CSE 421 Winter 2025

Lecture 21: Linear Duality 👿 / Relaxation 😌

Literal not LP

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throw new
NoPollEVFoundException

Hello.

My name is Owen! I’m uh. An undergrad who is obsessed with teaching and theory.

{Resume = null, Teaching Credentials = null}

I’ll be available for questions after lecture! (only 10 min I have crypto ☹)

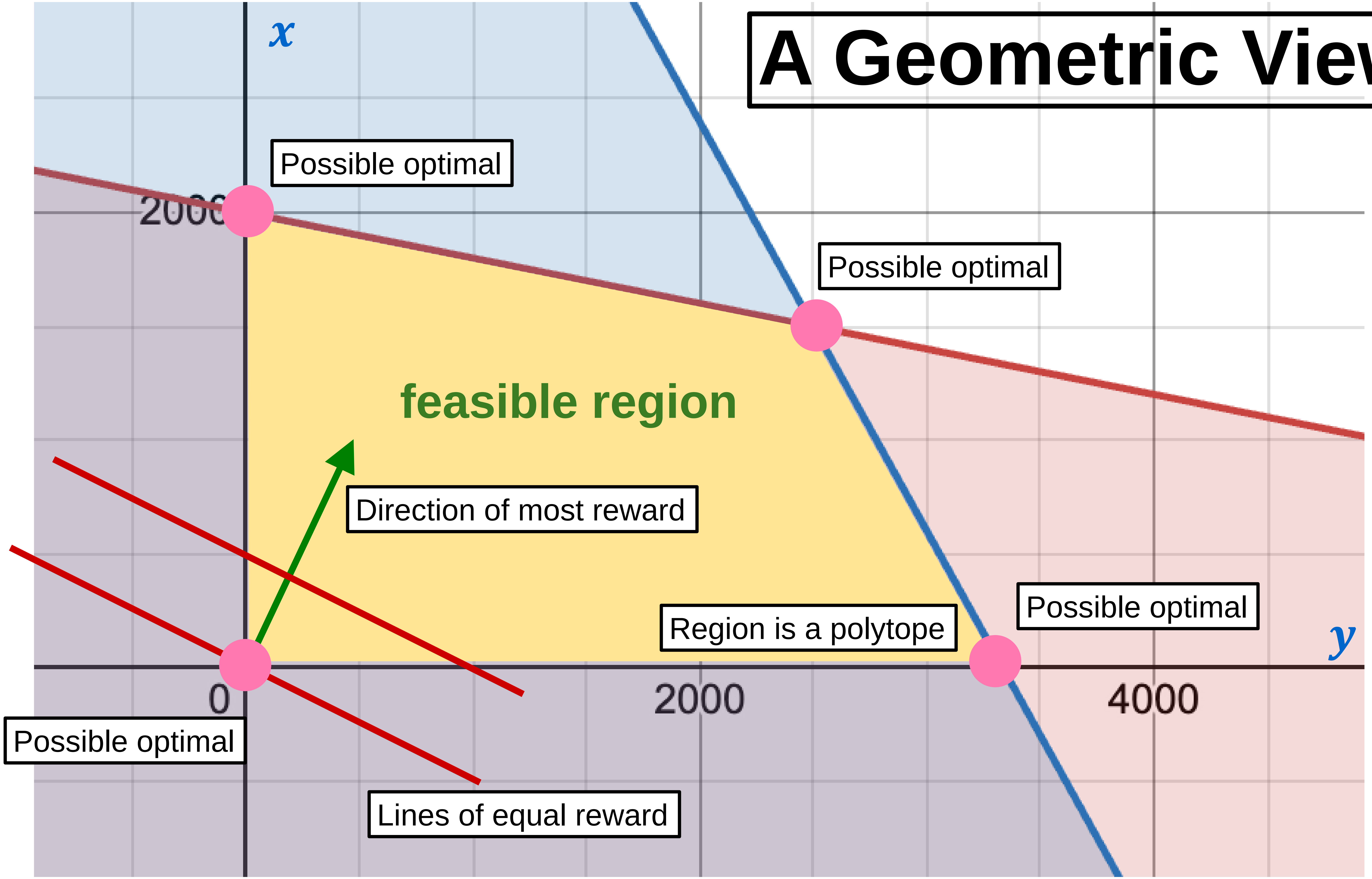
Topic of the Day.

- Linear Programs Review
- Linear Duality
- Boba Buying

A review of LPs

- We looked at a very general class of optimization problems
- We looked at few ways of **understanding** them:
  - **Geometric** View
  - **Linear Algebraic** View
- We looked at ways of **writing** them:
  - **Standard** Form
- We looked at ways of **applying** them to 421:
  - **Max Flow / Min Cut**
  - **Selling Boba.. lul**

A Geometric View



An Algebraic View

$$\begin{aligned} \textbf{maximize} \quad & c_1 x_1 + \cdots + c_n x_n \\ \textbf{subject to} \quad & a_{11} x_1 + \cdots + a_{1n} x_n \leq b_1 \\ & \vdots \\ & a_{m1} x_1 + \cdots + a_{mn} x_n \leq b_m \\ & x_1, \ldots, x_n \geq 0 \end{aligned}$$

(equivalently)

$$\begin{aligned} \textbf{maximize} \quad & c^\top x \\ \textbf{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

This is **standard form:**

- maximization
- $\leq$ inequalities with constant RHS
- nonnegative x's

Now back to Boba…


Sharetea

Setting.

