CSE 421 Winter 2025
Lecture 18:
Max Flow Applications

Nathan Brunelle

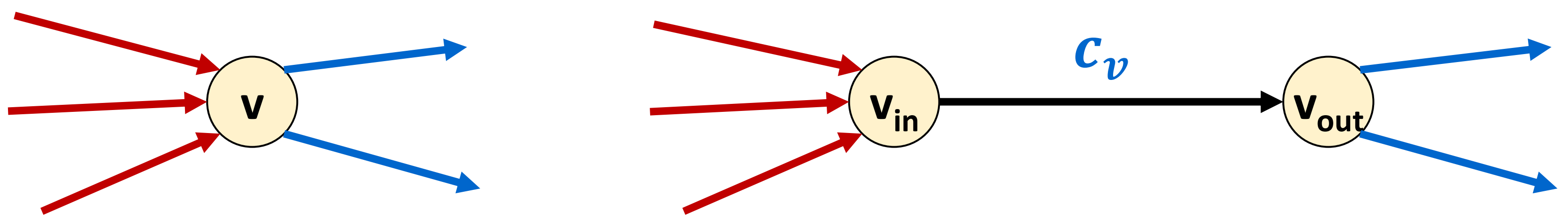
http://www.cs.uw.edu/421

Today – Reductions and Max Flow

- Max flow is primarily useful as the destination of a reduction
  - Given some problem that is not already a max flow problem
  - Use that to create a flow network
  - Run Edmonds Karp on that flow network
  - Use the flow assignment to solve your original problem
- Proving correctness:
  - Argue that the flow through your constructed network is maximal if and only if your final answer is correct
    - Valid flow assignment in the network ⇒ Valid answer to original problem
      - The flow we found is guaranteed to give us a feasible solution
    - Valid answer to original problem ⇒ Valid flow assignment in the network
      - We must have had the best feasible solution as a better one would have allowed more flow

Some general ideas for using MaxFlow/MinCut

- If no source/sink, add them with appropriate capacity depending on application
- Sometimes can have edges with no capacity limits
  - Infinite capacity (or, equivalently, very large integer capacity)
- Convert undirected graphs to directed ones
- Can remove unnecessary flow cycles in answers
- Another idea:
  - To use them for vertex capacities c_v
    - Make two copies of each vertex v named v_{in}, v_{out}



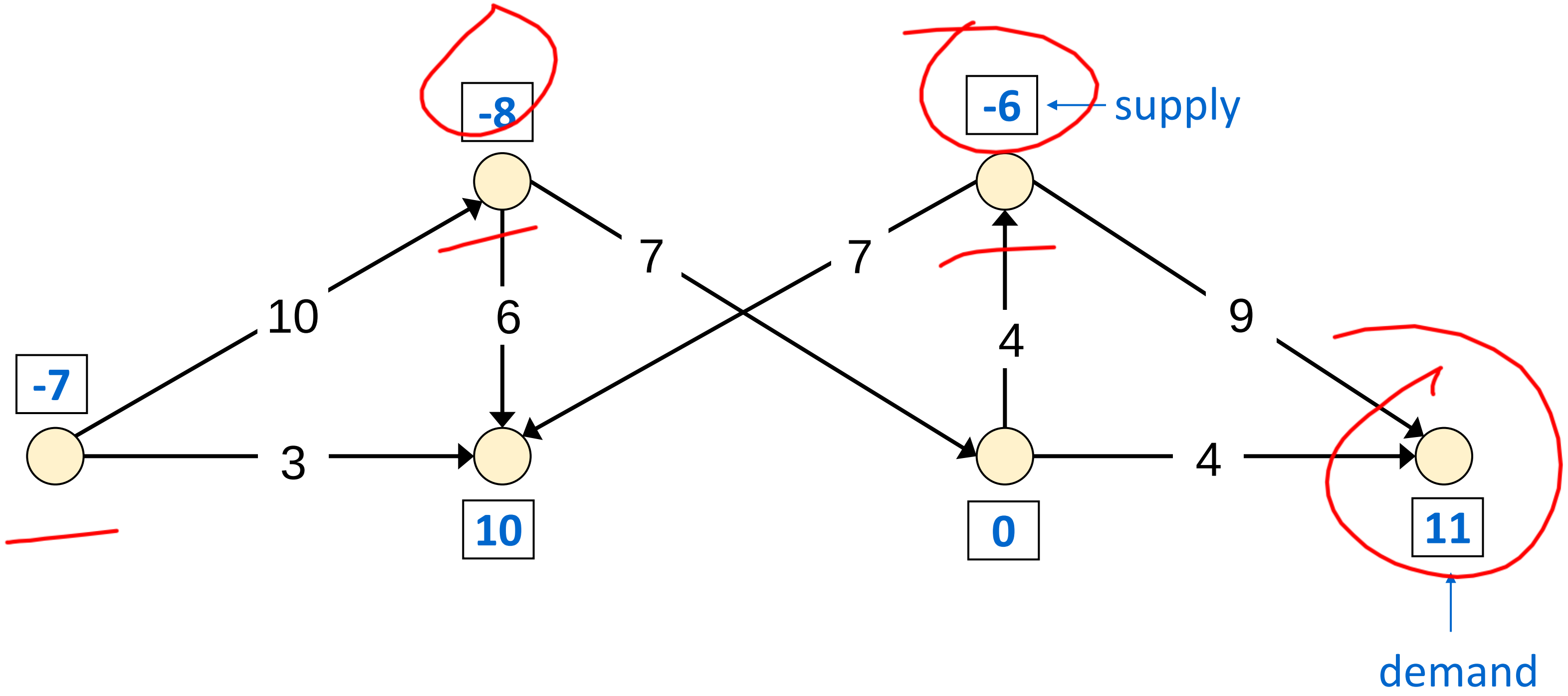
Correctness

- Valid flow ⇒ Valid answer
  - Claim: If we have flow matching the number of games remaining then x can win the season
  - The amount of flow going from each matchup edge to each team edge represents the number of times that team won in that matchup
  - Max flow equaling the number of games means we could assign winners to each remaining game
  - The capacity on the team-to-sink edges guarantees none of them won more games than x
- Valid answer ⇒ Valid flow
  - Claim: If we had a way for x to be champion, we could find flow through the graph to match the number of remaining games
  - Consider the collection of game outcomes which would cause this.
  - For each game, assign 1 unit of flow along the path $s, (t_i, t_j), w, t$ where w is the winner of the game
  - After doing this for all games we will not have violated any capacity constraints because we required that x would be the champion (and so no other team could have more wins)

Circulation with Demands

Nodes have either a “supply” (negative value) or “demand” (positive value)

We want to transport from our supply to our demand through a transportation network



Circulation with Demands

- Single commodity, directed graph $G = (V, E)$
- Each node v has an associated demand $d(v)$
  - Needs to receive an amount of the commodity: demand $d(v) > 0$
  - Supplies some amount of the commodity: "demand" $d(v) < 0$ (amount = $|d(v)|$)
- Each edge e has a capacity $c(e) \geq 0$.
- Nothing lost: $\sum_v d(v) = 0$.

