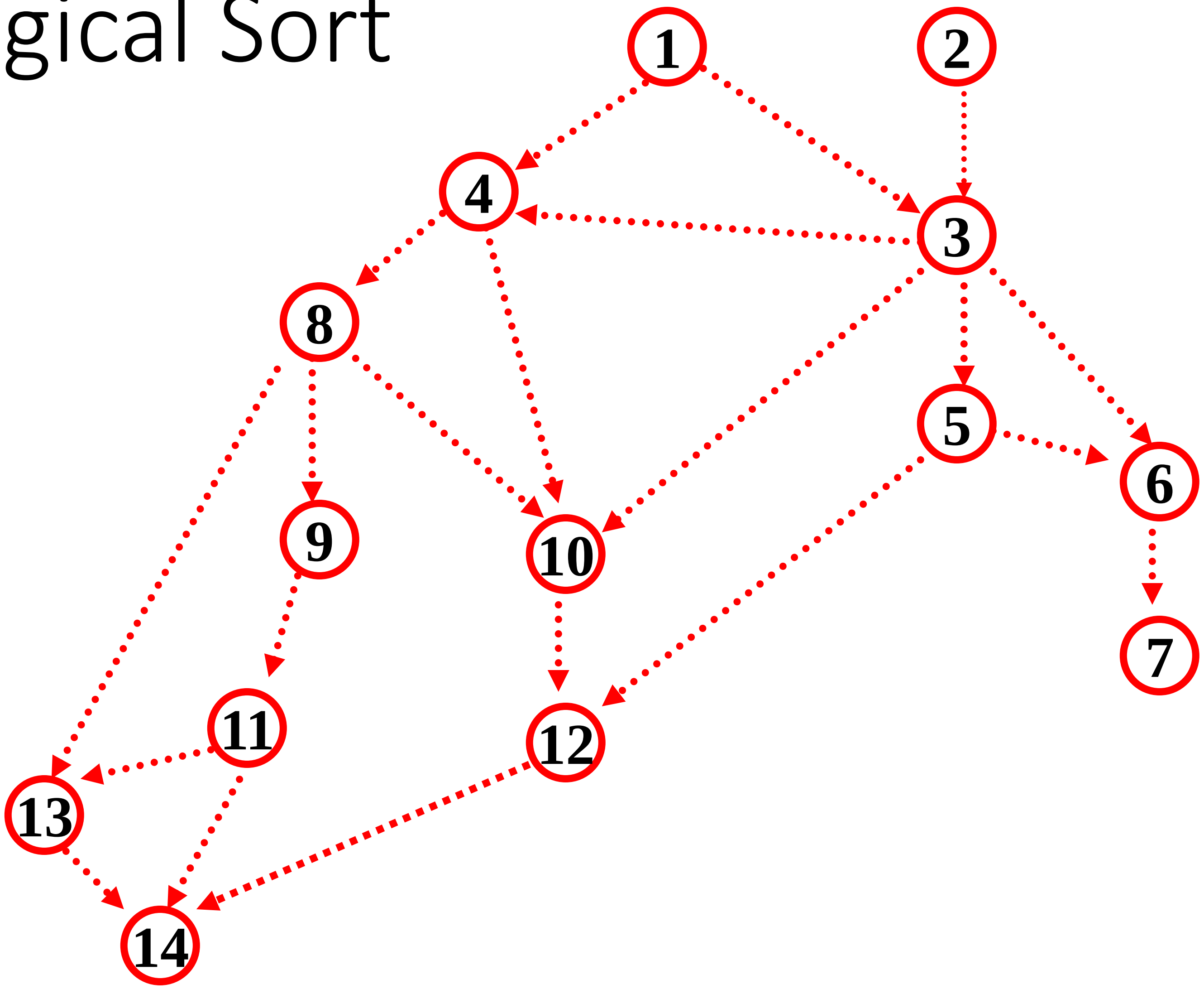


Implementing Topological Sort

- Go through all edges, computing array with in-degree for each vertex
  $O(m+n)$
- Maintain a list of vertices of in-degree $\mathbf{0}$
- Remove any vertex in list and number it
- When a vertex is removed, decrease in-degree of each neighbor by $\mathbf{1}$ and add them to the list if their degree drops to $\mathbf{0}$