- Glenn and I love boba. So much that we want to buy out the entirety of Nathan’s store’s stock!
- Our **Goal:** Buy all the tea and boba at minimal cost
  - Cause we are TAs and broke…

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- Our **Goal:** Buy all the tea and boba at minimal cost
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- Our **Strategy:** Offer the shop a rate for each ingredient
  - We offer **x cents / oz of boba, y cents / oz of tea**
- Our **Constraints:** Nathan wants us to buy through his standard and premium drinks
  - Our rates must behave as if we had bought standard drinks

Claim, solve with an LP.

$$\begin{aligned} \textbf{maximize} \quad & 20s + 40p \\ \textbf{subject to} \quad & 0.1s + 0.5p \leq 1000 \\ & 0.9s + 0.5p \leq 3000 \\ & s, p \geq 0 \end{aligned}$$

Original Problem

$$\begin{aligned} \textbf{minimize} \quad & 1000x + 3000y \\ \textbf{subject to} \quad & 0.1x + 0.9y \geq 20 \\ & 0.5x + 0.5y \geq 40 \\ & x, y \geq 0 \end{aligned}$$

Our Problem

Try this on your own at home!

Claim, solve with an LP.

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Hold on… this feels familiar?

Question: How is this different and similar to the boba problem on Wednesday?

2 min to discuss with those around you…

Claim, solve with an LP.

$$\begin{aligned} \textbf{minimize} \quad & 1000x + 3000y \\ \textbf{subject to} \quad & 0.1x + 0.9y \geq 20 \\ & 0.5x + 0.5y \geq 40 \\ & x, y \geq 0 \end{aligned}$$

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Claim: This is the "**dual**" problem to our boba problem from Wednesday.

- We are **buying** not selling
- We are **minimizing** not maximizing

But… are the solutions the same?

Short answer, **YES!!!!!**

Duality Definition

$$\begin{aligned} \textbf{minimize} \quad & b^T y \\ \textbf{subject to} \quad & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

The Dual

"Buying Boba"

The Primal

"Selling Boba"

$$\begin{aligned} \textbf{maximize} \quad & c^T x \\ \textbf{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

See slides for other formulation….

Properties of Duals

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        - In some sense, the dual is trying to minimize an upper bound on the primal. Think about the boba problem!

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Take 2 min to talk about why this might be true of the boba problem!

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- Property 2: **Weak Duality**
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- Property 3: **Strong Duality**
    - The **Optimal Dual = Optimal Primal (if the primal and dual have a solution)**.
        - It turns out that they are actually equal (assuming a solution exists).

But… what is a dual? Why care?

- Duals are useful **compliments** to LPs!
- They convey potentially new information and can also be used in creative proofs of seemingly different problems

Something cool?

Tool in our toolkit

But… what is a dual? Why care?

- **Intuitions** for the dual are sometimes not good… but to help motivate you guys to love duality, let’s look at some approaches to understand them!

Intuition #1: Games

- Primal: We want to maximize our reward!

Sell Boba for lots!