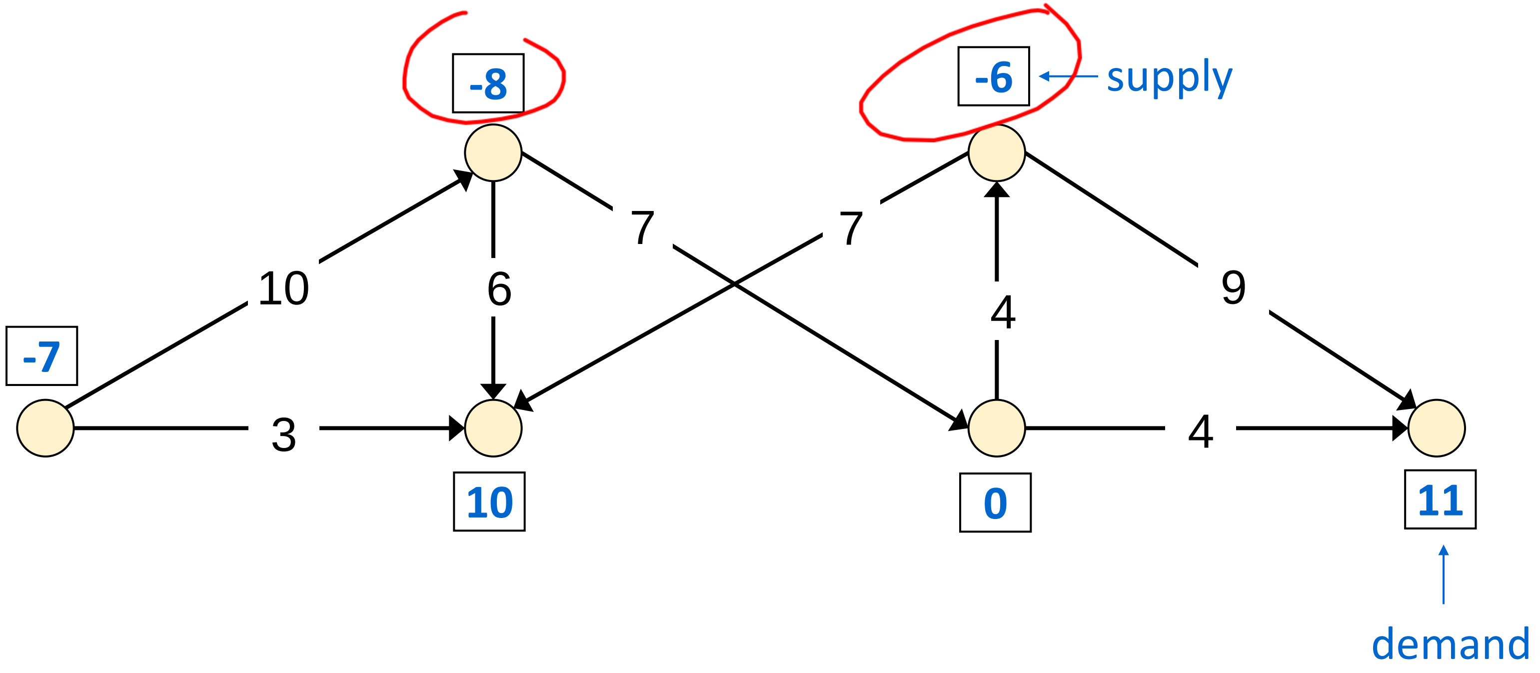
Defn: A circulation for (G, c, d) is a flow function $f: E \to \mathbb{R}$ meeting all the capacities, $0 \leq f(e) \leq c(e)$, and demands:
$\sum_{e \text{ into } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$.

Circulation with Demands: Given (G, c, d), does it have a circulation? If so, find it.

Circulation with Demands

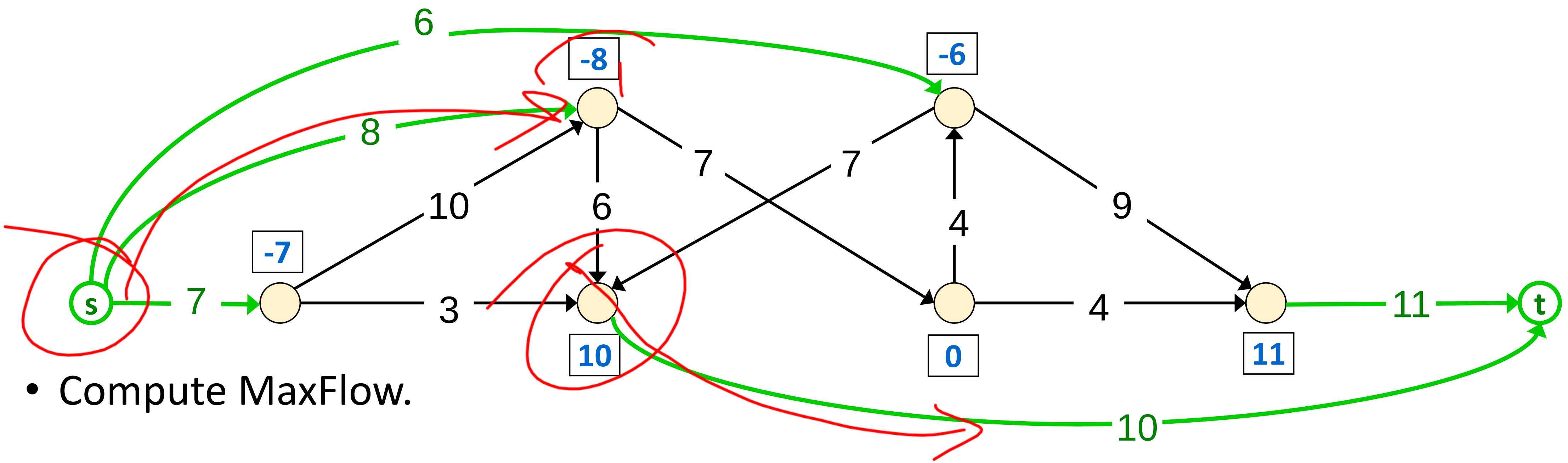
Defn: Total supply $\boldsymbol{D} = \sum_{v:\, d(v)<0} |d(v)| = -\sum_{v:\, d(v)<0} d(v)$.

Necessary condition: $\sum_{v:\, d(v)>0} d(v) = \boldsymbol{D}$ (no supply is lost)