Total cost: $O(m+n)$

Strongly Connected Components of Directed Graphs

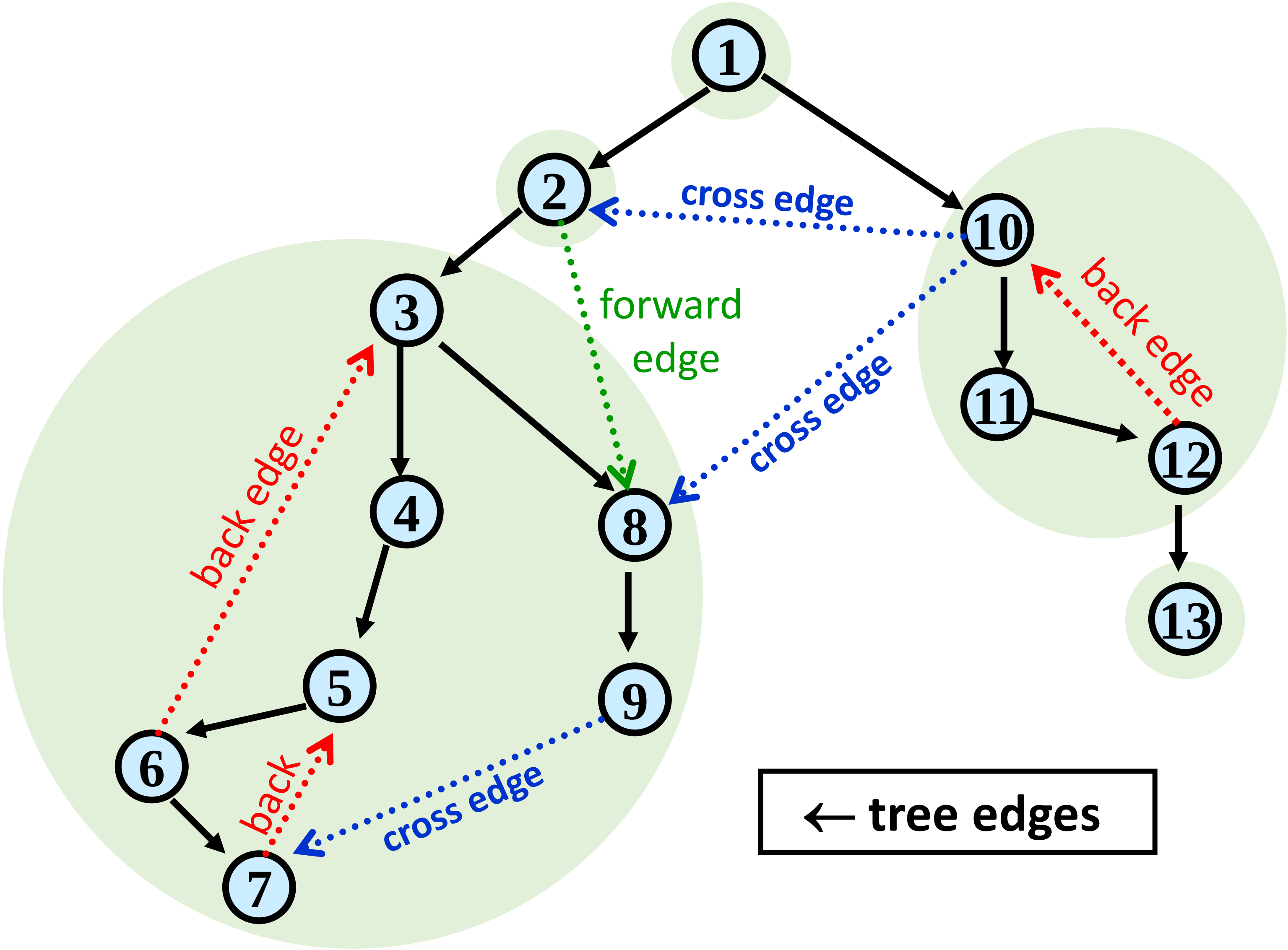
Defn: Vertices u and v are strongly connected iff they are on a directed cycle (there are paths from u to v and from v to u).

