# CSE 421 Winter 2025
# Lecture 17:
# Max Flow Running Time

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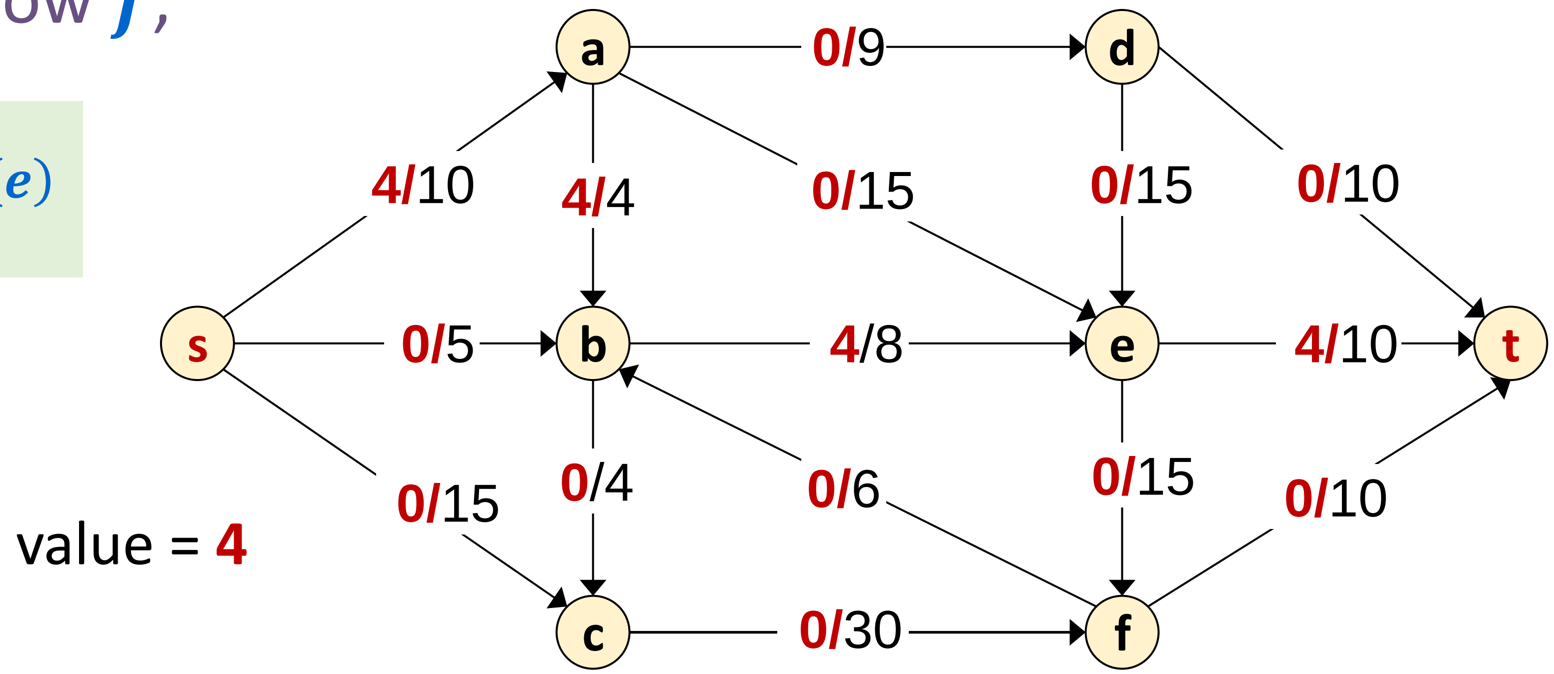
# Flows

**Defn:** An $s$-$t$ flow in a flow network is a function $f\colon\ E \rightarrow \mathbb{R}$ that satisfies:

- For each $e \in E$: $0 \leq\ f(e) \leq\ c(e)$ [capacity constraints]
- For each $v \in V - \{s, t\}$ : $\displaystyle\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$ [flow conservation]