Circulation with Demands using Network Flow

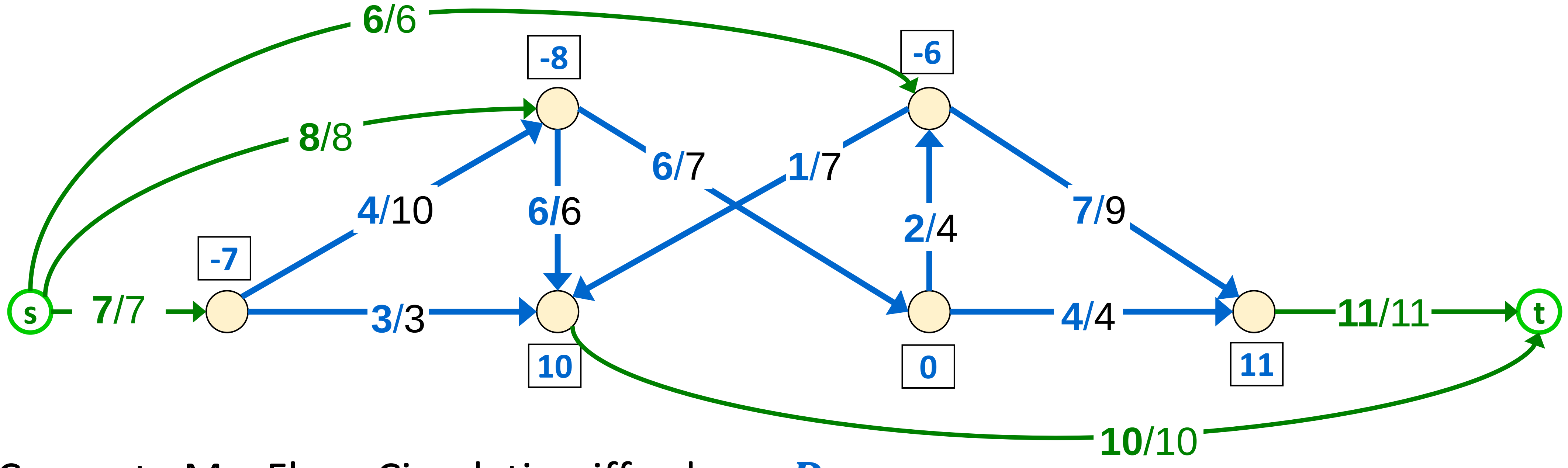
- Add new source s and sink t.
- Add edge (s, v) for all supply nodes v with capacity $|d(v)|$.
- Add edge (v, t) for all demand nodes v with capacity $d(v)$.



- Compute MaxFlow.

Circulation with Demands using Network Flow

- MaxFlow $\leq D$ based on cuts out of s or into t.
- If MaxFlow $= D$ then all supply/demands satisfied.

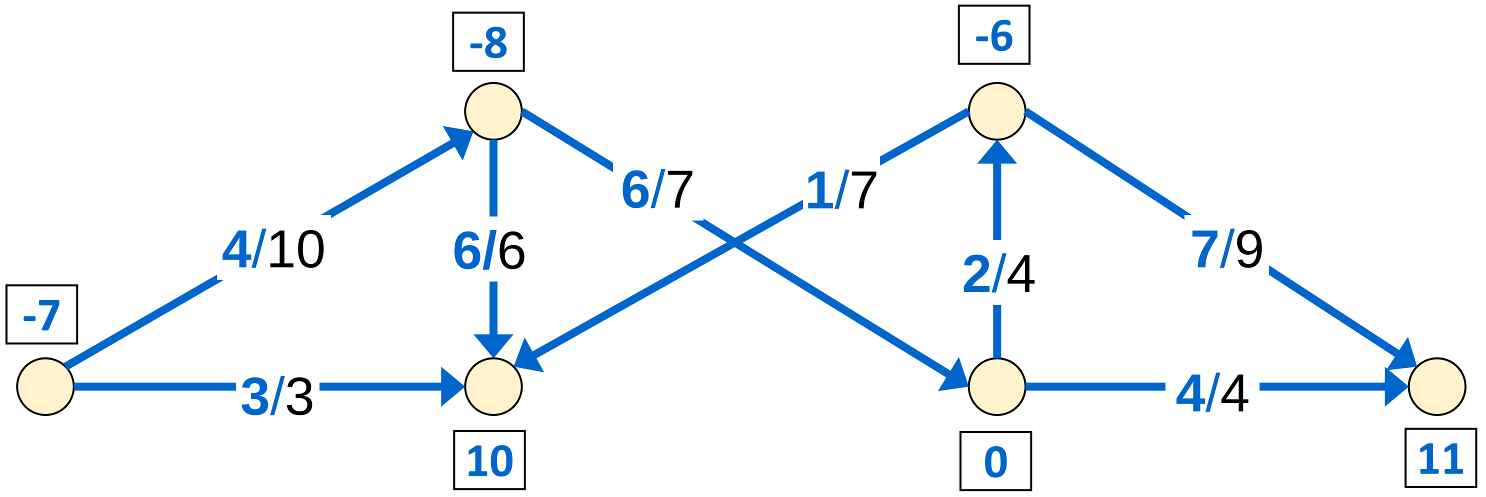


- Compute MaxFlow. Circulation iff value = D

Circulation with Demands using Network Flow

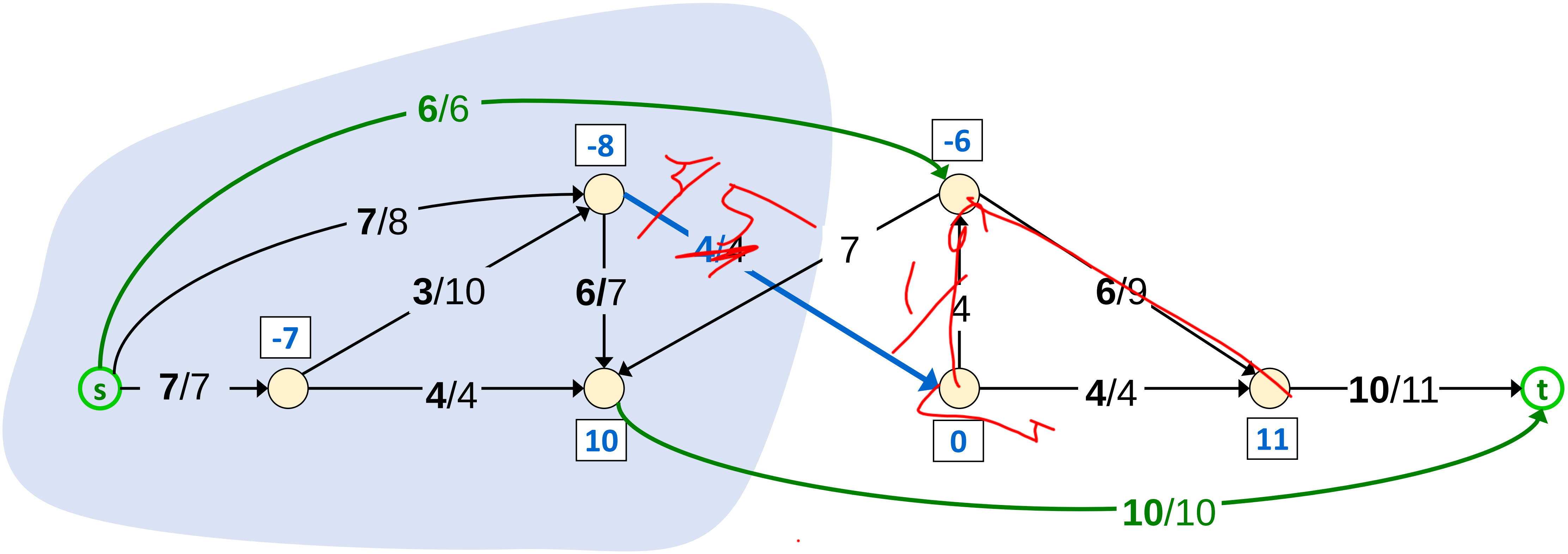
Circulation = flow on original edges

Circulations only need integer flows



Circulation with Demands using Network Flow

When does a circulation not exist? MaxFlow $< D$ iff MinCut $< D$.



Circulation with Demands using Network Flow

When does a circulation not exist? MaxFlow $< D$ iff MinCut $< D$.

Equivalent to excess supply on "source" side of cut smaller than cut capacity.