Defn: Can partition vertices of any directed graph into strongly connected components:

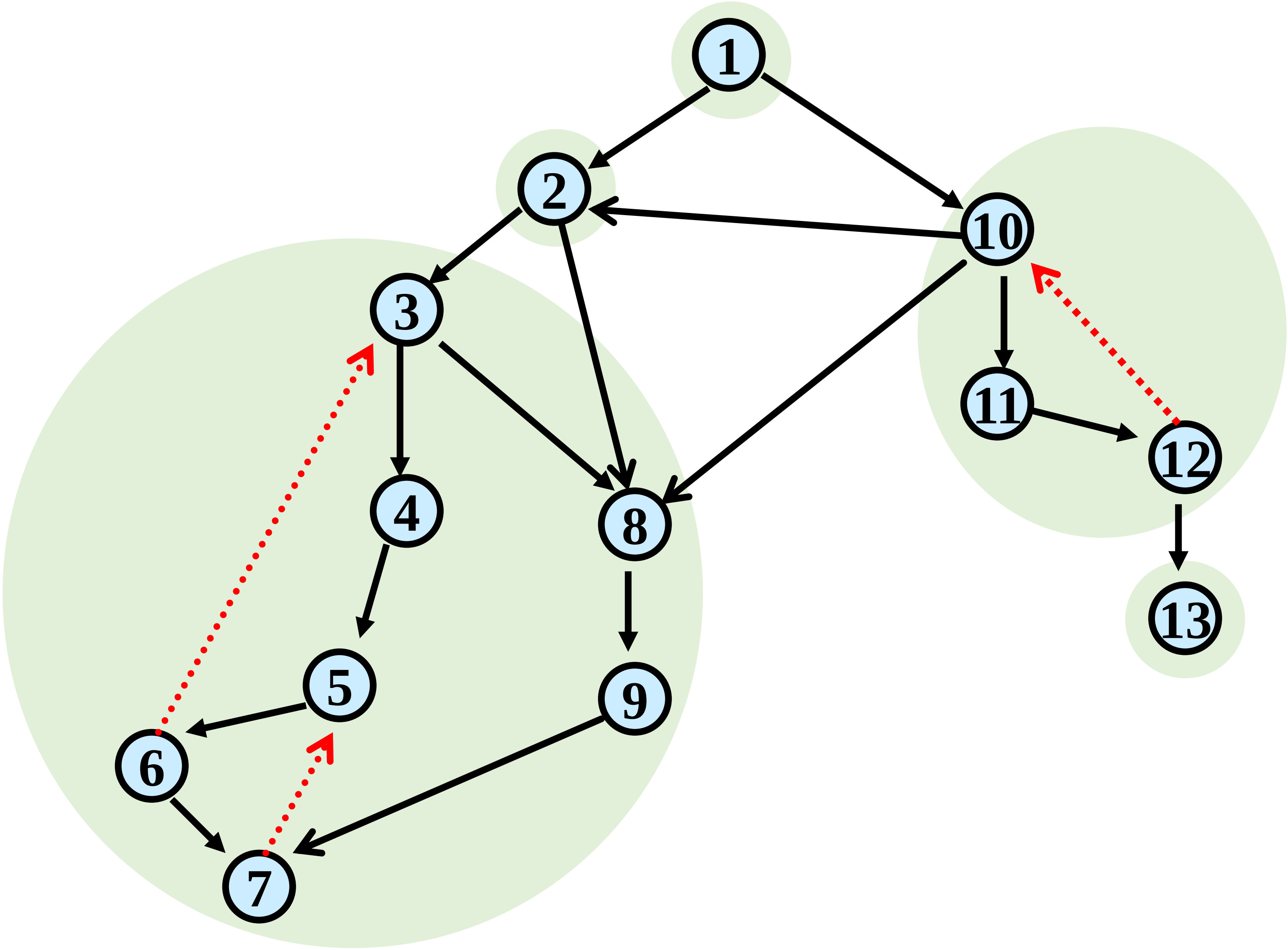
1. all pairs of vertices in the same component are strongly connected
2. can't merge components and keep property 1

- Strongly connected components can be stored efficiently just like connected components
- Can be found in $O(n+m)$ time using a DFS then a BFS
  - Do a depth-first sort, keeping track of the order nodes are marked "fully-explored"
  - Going in order from least recent to most recent, run connected components

