# CSE 421 Winter 2025
# Lecture 1: Intro, Stable Matching

Nathan Brunelle

http://www.cs.uw.edu/421

# Course Goals

Two Goals:

1. Learn specific noteworthy algorithms
2. Hone insights on how to design algorithms for novel problems

# What is an algorithm?

- a finite sequence of **mathematically rigorous instructions**, typically used to **solve** a class of specific problems or to perform a **computation**. [January 2025]
- a finite sequence of **well-defined instructions**, typically used to **solve** a class of specific problems or to perform a **computation**. [November 2021]
- a set of **instructions**, typically to **solve** a class of problems or perform a **computation**. [August 2019]
- an **unambiguous** specification of how to **solve** a class of problems. [September 2018]
- How it will sometimes feel

# Course Goals

~~Two Goals:~~ Three Goals

1. Learn specific noteworthy algorithms
2. Hone insights on how to design algorithms for novel problems
3. Have Fun!

**Hopefully not you…**

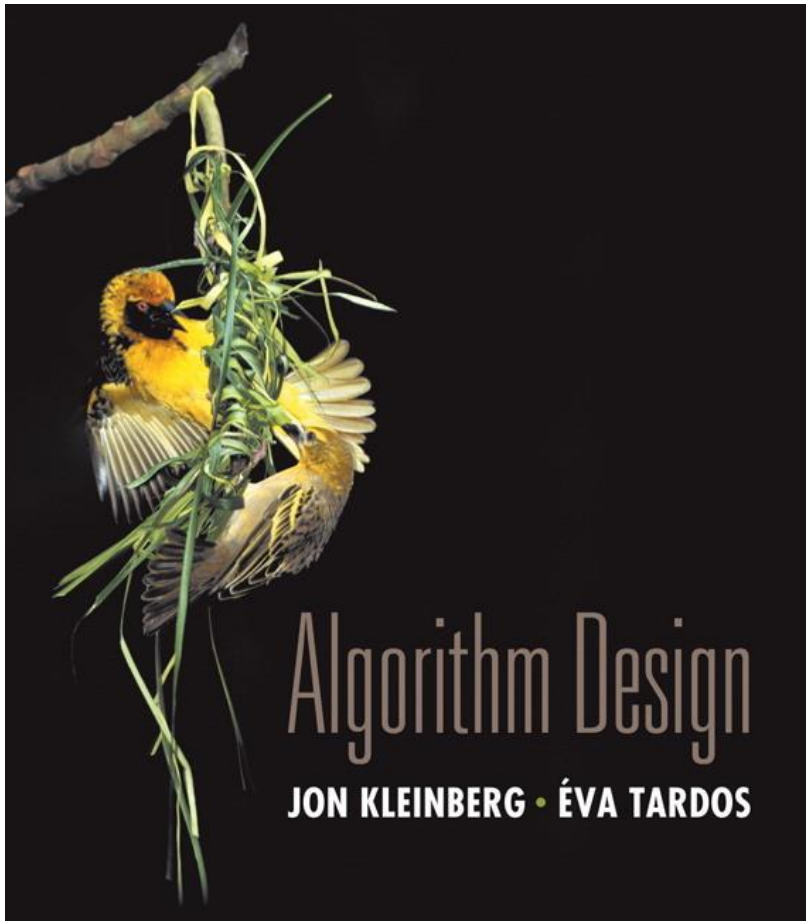
# Nathan Brunelle

- Born: Virginia Beach, VA
- Ugrad: Math and CS at University of Virginia
- Grad: CS at University of Virginia
- Taught at UVA for 6 years
  - Intro to programming (e.g. 121)
  - Discrete Math (e.g. 311)
  - Algorithms (e.g. 412)
  - Theory of Computation (e.g. 431)

# Our Amazing TAs!

# Course Info

- Text:
  - Kleinberg and Tardos Algorithm Design
    - Recommended but not required
  - Other supplements, which we'll make available
- Course Page:
  - http://www.cs.uw.edu/421