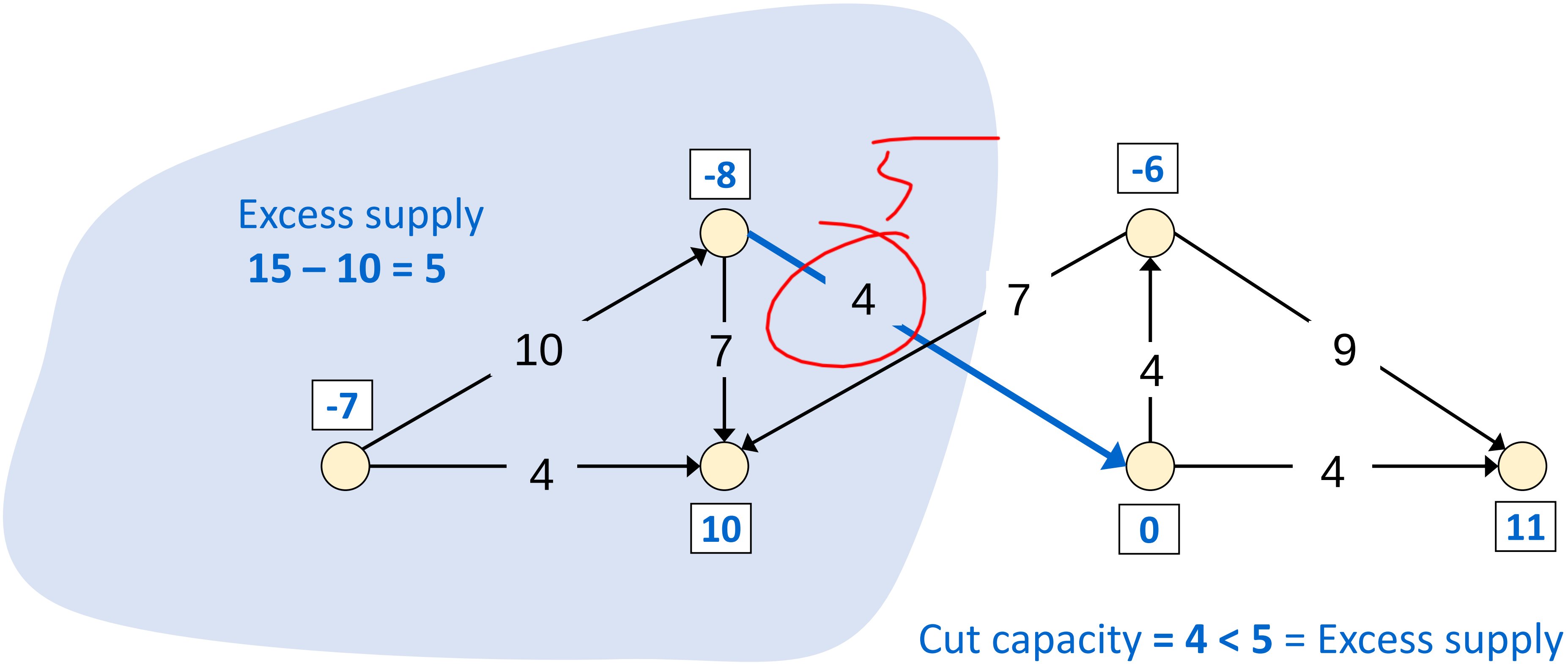


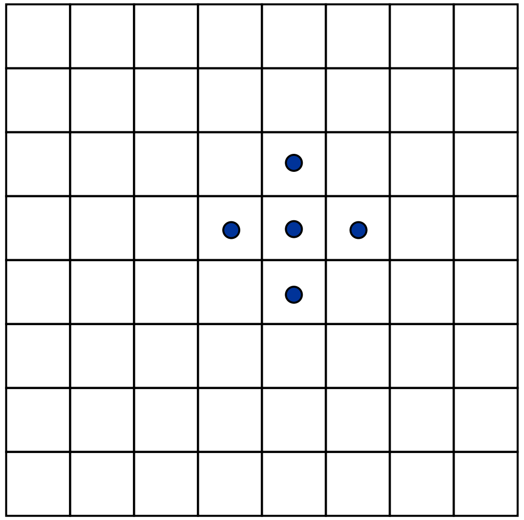
Image Segmentation

Image segmentation:

Given: an Image

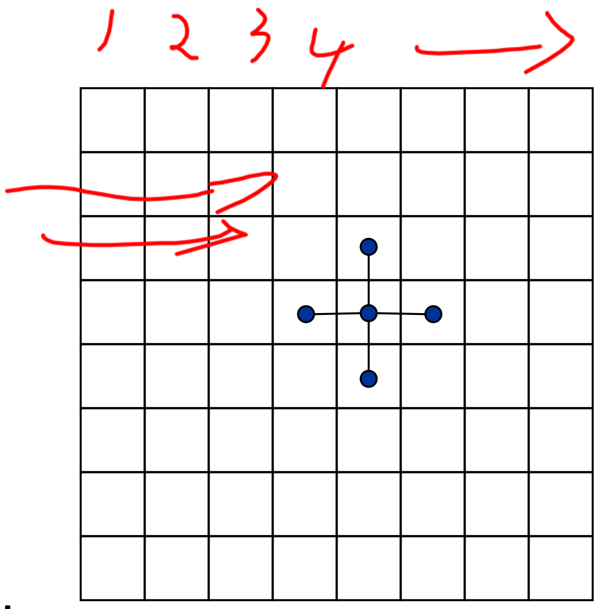
- a grid of pixels with RGB values

Divide image into coherent regions.



Example: Three people standing in front of complex background scene.
Identify each person as a coherent object.

Image Segmentation



Foreground / background segmentation:

Given: A grid V of pixels, E set of pairs of neighboring pixels.

- $a_i \geq 0$ is likelihood pixel i is foreground.
- $b_i \geq 0$ is likelihood pixel i is background.
- For $(i, j) \in E$, $p_{ij} \geq 0$ is separation penalty for labeling one of i and j as foreground, and the other as background.

Label each pixel in image as belonging to foreground (in A) or background (in B)

Goals: Maximize

Accuracy: if $a_i > b_i$ in isolation, prefer to label i in foreground.

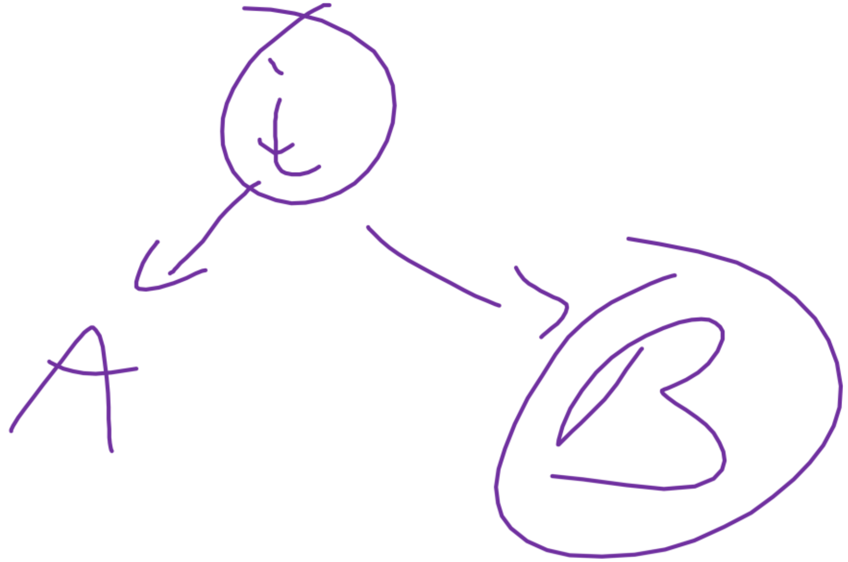
Smoothness: if many neighbors of i are labeled foreground, we should be inclined not to label i as background.

so… **Find:** partition (A, B) that maximizes $\sum_{i \in A} a_i + \sum_{i \in B} b_i - \sum_{(i,j) \in E,\ |A \cap \{i,j\}| = 1} p_{ij}$

Image Segmentation

Issues with formulating as min cut problem:

- Maximization.
- No source or sink.
- Undirected graph.