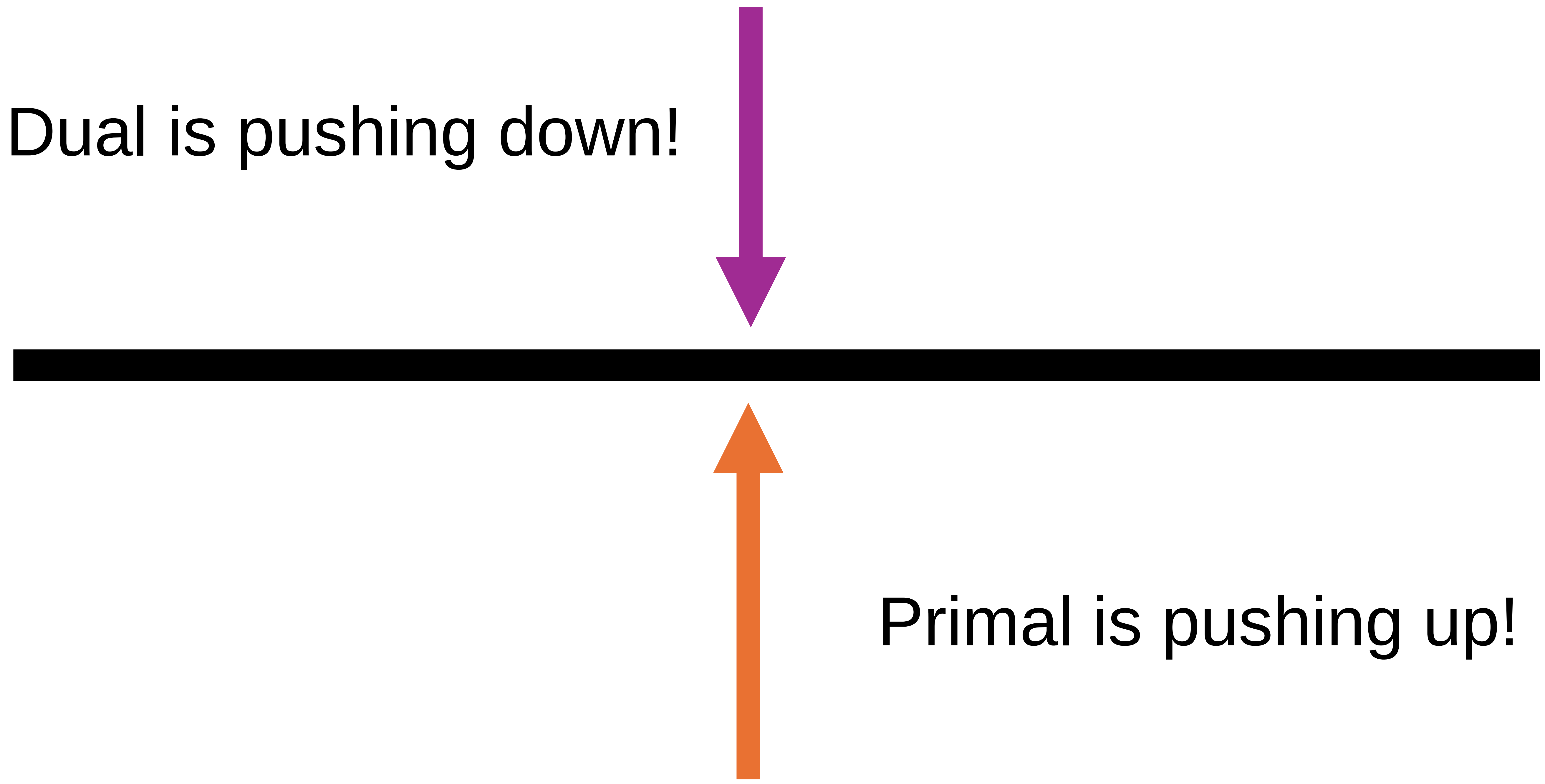
VS.

- Dual: We want to minimize their reward!

Buy Boba for Cheap!

I'm sure Nathan would sell boba for reasonable prices…

Intuition #2: Geometric



There are more concrete geometric intuitions…

More fun things…

Max Flow (Revisited)

Input: A flow network $G = (V, E)$, source s, sink t, and $c : E \to \mathbb{R}^{\geq 0}$

Goal:

maximize flow out of s

subject to respecting capacities and flow conservation

We want $x_e =$ flow on edge $e \in E$.

$$\textbf{maximize} \quad \sum_{e \text{ out of } s} (x_e)$$

$$\textbf{subject to} \quad 0 \leq x_e \leq c(e) \text{ for all } e \in E$$

$$\sum_{e \text{ out of } v} (x_e) = \sum_{e \text{ into } v} (x_e) \text{ for all } v \in V \setminus \{s, t\}$$