**Defn:** The value of flow $f$,

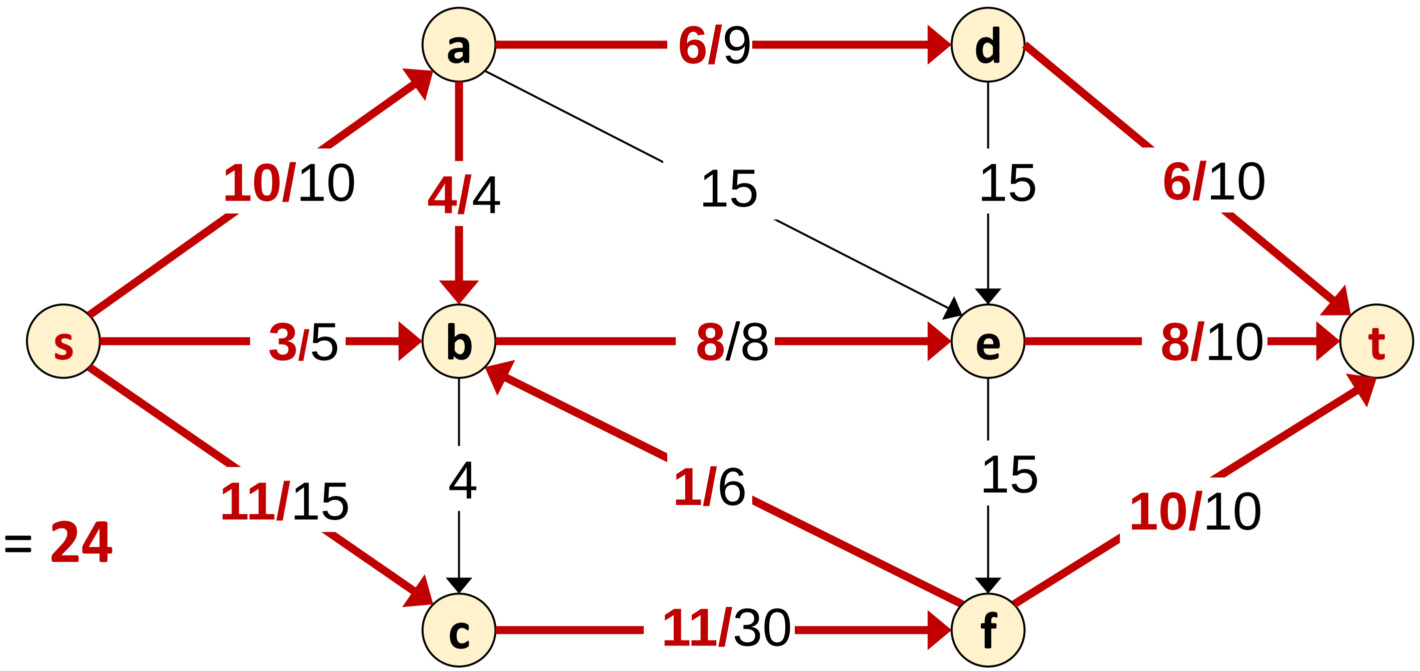
$$v(f) = \sum_{e \text{ out of } s} f(e)$$



# Maximum Flow Problem

**Given:** a flow network

**Find:** an $s$-$t$ flow of maximum value

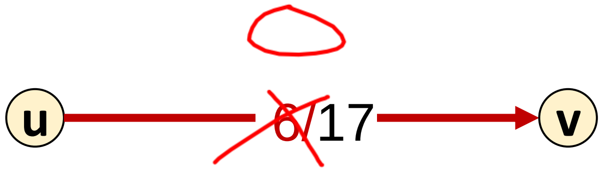



value = **24**

# Residual Graphs and Augmenting Paths

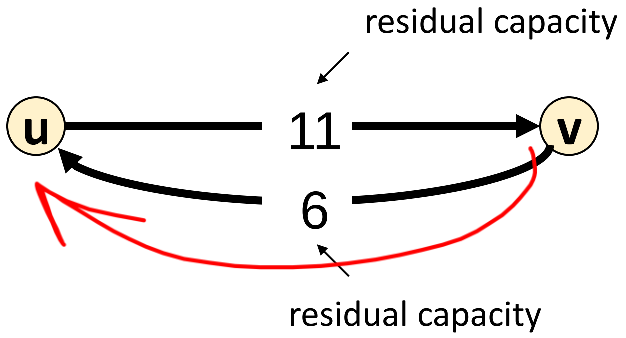
Residual edges of two kinds:

- Forward: $\boldsymbol{e = (u, v)}$ with capacity $\boldsymbol{c_f(e) = c(e) - f(e)}$
  - Amount of extra flow we can add along $\boldsymbol{e}$
- Backward: $\boldsymbol{e^{\mathrm{R}} = (v, u)}$ with capacity $\boldsymbol{c_f(e) = f(e)}$
  - Amount we can reduce/undo flow along $\boldsymbol{e}$




Residual graph: $\boldsymbol{G_f = (V, E_f)}$.

- Residual edges with residual capacity $\boldsymbol{c_f(e) > 0}$.
- $\boldsymbol{E_f = \{e : f(e) < c(e)\} \cup \{e^{\mathrm{R}} : f(e) > 0\}}$.




Augmenting Path: Any $\boldsymbol{s}$-$\boldsymbol{t}$ path $\boldsymbol{P}$ in $\boldsymbol{G_f}$.     Let **bottleneck**($\boldsymbol{P}$)$= \min_{e \in P} \boldsymbol{c_f(e)}$.