But maximizing $\sum_{i \in A} a_i + \sum_{i \in B} b_i - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}|=1'}} p_{ij}$

The sum in red is a constant

is equivalent to maximizing $\sum_{i \in V} a_i - \sum_{i \in B} a_i + \sum_{i \in V} b_i - \sum_{i \in a} b_i - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}|=1'}} p_{ij}$

or, alternatively, minimizing $\sum_{i \in A} b_i + \sum_{i \in B} a_i + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}|=1'}} p_{ij}$

Image Segmentation

Minimize $\sum_{i \in A} b_i + \sum_{i \in B} a_i + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}|=1}} p_{ij}$

Formulate as min cut problem.

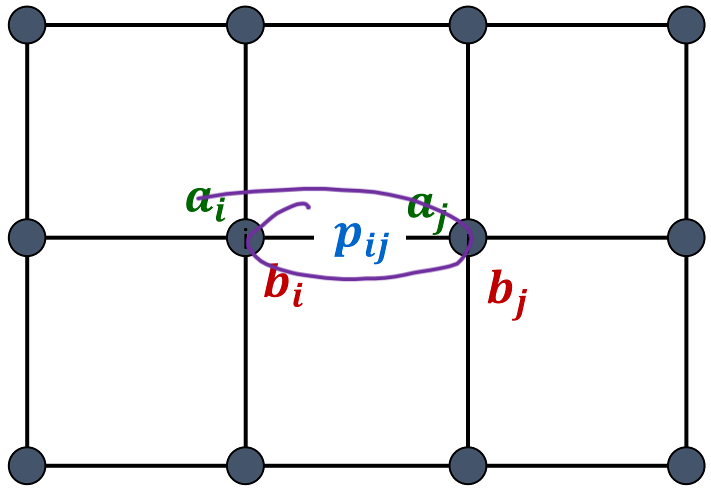


Image Segmentation

Minimize $\sum_{i \in A} b_i + \sum_{i \in B} a_i + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}|=1}} p_{ij}$

Formulate as min cut problem.

- Add source s to correspond to foreground, edges (s, i) with capacity a_i; add sink t to correspond to background, edges (j, t) with capacity b_j.

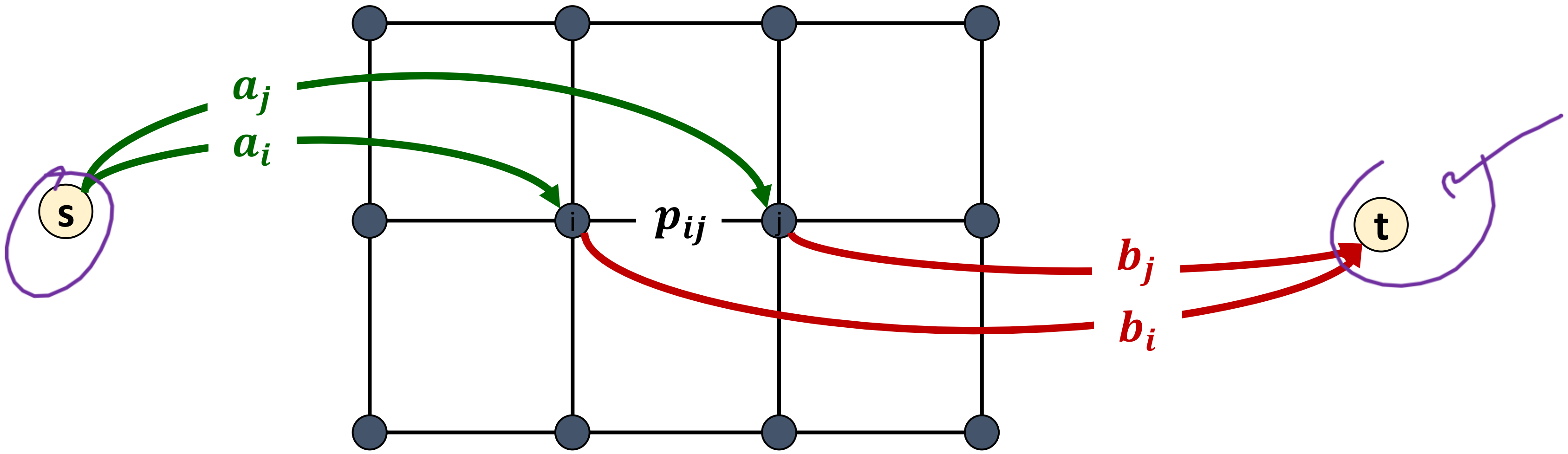


Image Segmentation

Minimize $\sum_{i \in A} b_i + \sum_{i \in B} a_i + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$

Formulate as min cut problem.

- Add source s to correspond to foreground, edges (s, i) with capacity a_i; add sink t to correspond to background, edges (j, t) with capacity b_j.
- Use two anti-parallel edges instead of undirected edge, capacity p_{ij}.
- $G' = (V', E')$.