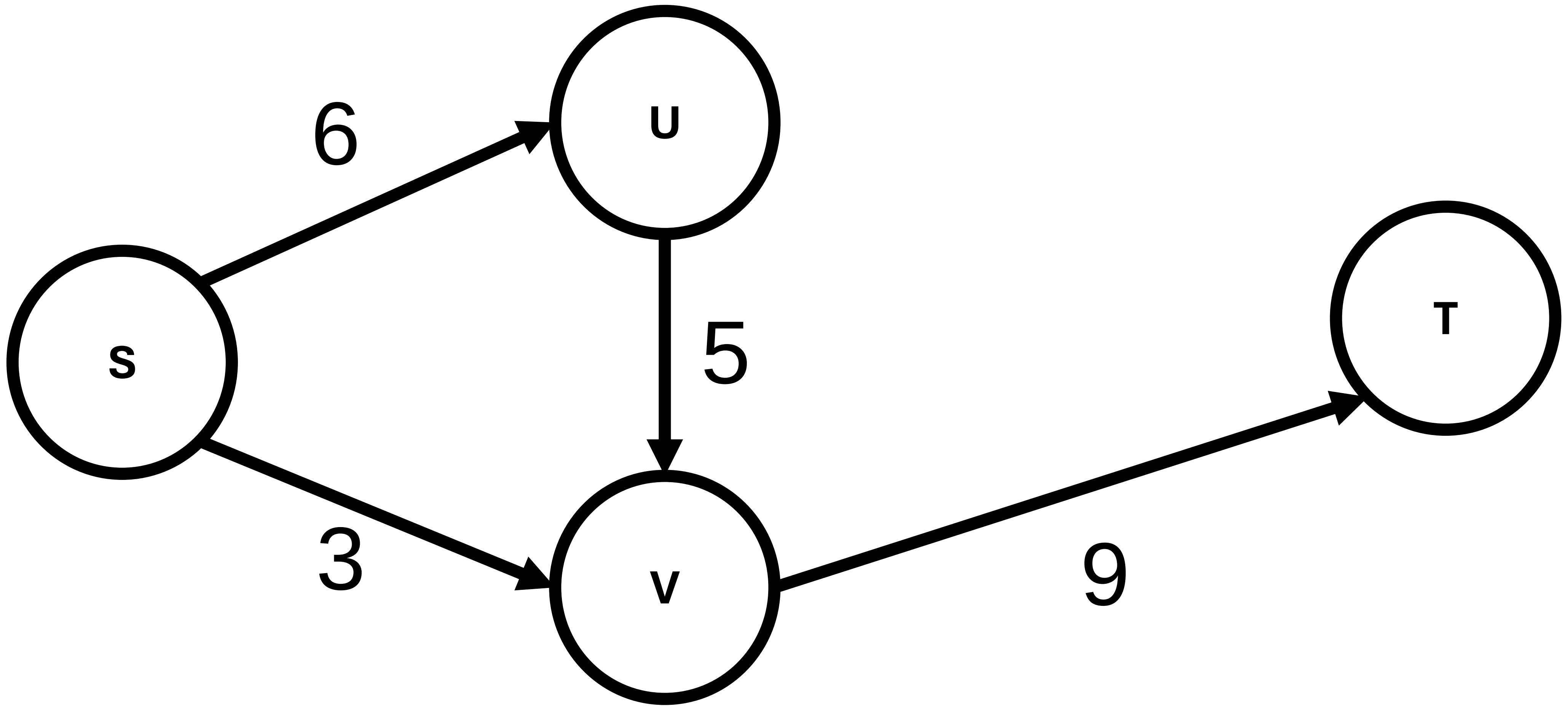
Max Flow (Revisited)

maximize $\sum_{e \text{ out of } s}(x_e)$
subject to $0 \leq x_e \leq c(e)$ for all $e \in E$
$\sum_{e \text{ out of } v}(x_e) = \sum_{e \text{ into } v}(x_e)$ for all $v \in V \setminus \{s, t\}$

In standard form:

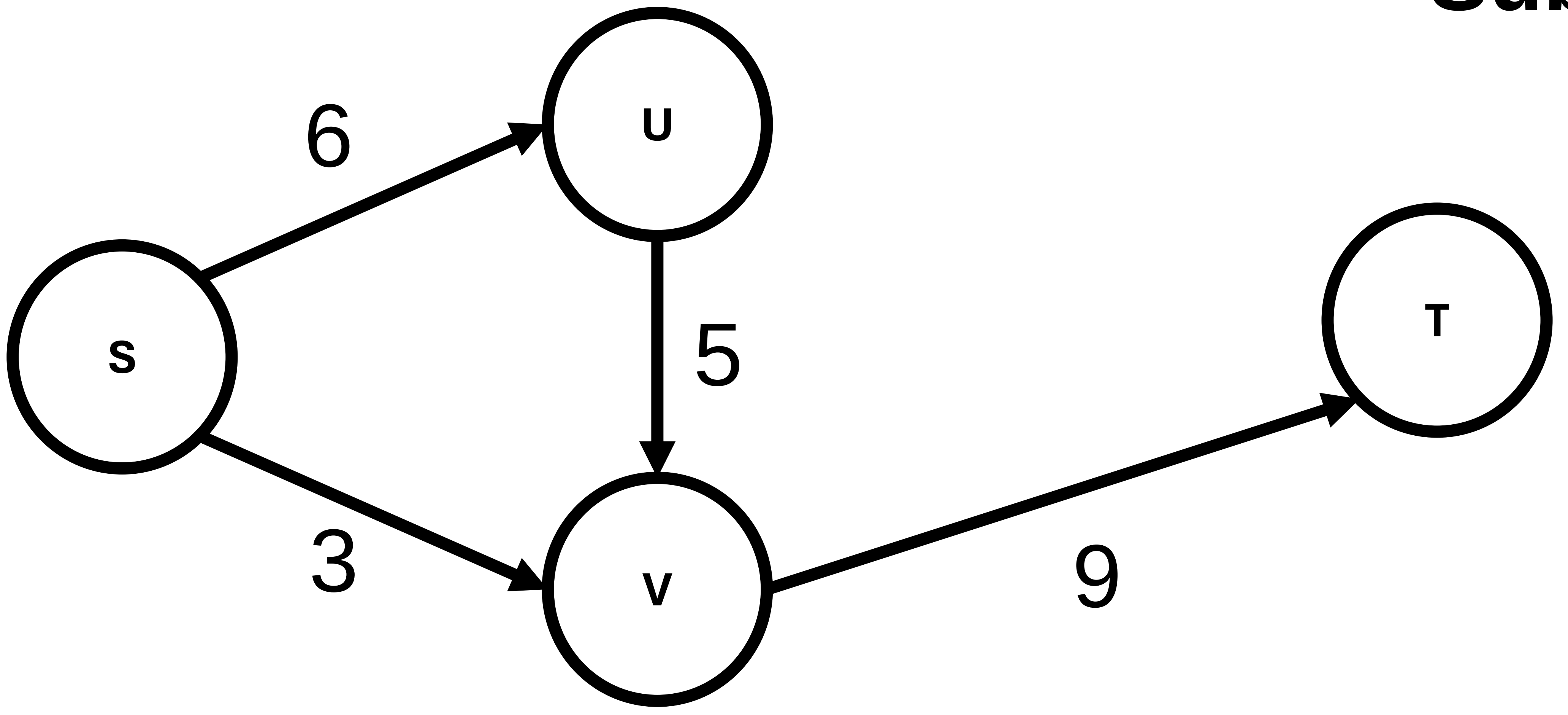
maximize $\sum_{e \text{ out of } s}(x_e)$
subject to $x_e \leq c(e)$ for all $e \in E$
$\sum_{e \text{ out of } v}(x_e) - \sum_{e \text{ into } v}(x_e) \leq 0$ for all $v \in V \setminus \{s, t\}$
$\sum_{e \text{ into } v}(x_e) - \sum_{e \text{ out of } v}(x_e) \leq 0$ for all $v \in V \setminus \{s, t\}$
$x_e \geq 0$ for all $e \in E$