**Ford-Fulkerson idea:** Repeat "find an augmenting path $\boldsymbol{P}$ and increase flow by **bottleneck**($\boldsymbol{P}$)" until none left.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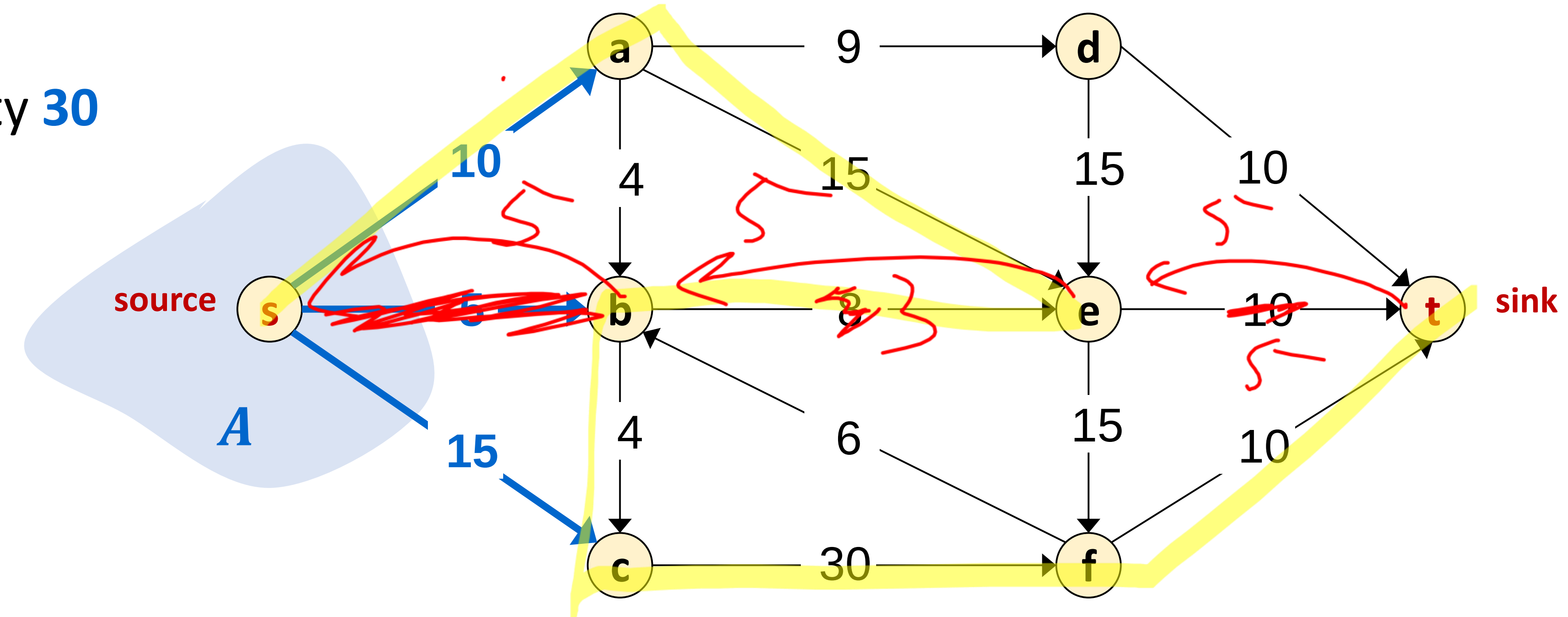
# Cuts

**Defn:** An $s$-$t$ cut is a partition $(A, B)$ of $V$ with $s \in A$ and $t \in B$.

The capacity of cut $(A, B)$ is

$$c(A, B) = \sum_{e \text{ out of } A} c(e)$$

capacity **30**



# Cuts

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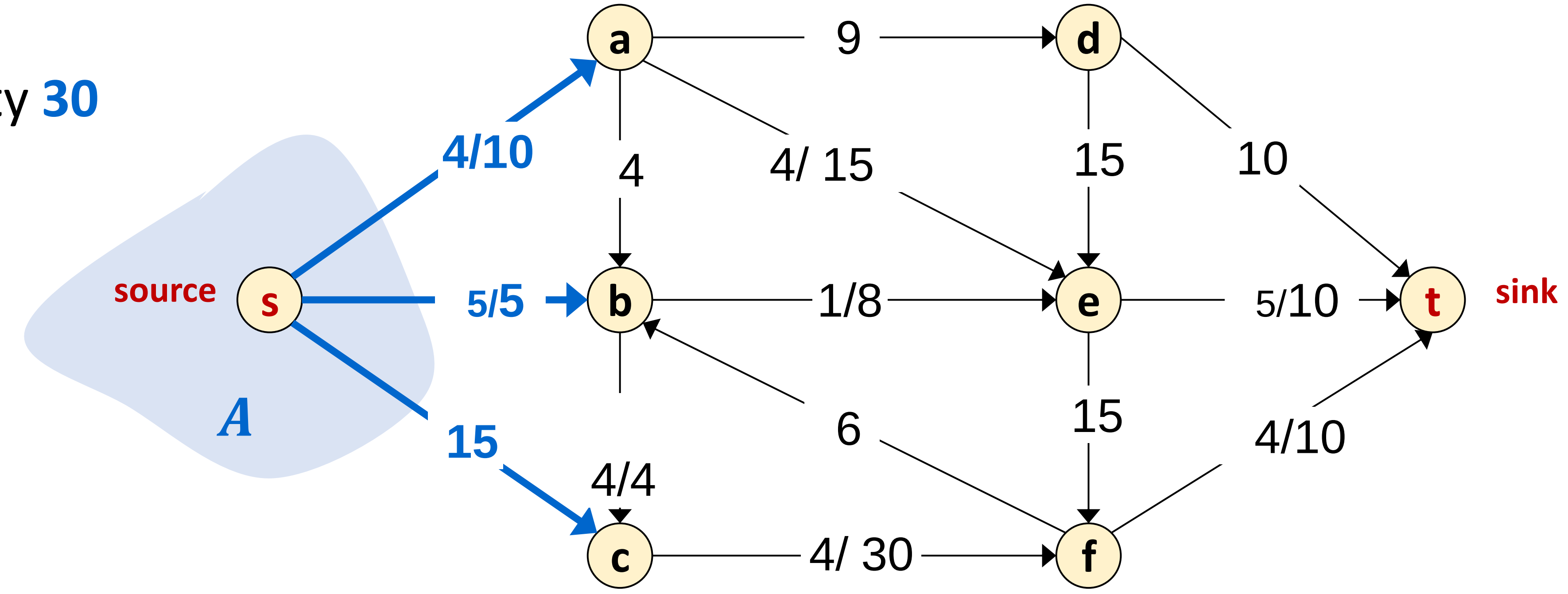