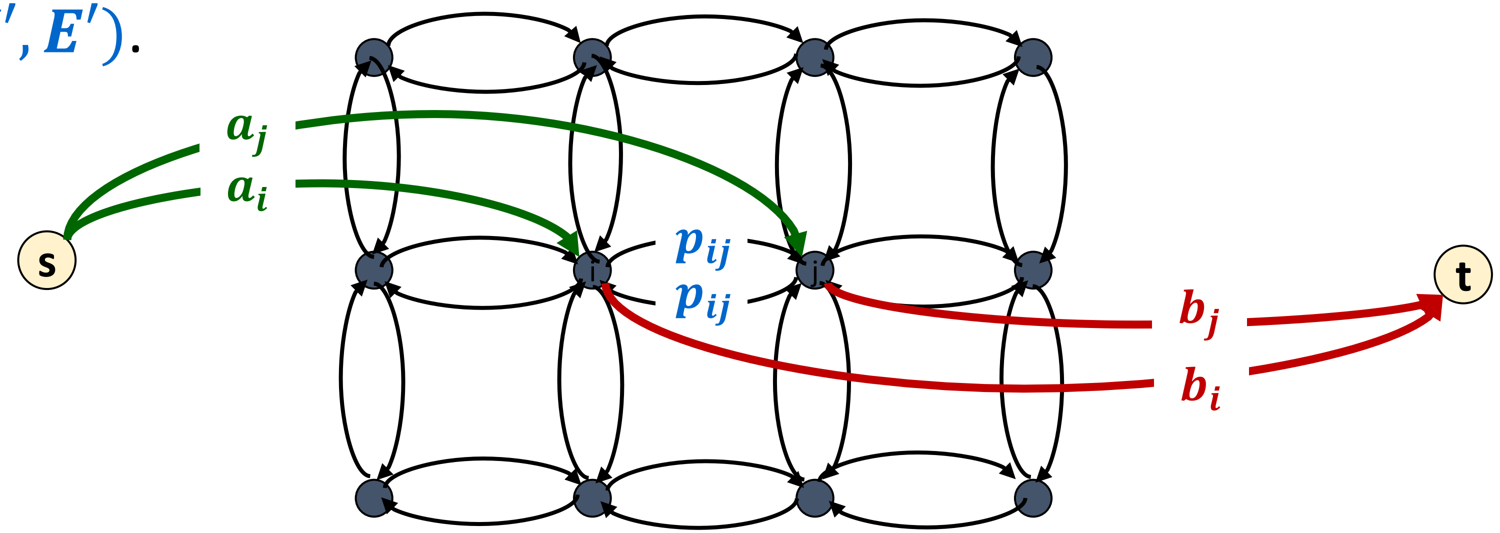
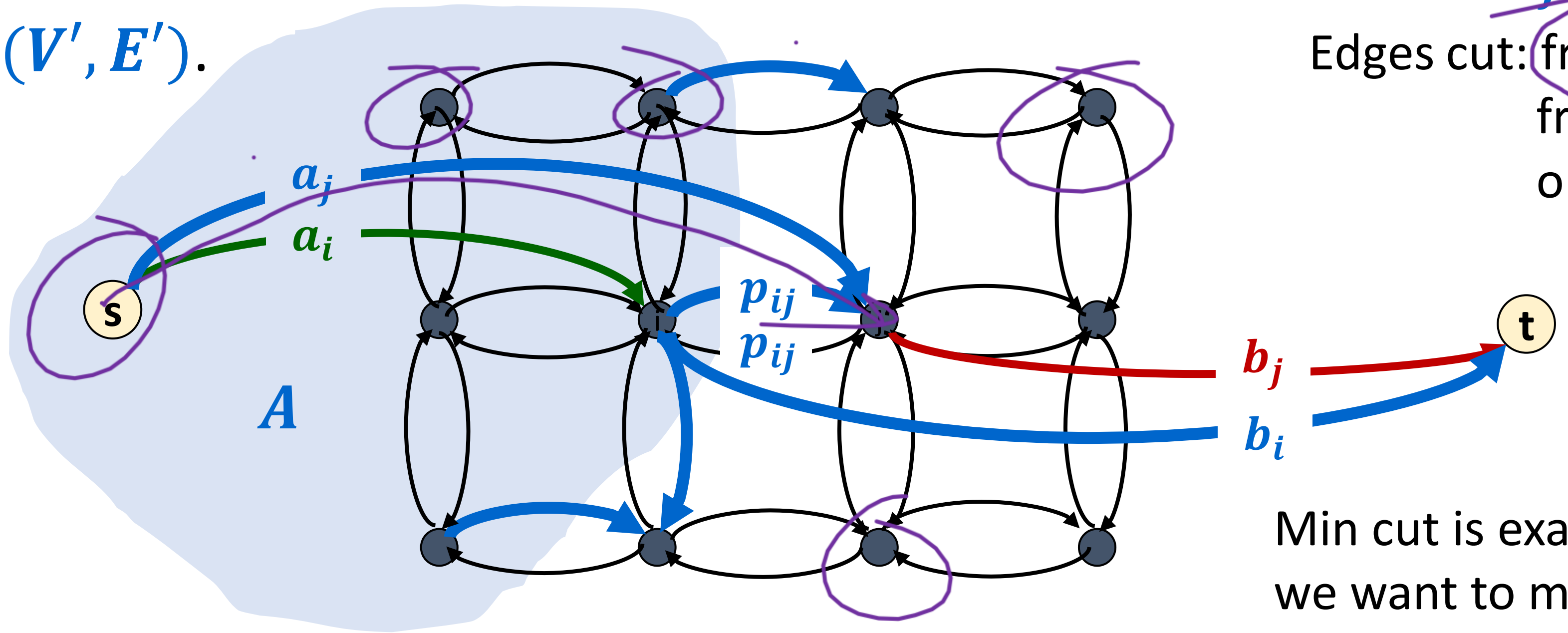


Image Segmentation

Minimize $\sum_{i \in A} b_i + \sum_{i \in B} a_i + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$

Formulate as min cut problem.

- Add source s to correspond to foreground, edges (s, i) with capacity a_i; add sink t to correspond to background, edges (j, t) with capacity b_j.
- Use two anti-parallel edges instead of undirected edge, capacity p_{ij}.
- $G' = (V', E')$.



Edges cut: from s leaving A
from A to t
one direction for p_{ij}

Min cut is exactly the quantity we want to minimize!

Baseball Elimination

- Though you probably don’t care at all about baseball or sports in general, the way that the solution works is interesting.
  - This particular problem is a bit old style since baseball scheduling doesn’t work this way any more…
- Near the end of a season
  - Sportswriters use simple notions to tell which teams can be eliminated from getting a top place finish:
    - “magic number”, “elimination number”, etc.
- These are not accurate
  - We can do better with network flow

Baseball Elimination: Scenario

Team i	Wins w_i	Losses l_i	To play r_i	Against = r_{ij}			
				Tex	Hou	Sea	Oak
Huskies	**83**	**71**	**8**	**-**	**1**	**6**	**1**
Wolverines	**80**	**79**	**3**	**1**	**-**	**0**	**2**
Buckeyes	**78**	**78**	**6**	**6**	**0**	**-**	**0**
Ducks	**77**	**82**	**3**	**1**	**2**	**0**	**-**

- Which teams have a chance of finishing the season with most wins?
  - Ducks eliminated since they can finish with at most 80 wins, but the Huskies already have 83.
  - If $w_i + r_i < w_j \Rightarrow$ team i eliminated.
  - Only reason sports broadcasters appear to be aware of.
  - Sufficient, but not necessary!

Baseball Elimination: Scenario

Team i	Wins w_i	**Losses** l_i	**To play** r_i	**Against =** r_{ij}			
				Tex	Hou	Sea	Oak
Texas	**83**	**71**	**8**	**-**	**1**	**6**	**1**
Houston	**80**	**79**	**3**	**1**	**-**	**0**	**2**
Seattle	**78**	**78**	**6**	**6**	**0**	**-**	**0**
Oakland	**77**	**82**	**3**	**1**	**2**	**0**	**-**

- Which teams have a chance of finishing the season with most wins?
  - Wolverines can win **83** games, but are still eliminated . . .
  - If the Huskies don't get to **84** wins then the Buckeyes will get 6 more wins and finish with **84** wins.

- The answer depends on more than current wins and # of remaining games
  - It also depends on all the games that are being played.