Max Flow Dual



Take 2-3 min to make the LP with those around you

Max Flow Dual



Maximize $x_1 + x_2$

Subject to $x_1 \leq 6, x_2 \leq 3, x_3 \leq 5, x_4 \leq 9$

$x_1 - x_3 \leq 0$

$x_3 - x_1 \leq 0$

$x_4 - x_3 - x_2 \leq 0$

$x_3 + x_2 - x_4 \leq 0$

$\sum_{e \text{ out of } s}(x_e)$

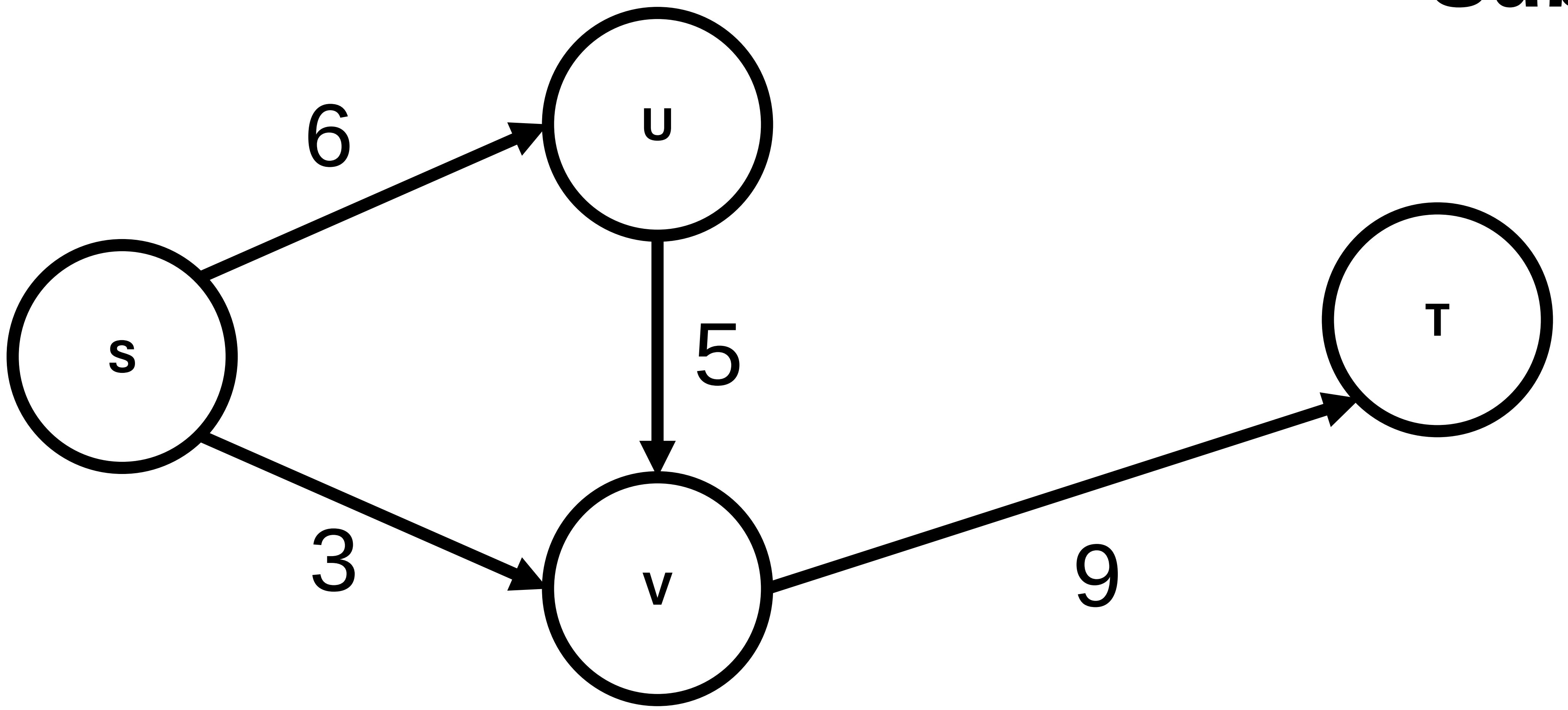
$x_e \leq c(e)$ for all $e \in E$

$\sum_{e \text{ out of } v}(x_e) - \sum_{e \text{ into } v}(x_e) \leq 0$ for all $v \in V \setminus \{s,t\}$