# Minimum Cut Problem

**Defn:** An $s$-$t$ cut is a partition $(A, B)$ of $V$ with $s \in A$ and $t \in B$.

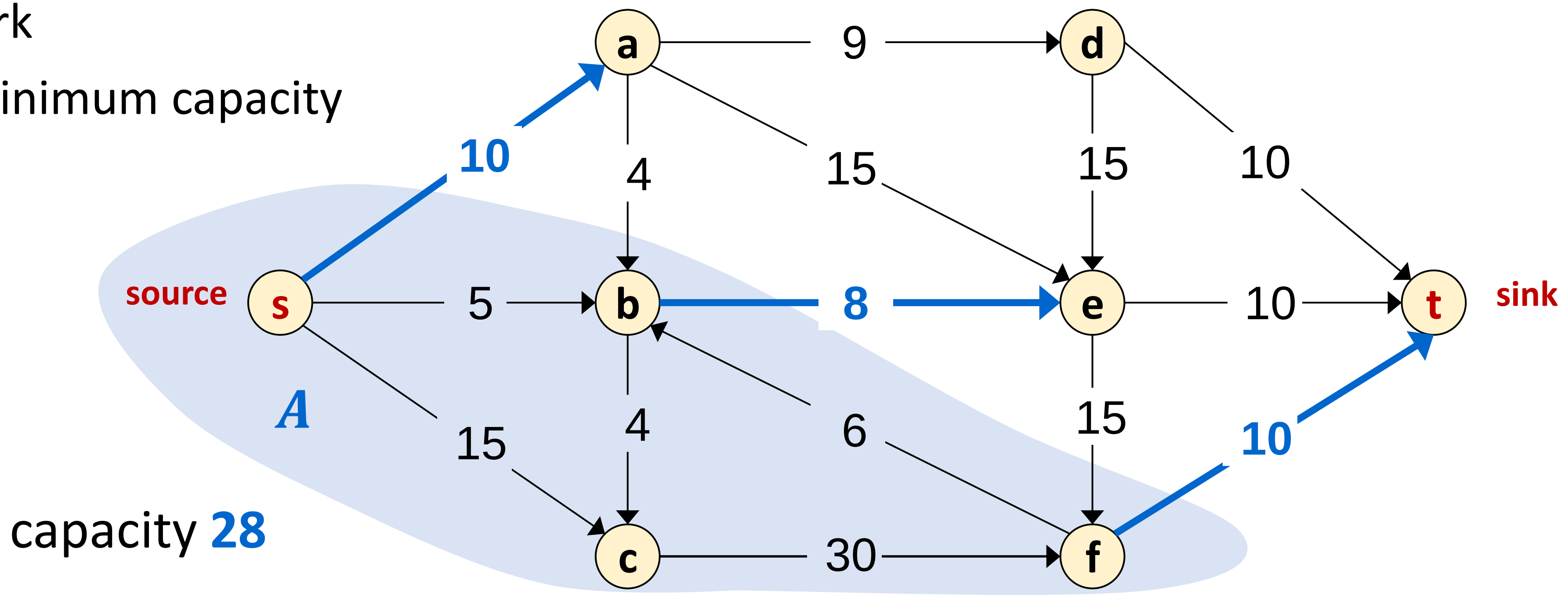
The capacity of cut $(A, B)$ is

$$c(A,B) = \sum_{e \text{ out of } A} c(e)$$

**Minimum s-t cut problem:**

**Given:** a flow network

**Find:** an $s$-$t$ cut of minimum capacity



# Flows and Cuts

Let $\boldsymbol{f}$ be any $\boldsymbol{s}$-$\boldsymbol{t}$ flow and $(\boldsymbol{A}, \boldsymbol{B})$ be any $\boldsymbol{s}$-$\boldsymbol{t}$ cut:

**Flow Value Lemma:** The net value of the flow sent across $(\boldsymbol{A}, \boldsymbol{B})$ equals $\boldsymbol{v}(\boldsymbol{f})$.