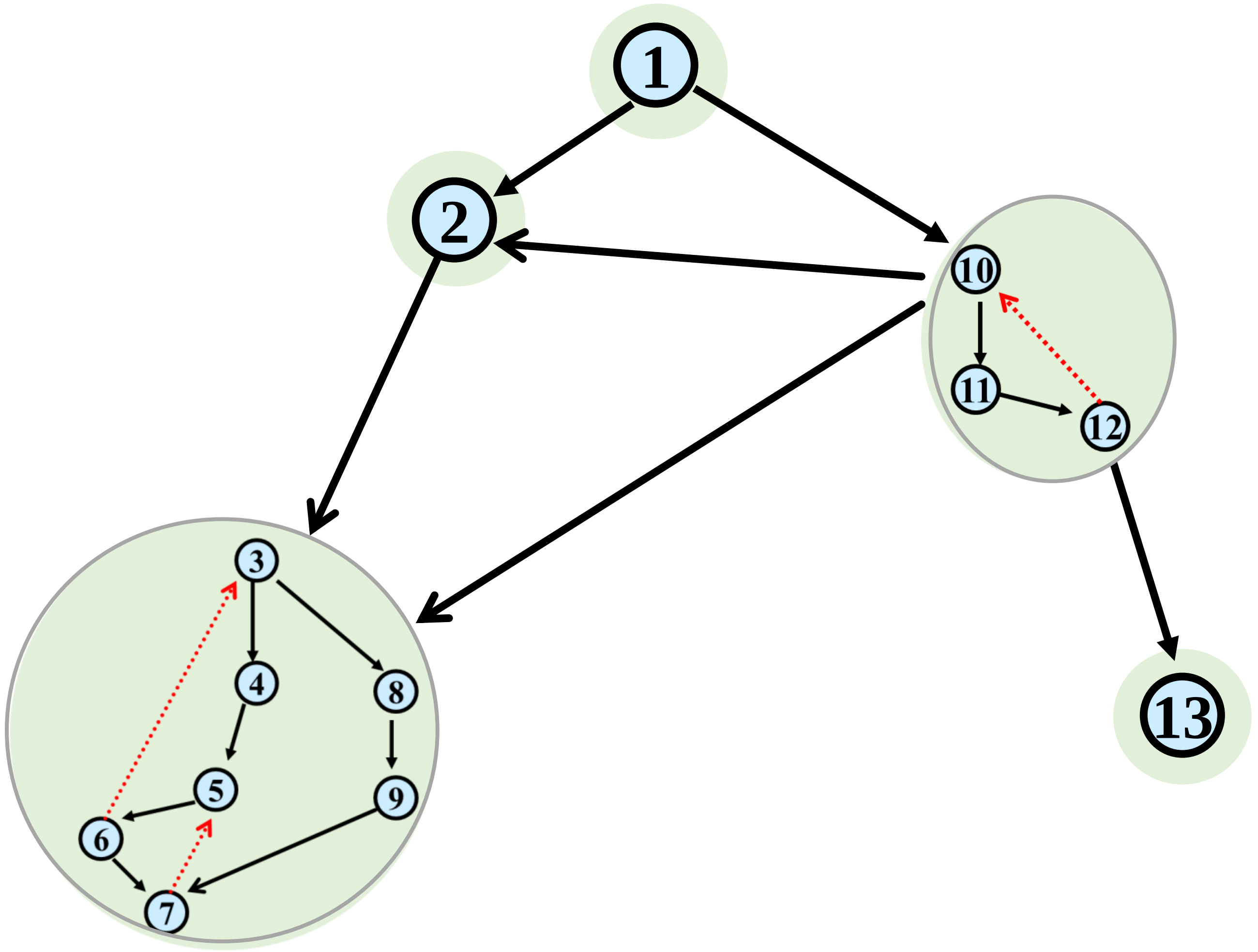
Strongly Connected Components



Strongly Connected Components



Strongly Connected Components



Strongly-Connected Components Usage

Common algorithmic paradigm for general directed graphs:

- Process strongly connected components one-by-one in the order given by topological sort of the DAG you get from shrinking them.