$\sum_{e \text{ into } v}(x_e) - \sum_{e \text{ out of } v}(x_e) \leq 0$ for all $v \in V \setminus \{s,t\}$

$x_e \geq 0$ for all $e \in E$

Max Flow Dual



Minimize $6y_1+3y_2+5y_3+9y_4$

Subject to $y_1+y_5-y_6\geq 1$

$y_2+y_8-y_7\geq 1$

$y_3+y_6+y_8-y_5-y_7\geq 0$

$y_4+y_7-y_8\geq 0$

$\sum_{e \text{ out of } s}(x_e)$

$x_e \leq c(e)$ for all $e \in E$

$\sum_{e \text{ out of } v}(x_e)-\sum_{e \text{ into } v}(x_e)\leq 0$ for all $v \in V \setminus \{s,t\}$

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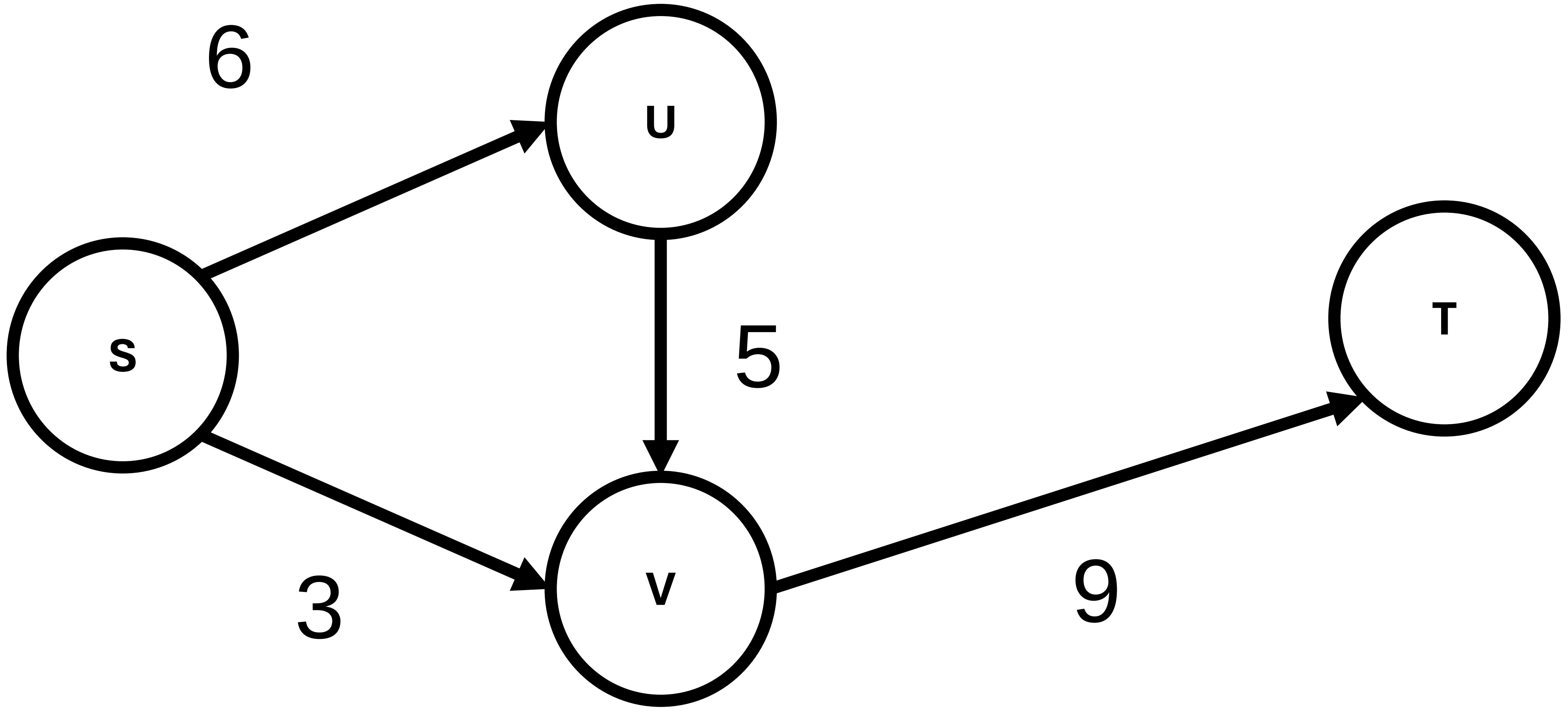