**Intuition**: All flow coming from $s$ must eventually reach $t$, and so must cross that cut

**Weak Duality:** The value of the flow is at most the capacity of the cut; i.e., $\boldsymbol{v}(\boldsymbol{f}) \leq \boldsymbol{c}(\boldsymbol{A}, \boldsymbol{B})$.

**Intuition**: Since all flow must cross any cut, any cut's capacity is an upper bound on the flow

**Corollary:** If $\boldsymbol{v}(\boldsymbol{f}) = \boldsymbol{c}(\boldsymbol{A}, \boldsymbol{B})$ then $\boldsymbol{f}$ is a maximum flow and $(\boldsymbol{A}, \boldsymbol{B})$ is a minimum cut.

**Intuition**: If we find a cut whose capacity matches the flow, we can't push more flow through that cut because it's already at capacity. We additionally can't find a smaller cut, since that flow was achievable.

# Max-Flow Min-Cut Theorem

**Augmenting Path Theorem:** Flow $\boldsymbol{f}$ is a max flow $\Leftrightarrow$ there are no augmenting paths wrt $\boldsymbol{f}$

**Max-Flow Min-Cut Theorem:** The value of the max flow equals the value of the min cut.

[Elias-Feinstein-Shannon 1956, Ford-Fulkerson 1956] **"Maxflow = Mincut"**

**Proof:** We prove both together by showing that all of these are equivalent:

(i) There is a cut $(\boldsymbol{A}, \boldsymbol{B})$ such that $\boldsymbol{v}(\boldsymbol{f}) = \boldsymbol{c}(\boldsymbol{A}, \boldsymbol{B})$.

(ii) Flow $\boldsymbol{f}$ is a max flow.

(iii) There is no augmenting path w.r.t. $\boldsymbol{f}$.

(i) $\Rightarrow$ (ii): Comes from weak duality lemma.

(ii) $\Rightarrow$ (iii): (by contradiction)
If there is an augmenting path w.r.t. flow $\boldsymbol{f}$ then we can improve $\boldsymbol{f}$. Therefore $\boldsymbol{f}$ is not a max flow.

(iii) $\Rightarrow$ (i): We will use the residual graph to identify a cut whose capacity matches the flow