# Communication

- EdStem Discussion board
  - Your first stop for questions about course content & assignments
  - Course announcements will be made there
    - Announcements will be forwarded to your email as well

# Course Meetings

- Lecture
  - Materials posted (slides before class, inked slides after)
  - Recorded using Panopto
  - Ask questions, focus on key ideas (rarely coding details)
- Section
  - Practice problems!
  - Answer content/homework questions
- Office hours
  - Use them: *please visit us!*

# Tasks

- 8 Weekly individual homework exercises (60%)
  - Mechanical Problems (1 per each)
    - Apply knowledge of a specific algorithm
  - Long-Form Problems (3 per each)
    - Design and analyze a new algorithm
- Midterm and final exam (40%)
  - Midterm: 15%
    - Wednesday Feb 19, 6:00pm-7:30pm
  - Final: 25%
    - Monday March 17, 2:30pm-5:20pm
    - cumulative

# Grading

- Homework:
  - Each mechanical problem is worth 10 points
  - Each long-form problem is worth 25 points
  - We count your 7 best mechanical problems (1 dropped)
  - We count your 20 best long-form problems (4 dropped)
  - Score is therefore out of 570 points
- Late Work:
  - Unless you talk to Nathan, nothing is accepted 48 hours after the deadline
  - You have 10 late problem-days. After those have been used, late problems will receive a 50% grade penalty (multiplicative)

# Collaboration

- Try it yourself first
- Collaborate with classmates (no external interactive help on assignments permitted)
  - Collaboration is “whiteboard only”
  - Looking for a collaborator?
- Cite your sources!

# Getting Started (Your TODO List)

- Make sure you are on Ed (a.k.a. EdStem)!
- Attend your first Quiz Section Thursday!
- Homework 1 will be out Wednesday
  - You will have enough to start on it after section Thursday
  - Start thinking about it right away after that
- Sign up for CSE 490D
- Attend lecture and participate
  - Students who participate do better on average

# Matching Medical Residents to Hospitals

**Goal:** Given a set of preferences among hospitals and medical school residents (graduating medical students), design a *self-reinforcing* admissions process.

Unstable pair: applicant $x$ and hospital $y$ are *unstable* if:

- $x$ prefers $y$ to their assigned hospital.
- $y$ prefers $x$ to one of its admitted residents.

Stable assignment. Assignment with no unstable pairs.

- Natural and desirable condition.
- Individual self-interest will prevent any applicant/hospital side deal from being made.

# Simplification: Stable Matching Problem

**Goal:** Given two groups of $n$ people each, find a "suitable" matching.

- Participants rate members from opposite group.
- Each person lists members from the other group in order of preference from best to worst.

favorite ↓ least favorite ↓

| | 1st | 2nd | 3rd |
|---|---|---|---|
| X | A | B | C |
| Y | B | A | C |
| Z | A | B | C |

*Group **P** Preference Profile*

favorite ↓ least favorite ↓

| | 1st | 2nd | 3rd |
|---|---|---|---|
| A | Y | X | Z |
| B | X | Y | Z |
| C | X | Y | Z |

*Group **R** Preference Profile*

# Stable Matching Problem

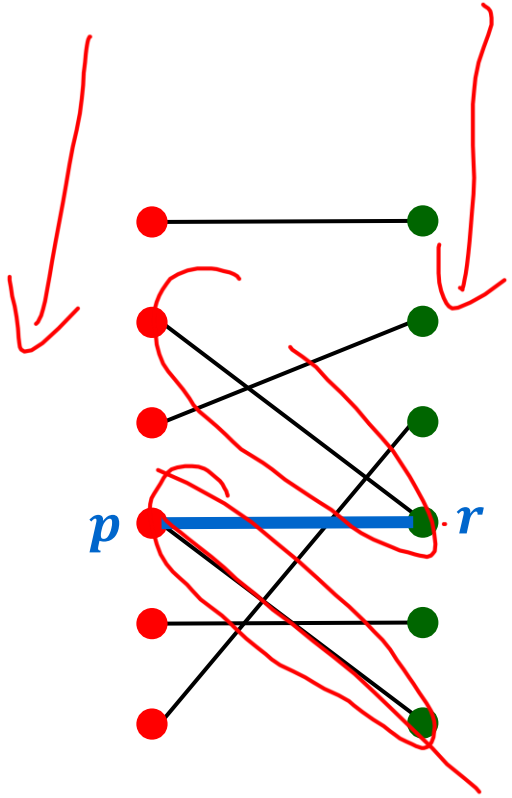
**Perfect matching:** everyone is matched to precisely one person from the other group