Baseball Elimination

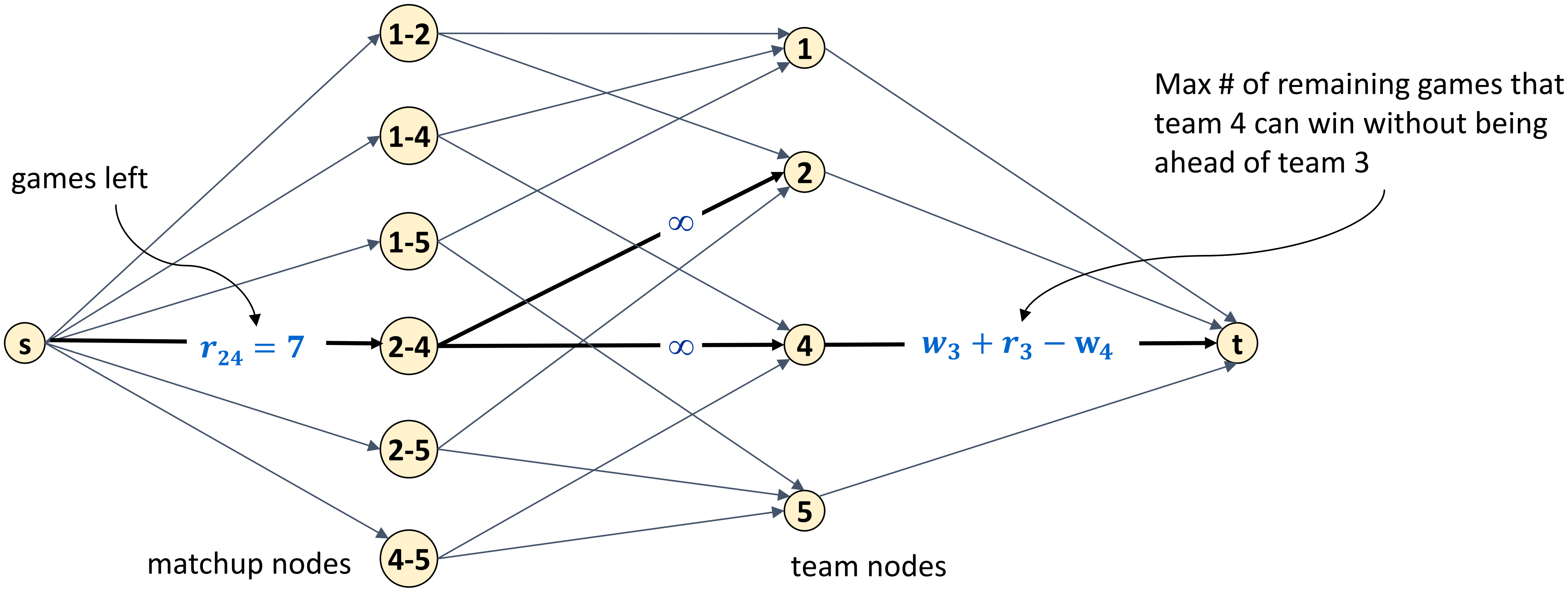
Baseball elimination problem:

- Set of teams $\boldsymbol{S}$.
- Distinguished team $\boldsymbol{z} \in \boldsymbol{S}$.
- Team $\boldsymbol{x}$ has won $\boldsymbol{w_x}$ games already.
- Teams $\boldsymbol{x}$ and $\boldsymbol{y}$ play each other $\boldsymbol{r_{xy}}$ additional times.
- Is there any outcome of the remaining games in which team $\boldsymbol{z}$ finishes with the most (or tied for the most) wins?

Baseball Elimination: Max Flow Formulation

Can team 3 finish with most wins?

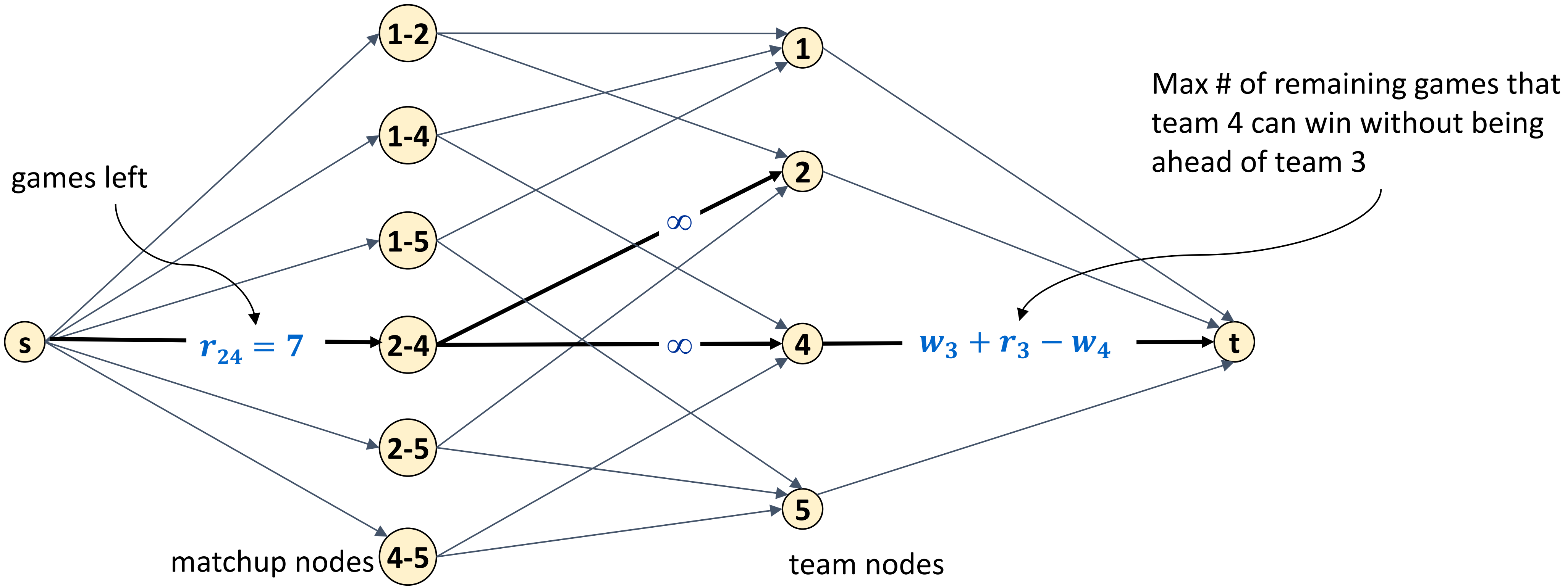
- Assume team 3 wins all remaining games $\Rightarrow \boldsymbol{w_3 + r_3}$ wins.
- Divide up remaining games so that all teams have $\leq \boldsymbol{w_3 + r_3}$ wins.



Baseball Elimination: Max Flow Formulation

Theorem: Team 3 is not eliminated iff max flow equals capacity leaving source.

- Integrality implies that each remaining $\boldsymbol{x}$-$\boldsymbol{y}$ game counts as a win for $\boldsymbol{x}$ or $\boldsymbol{y}$.
- Capacity on $(\boldsymbol{x}, \boldsymbol{t})$ edge ensures no team wins too many games.



Baseball Elimination: Explanation for Sports Writers

Team i	Wins w_i	Losses l_i	To play r_i	Against = r_{ij}				
				NY	Bal	Bos	Tor	Det
NY	75	59	28	-	3	8	7	3
Baltimore	71	63	28	3	-	2	7	4
Boston	69	66	27	8	2	-	0	0
Toronto	63	72	27	7	7	0	-	-
Detroit	49	86	27	3	4	0	0	-

AL East: August 30, 1996

Which teams have a chance of finishing the season with most wins?