Optimal Solution:

$y_1 = 0$ $y_2 = 1$ $y_3 = 1$ $y_4 = 0$

$y_5 = 1$ $y_6 = 0$ $y_7 = 0$ $y_8 = 0$

$\sum_{e \text{ out of } s}(x_e)$

$x_e \leq c(e)$ for all $e \in E$

$\sum_{e \text{ out of } v}(x_e) - \sum_{e \text{ into } v}(x_e) \leq 0$ for all $v \in V \setminus \{s, t\}$

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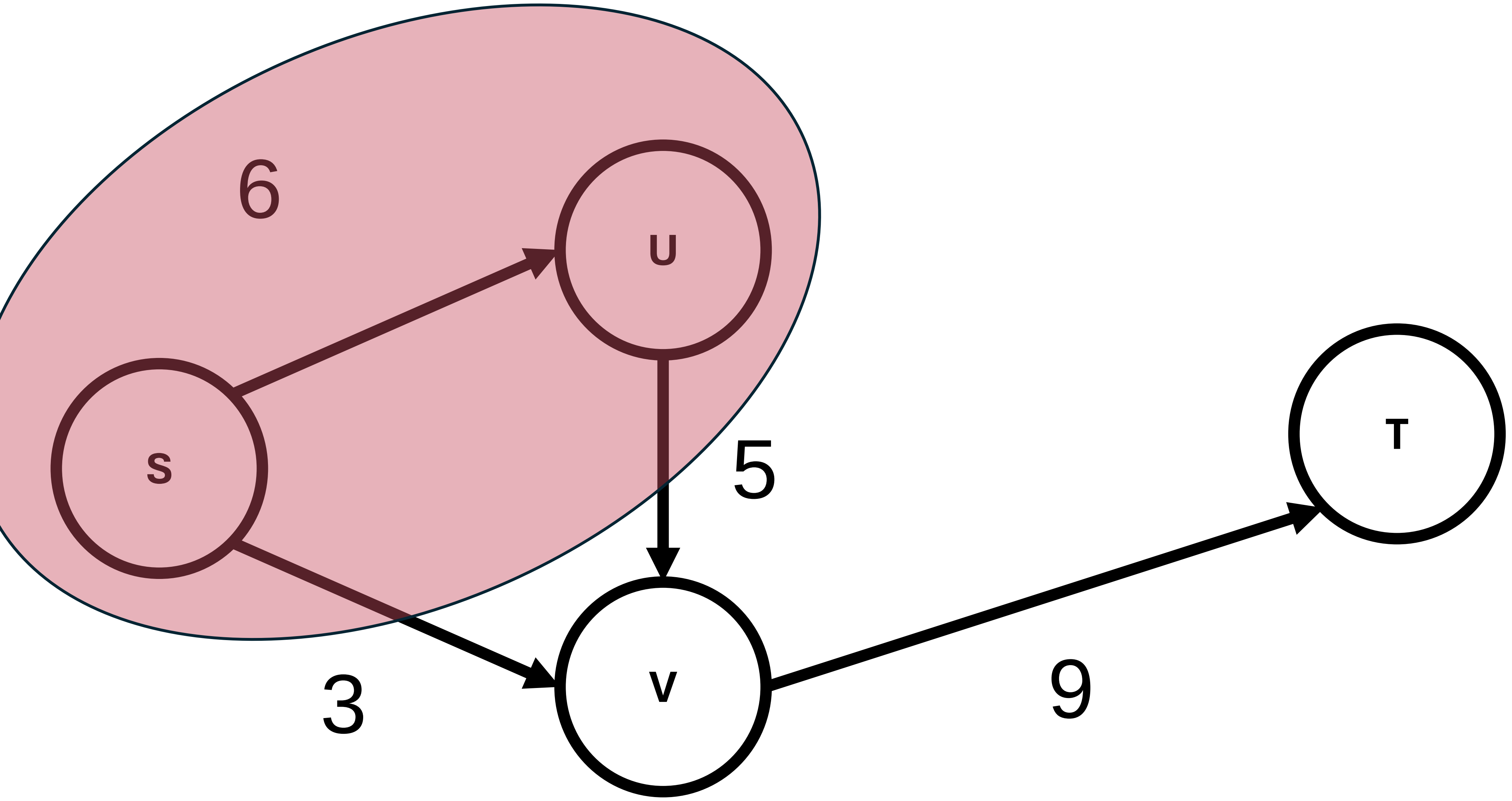
Max Flow Dual



This is just the minimum cut!