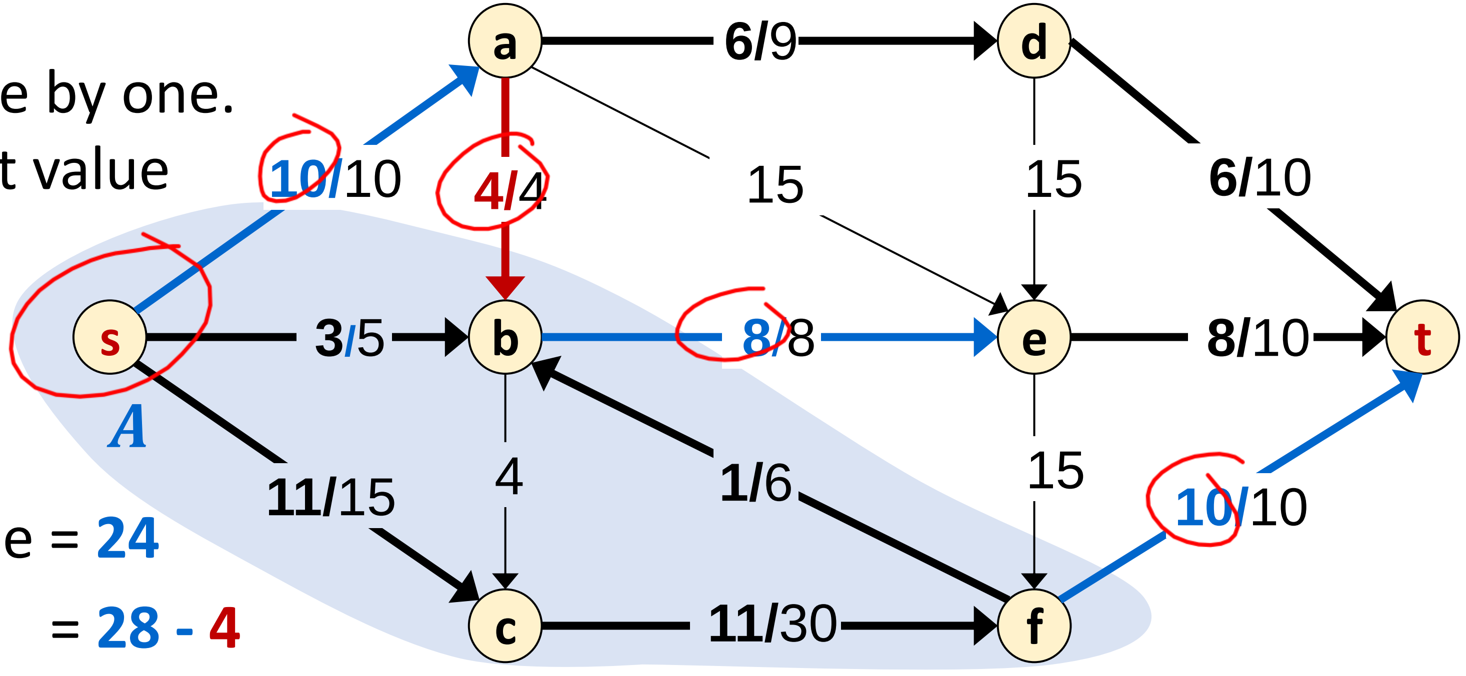
# Flow Value Lemma – Idea

**Flow Value Lemma:** Let $f$ be any $s$-$t$ flow and $(A, B)$ be any $s$-$t$ cut. The net value of the flow sent across the cut equals $v(f)$:

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) = v(f)$$

Why is it true?

- Add vertices to $s$ side one by one.
- By flow conservation, net value doesn't change




value = 24

= 28 - 4

# Flow Value Lemma – Proof

**Flow Value Lemma:** Let $f$ be any $s$-$t$ flow and $(A, B)$ be any $s$-$t$ cut. The net value of the flow sent across the cut equals $v(f)$:

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) = v(f)$$

**Proof:**

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

$$= \sum_{e \text{ out of } s} f(e) - \sum_{e \text{ into } s} f(e) + \sum_{v \in A - \{s\}} \left[ \sum_{e \text{ out of } v} f(e) - \sum_{e \text{ into } v} f(e) \right]$$

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$

$= \mathbf{0}$. No edges into $s$ since it is a source

$= \mathbf{0}$ by flow conservation.

Contributions from internal edges of $A$ cancel.

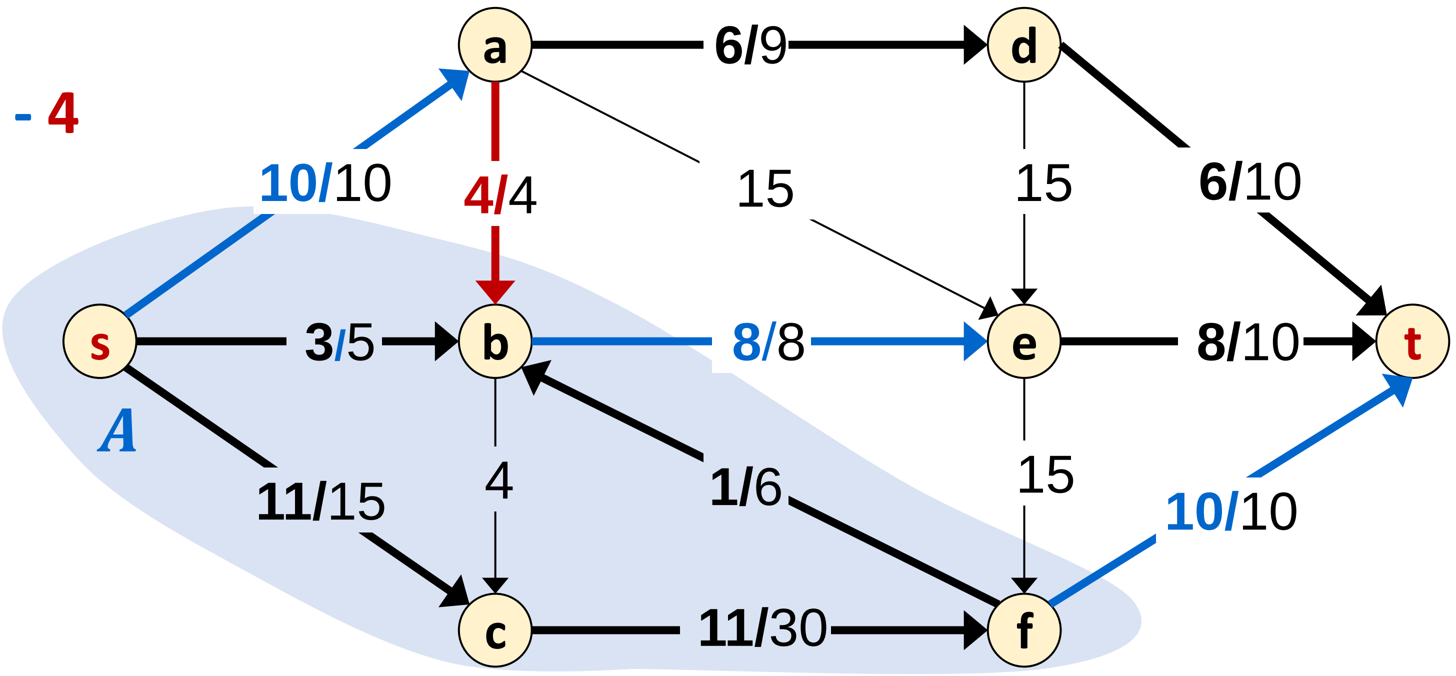
# Weak Duality - Idea

(i) ⇒ (ii)

**Weak Duality:** Let $f$ be any $s$-$t$ flow and $(A, B)$ be any $s$-$t$ cut. The value of the flow is at most the capacity of the cut; i.e., $v(f) \leq c(A, B)$:

Value of flow = **24** = **28** - **4**

Capacity of cut = **28**



# Weak Duality - Proof

(i) ⇒ (ii)

**Weak Duality:** Let $f$ be any $s$-$t$ flow and $(A, B)$ be any $s$-$t$ cut. The value of the flow is at most the capacity of the cut; i.e., $v(f) \leq c(A, B)$.