**Stability:** self-reinforcing, i.e. no pair has incentive to defect from their assignment.

- For a matching $M$, an unmatched pair $p$-$r$ from different groups is *unstable* if $p$ and $r$ prefer each other to current partners.
- Unstable pair $p$-$r$ could each improve by ignoring the assignment.

**Stable matching:** perfect matching with no unstable pairs.

**Stable matching problem:** Given the preference lists of $n$ people from each of two groups, find a stable matching between the two groups if one exists.



# Stable Matching Problem

**Q:** Is matching (X,C), (Y,B), (Z,A) stable?

favorite ↓ least favorite ↓

| | 1st | 2nd | 3rd |
|---|---|---|---|
| X | A | B | C |
| Y | B | A | C |
| Z | A | B | C |

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| | 1st | 2nd | 3rd |
|---|---|---|---|
| A | Y | X | Z |
| B | X | Y | Z |
| C | X | Y | Z |

*Group **R** Preference Profile*

# Stable Matching Problem

**Q:** Is matching (X,C), (Y,B), (Z,A) stable?

**A:** No. B and X prefer each other.

favorite ↓ least favorite ↓

| | 1st | 2nd | 3rd |
|---|---|---|---|
| X | A | B | C |
| Y | B | A | C |
| Z | A | B | C |

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| | 1st | 2nd | 3rd |
|---|---|---|---|
| A | Y | X | Z |
| B | X | Y | Z |
| C | X | Y | Z |

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| X | A | B | C |
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| Z | A | B | C |

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| | 1st | 2nd | 3rd |
|---|---|---|---|
| A | Y | X | Z |
| B | X | Y | Z |
| C | X | Y | Z |

*Group **R** Preference Profile*

# Stable Matching Problem

**Q:** Is matching (X,A), (Y,B), (Z,C) stable?

**A:** Yes

favorite ↓ least favorite ↓

| | 1st | 2nd | 3rd |
|---|---|---|---|
| X | A | B | C |
| Y | B | A | C |
| Z | A | B | C |

*Group **P** Preference Profile*

favorite ↓ least favorite ↓

| | 1st | 2nd | 3rd |
|---|---|---|---|
| A | Y | X | Z |
| B | X | Y | Z |
| C | X | Y | Z |

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# Variant: "Stable Roommate" Problem (one set rather than 2)

**Q.** Do stable matchings always exist?

**A.** Not exactly obvious…

**Stable roommate problem:**

- $2n$ people; each person ranks others from $1$ to $2n - 1$.
- Assign roommate pairs so that no unstable pairs.

| | 1st | 2nd | 3rd |
|---|---|---|---|
| A | B | C | D |
| B | C | A | D |
| C | A | B | D |
| D | A | B | C |

(A,B), (C,D) ⇒ B-C unstable
(A,C), (B,D) ⇒ A-B unstable
(A,D), (B,C) ⇒ A-C unstable

**Observation:** Stable matchings do not always exist for stable roommate problem.

# Propose-And-Reject Algorithm

**Propose-and-reject algorithm:** [Gale-Shapley 1962]

Intuitive method that guarantees to find a stable matching.