- Detroit could finish season with **49 + 27 = 76** wins.

Certificate of elimination. T = {NY, Bal, Bos, Tor}

- Have already won $w(T)$ =**75+71+69+63=278** games.
- Must win at least $r(T)$ = **3+8+7+2+7=27** more among themselves.
- Average team in T wins at least **305/4** > **76** games.

Baseball Elimination: Explanation for Sports Writers

Defn: Given a set T of teams define

- $w(T) = \sum_{x \in T} w_x$ total number of wins for teams in T
- $r(T) = \sum_{\{x,y\} \subseteq T} r_{xy}$ total remaining games between teams in T

We say that T eliminates team z iff $\frac{w(T)+r(T)}{|T|} > w_z + r_z$ since an average team in T will win more than $w_z + r_z$ games.

Theorem [Hoffman-Rivlin 1967]: Team z is eliminated
$\Leftrightarrow$ there is some set T of teams that eliminates z (as defined above).

Proof: $\Leftarrow$ Shown above

$\Rightarrow$ Choose T to be the set of teams on the source side of the min cut...

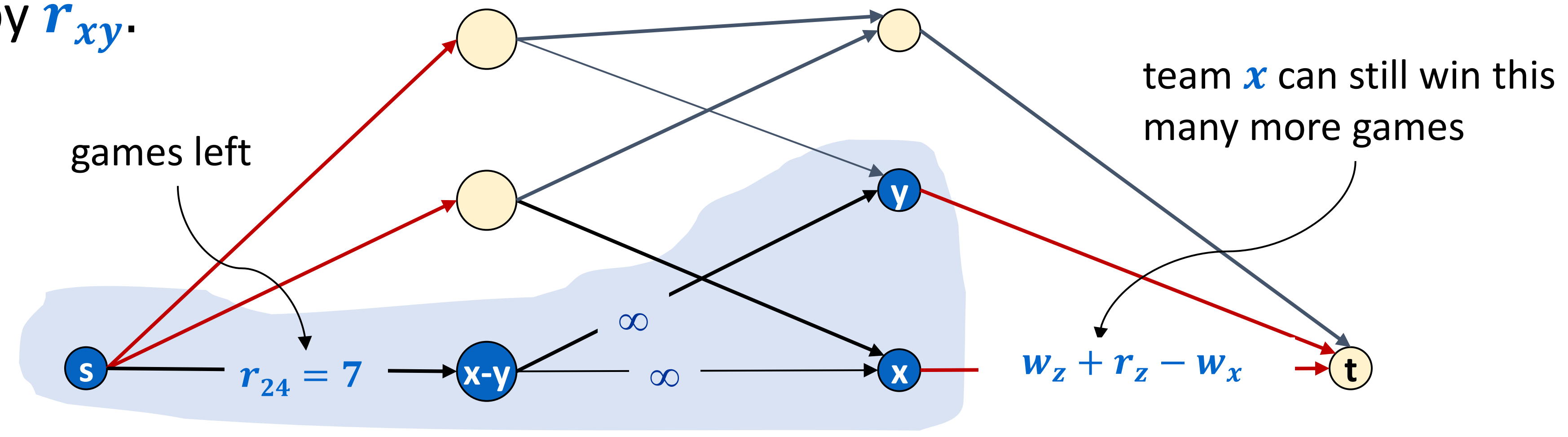
Baseball Elimination: Explanation for Sports Writers

Proof of ⇒: Assume that z is eliminated

Let T = team nodes in A for minimum cut (A, B) with capacity $< \sum_{xy} r_{xy}$.

Claim: x–$y \in A \iff$ both $x \in A$ and $y \in A$ (equivalently $x \in T$ and $y \in T$).

- infinite capacity edges ensure that if x–$y \in A$ then $x \in A$ and $y \in A$
- if $x \in A$ and $y \in A$ but x–$y \notin A$, then adding x–y to A decreases cut capacity by r_{xy}.



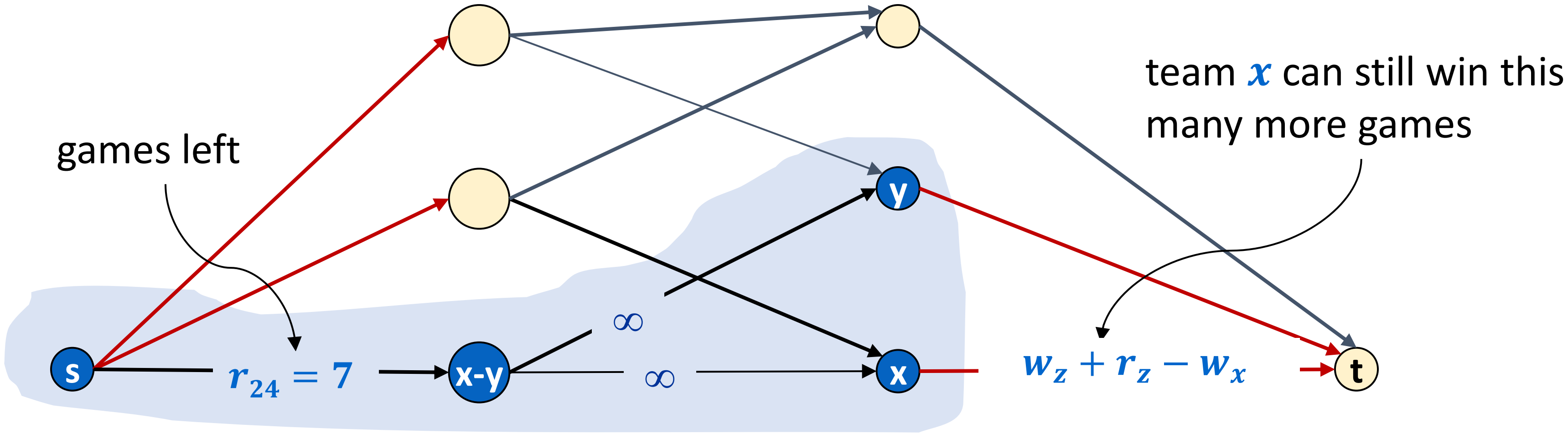
Baseball Elimination: Explanation for Sports Writers

Proof of ⇒: Assume that z is eliminated.

Let T = team nodes in A for minimum cut (A, B) with capacity $< \sum_{xy} r_{xy}$.

Claim: x−$y \in A \iff$ both $x \in A$ and $y \in A$ (equivalently $x \in T$ and $y \in T$).

Then $c(A, B) = \sum_{xy} r_{xy} - r(T) + |T|(w_z + r_z) - w(T)$



Baseball Elimination: Explanation for Sports Writers

Proof of ⇒: Assume that z is eliminated.

Let T = team nodes in A for minimum cut (A, B) with capacity $< \sum_{xy} r_{xy}$.

Claim: x–$y \in A \iff$ both $x \in A$ and $y \in A$ (equivalently $x \in T$ and $y \in T$).

Then $c(A, B) = \sum_{xy} r_{xy} - r(T) + |T|(w_z + r_z) - w(T)$

Now $c(A, B) < \sum_{xy} r_{xy}$ implies that $r(T) - |T|(w_z + r_z) + w(T) > 0$.

Rearranging, we have $r(T) + w(T) > |T|(w_z + r_z)$ so $\frac{w(T)+r(T)}{|T|} > w_z + r_z$ which means that T eliminates z.