**Proof:**

$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) \\
&\leq \sum_{e \text{ out of } A} f(e) && \text{since } f(e) \geq 0 \\
&\leq \sum_{e \text{ out of } A} c(e) && \text{since } f(e) \leq c(e) \\
&= c(A, B)
\end{aligned}
$$

∎

# Proof of Max-Flow Min-Cut Theorem

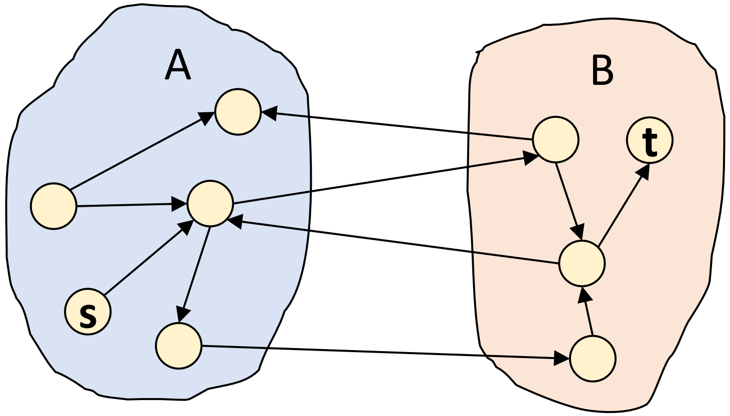
(iii) ⇒ (i):

**Claim:** If there is no augmenting path w.r.t. $\boldsymbol{f}$, there is a cut $(\boldsymbol{A}, \boldsymbol{B})$ s.t. $\boldsymbol{v}(\boldsymbol{f}) = \boldsymbol{c}(\boldsymbol{A}, \boldsymbol{B})$.

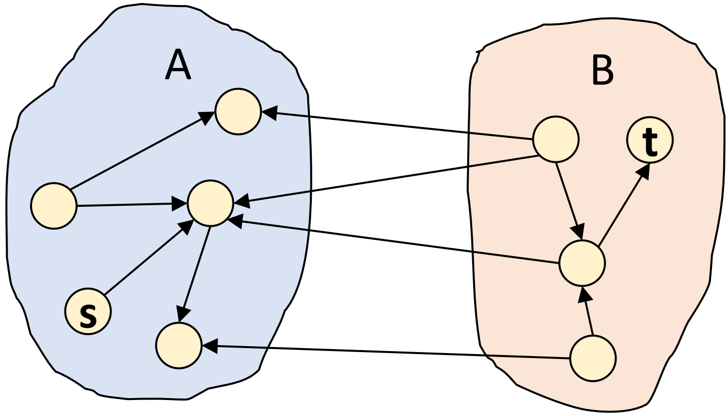
**Proof of Claim:** Let $\boldsymbol{f}$ be a flow with no augmenting paths.

Let $\boldsymbol{A}$ be the set of vertices reachable from $\boldsymbol{s}$ in residual graph $\boldsymbol{G_f}$.

- By definition of $\boldsymbol{A}$, $\boldsymbol{s} \in \boldsymbol{A}$.
- Since no augmenting path ($\boldsymbol{s}$-$\boldsymbol{t}$ path in $\boldsymbol{G_f}$), $\boldsymbol{t} \notin \boldsymbol{A}$.



original network



residual graph

# Proof: Identifying the Min Cut

(iii) $\Rightarrow$ (i):

**Claim:** If there is no augmenting path w.r.t. $\boldsymbol{f}$, there is a cut $(\boldsymbol{A}, \boldsymbol{B})$ s.t. $\boldsymbol{v}(\boldsymbol{f}) = \boldsymbol{c}(\boldsymbol{A}, \boldsymbol{B})$.

**Proof of Claim:** Let $\boldsymbol{f}$ be a flow with no augmenting paths.

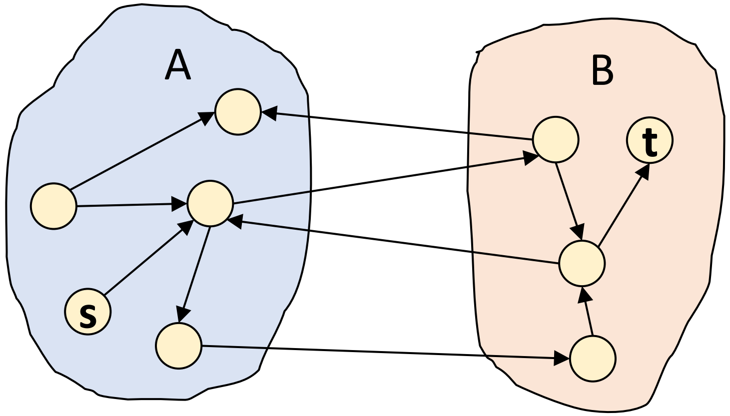
Let $\boldsymbol{A}$ be the set of vertices reachable from $\boldsymbol{s}$ in residual graph $\boldsymbol{G_f}$.