- Members of one group $P$ make *proposals*, the other group $R$ *receives* proposals

```
Initialize each person to be free.
while (some p in P is free) {
    Choose some free p in P
    r = 1st person on p's preference list to whom p has not yet proposed
    if (r is free)
        tentatively match (p,r)   //p and r both engaged, no longer free
    else if (r prefers p to current tentative match p')
        replace (p',r) by (p,r)   //p now engaged, p' now free
    else
        r rejects p
}
```

# Propose and Reject Algorithm Example

```
Initialize each person to be free.
while (some p in P is free) {
    Choose some free p in P
    r = 1st person on p's preference list to whom p has not yet proposed
    if (r is free)
        tentatively match (p,r)    //p and r both engaged, no longer free
    else if (r prefers p to current tentative match p')
        replace (p',r) by (p,r)    //p now engaged, p' now free
    else
        r rejects p
}
```

Tentative Matches:

| | |
|---|---|
| X | |
| Y | |
| Z | |

favorite ↓ least favorite ↓

| | 1st | 2nd | 3rd |
|---|---|---|---|
| X | A | B | C |
| Y | B | A | C |
| Z | A | B | C |

*Group **P** Preference Profile*

favorite ↓ least favorite ↓

| | 1st | 2nd | 3rd |
|---|---|---|---|
| A | Y | X | Z |
| B | X | Y | Z |
| C | X | Y | Z |

*Group **R** Preference Profile*

# Why Does This Work?

- What do we need to know before we're convinced that this algorithm is "correct"?

# Why Does This Work?

- What do we need to know before we're convinced that this algorithm is "correct"?
  - That is terminates (no infinite loop)
  - That it produces a stable matching
    - It's perfect (everyone gets paired with exactly one partner)
    - It's stable (no unmatched pair mutually prefer each other)

# Proof of Correctness: Termination (not obvious from the code)

**Observation 1:** Members of $P$ propose in decreasing order of preference.

**Claim:** The Gale-Shapley Algorithm terminates after at most $n^2$ iterations.

**Proof:** Proposals are never repeated (by Observation 1) and there are only $n^2$ possible proposals. ■

It could be nearly that bad…

General form of this example will take $n(n-1)+1$ proposals.

| | 1st | 2nd | 3rd | 4th | 5th |
|---|---|---|---|---|---|
| V | A | B | C | D | E |
| W | B | C | D | A | E |
| X | C | D | A | B | E |
| Y | D | A | B | C | E |
| Z | A | B | C | D | E |

Preference Profile for P

| | 1st | 2nd | 3rd | 4th | 5th |
|---|---|---|---|---|---|
| A | W | X | Y | Z | V |
| B | X | Y | Z | V | W |
| C | Y | Z | V | W | X |
| D | Z | V | W | X | Y |
| E | V | W | X | Y | Z |

Preference Profile for R

# Proof of Correctness: Perfection

**Observation 2:** Once a member of $\boldsymbol{R}$ is matched, they never become free; they only "trade up."

**Claim:** Everyone gets matched.

**Proof:**

- If no proposer is free then everyone is matched.
- After some $\boldsymbol{p}$ proposes to the last person on their list, all the $\boldsymbol{r}$ in $\boldsymbol{R}$ have been proposed to by someone (by $\boldsymbol{p}$ at least).
- By Observation 2, every $\boldsymbol{r}$ in $\boldsymbol{R}$ is matched at that point.
- Since $|\boldsymbol{P}| = |\boldsymbol{R}|$ every $\boldsymbol{p}$ in $\boldsymbol{P}$ is also matched. ■

# Proof of Correctness: Stability

**Claim:** No unstable pairs in the final Gale-Shapley matching $M$

**Proof:** Consider a pair $p$-$r$ not matched by $M$

Case 1: $p$ never proposed to $r$.

$\Rightarrow p$ prefers $M$-partner to $r$.

$\Rightarrow p$-$r$ is not unstable for $M$.

Case 2: $p$ proposed to $r$.

$\Rightarrow r$ rejected $p$ (right away or later when trading up)

$\Rightarrow r$ prefers $M$-partner to $p$.

$\Rightarrow p$-$r$ is not unstable for $M$. ∎

# Summary

**Stable matching problem:** Given $n$ people in each of two groups, and their preferences, find a stable matching if one exists.

Stable: No pair of people both prefer to be with each other rather than with their assigned partner

**Gale-Shapley algorithm:** Guarantees to find a stable matching for *any* problem instance.

⇒ Stable matching always exists!