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$y_1 = 0$ $y_2 = 1$ $y_3 = 1$ $y_4 = 0$

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$x_e \geq 0$ for all $e \in E$

Claim:

The LP Dual of Maximum Flow is the Minimum Cut!

(exam) Tips to compute duality

- Convert the primal into standard form first!

- Use the formula!
  - I recommend the linear algebra way because it is easier to remember (just transpose, swap, swap)

Original LP → Standard Form → Dual Form

Standard Form

maximize $c^\top x$

subject to $Ax \leq b$

$x \geq 0$

Dual Form

minimize $b^T \boldsymbol{y}$

subject to $A^T \boldsymbol{y} \geq c$

$\boldsymbol{y} \geq 0$

(exam) Tips to compute duality

- Convert the primal into standard form first!

- Use the formula!
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- Remember that the variables of your dual are fundamentally **not** the same as your primal.

Boba Problem: Amount of drinks **vs** the rate of ingredients

(exam) Tips to compute duality

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  - I recommend the linear algebra way because it is easier to remember (just transpose, swap, swap)

- Remember that the variables of your dual are fundamentally **not** the same as your primal.

- Remember the dual of an LP is its own LP, meaning it can be converted to standard form as well!

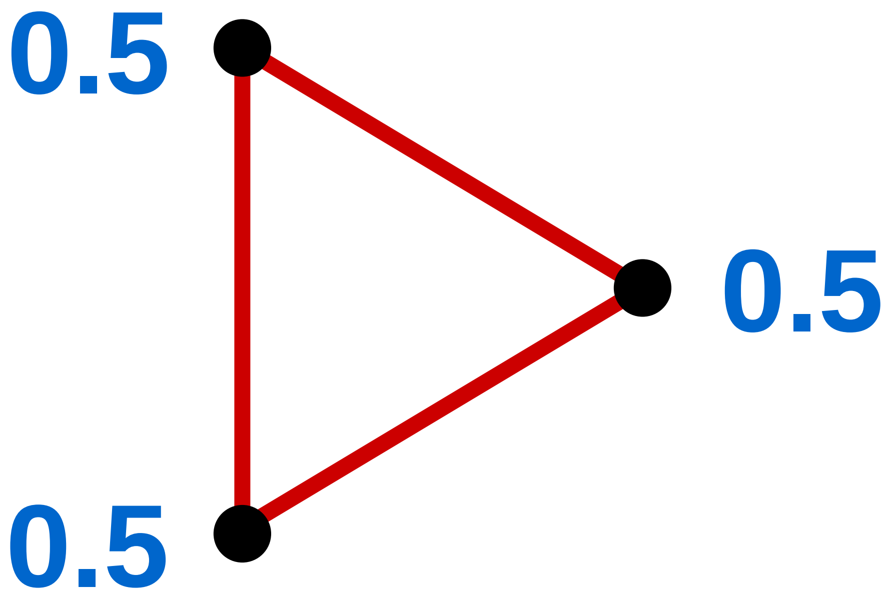
Sneak peek: Vertex Cover?

LP Relaxation: Instead of $\boldsymbol{x_v} = 0$ or 1 (out/in), have $\mathbf{0} \le \boldsymbol{x_v} \le \mathbf{1}$

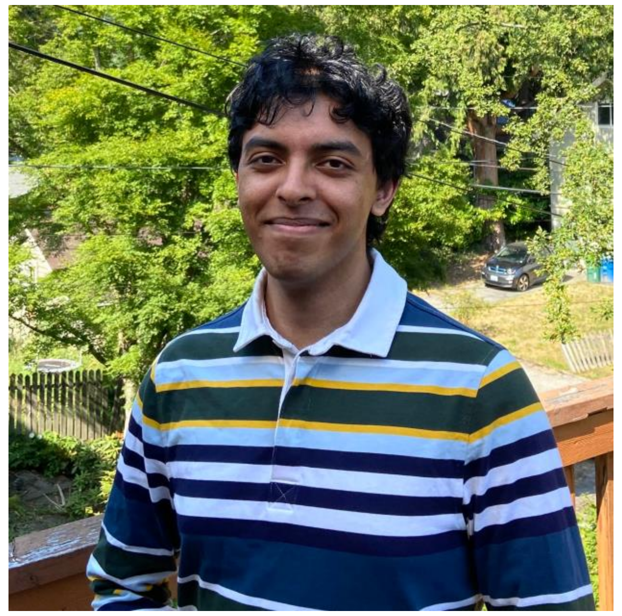
minimize	$\sum_{v \in V}(\boldsymbol{x_v})$
subject to	$\boldsymbol{x_u} + \boldsymbol{x_v} \ge \mathbf{1}$ for all edges $(u, v) \in E$
	$\mathbf{0} \le \boldsymbol{x_v} \le \mathbf{1}$ for all vertices $v \in V$

Might give **"fractional solutions"**:

LP optimum = 1.5, true optimum = 2








Back to your regularly scheduled program….

Have a good weekend.