- By definition of $\boldsymbol{A}$, $\boldsymbol{s} \in \boldsymbol{A}$.
- Since no augmenting path ($\boldsymbol{s}$-$\boldsymbol{t}$ path in $\boldsymbol{G_f}$), $\boldsymbol{t} \notin \boldsymbol{A}$.

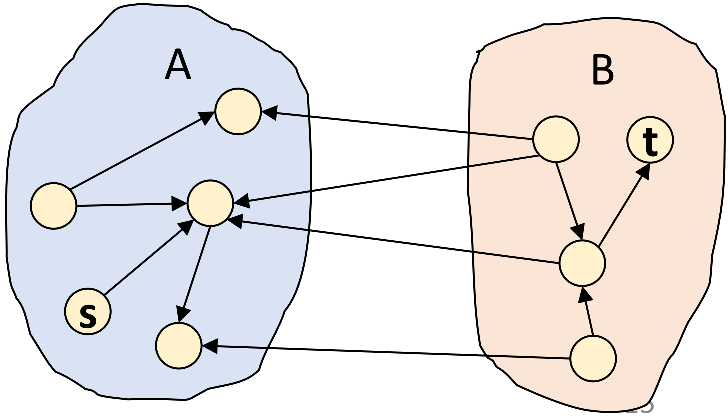
Then

$$\boldsymbol{v}(\boldsymbol{f}) = \sum_{\boldsymbol{e} \text{ out of } \boldsymbol{A}} \boldsymbol{f}(\boldsymbol{e}) - \sum_{\boldsymbol{e} \text{ into } \boldsymbol{A}} \boldsymbol{f}(\boldsymbol{e})$$

(by Flow-Value Lemma)




original network




residual graph

# Identifying the Min Cut: No Inflow

**(iii) $\Rightarrow$ (i):**

**Claim:** If there is no augmenting path w.r.t. $f$, there is a cut $(A, B)$ s.t. $v(f) = c(A, B)$.

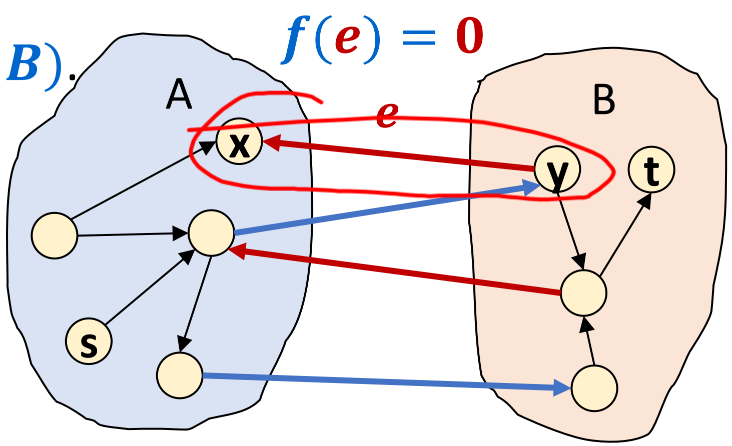
**Proof of Claim:** Let $f$ be a flow with no augmenting paths.

Let $A$ be the set of vertices reachable from $s$ in residual graph $G_f$.

- By definition of $A$, $s \in A$.
- Since no augmenting path ($s$-$t$ path in $G_f$), $t \notin A$.

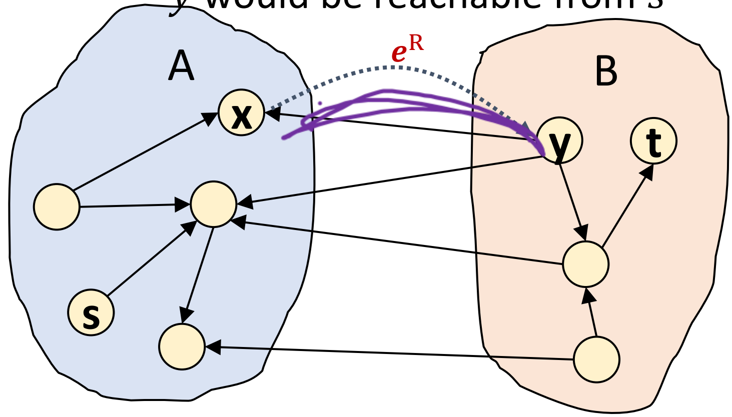
Then
$$v(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$
$$= \sum_{e \text{ out of } A} f(e)$$

(**By contradiction:** If an edge going into $A$ had flow then the backward edge would be in the residual graph, so the edge should not cross the cut)




original network

$e^R$ can't exist because then $y$ would be reachable from $s$




residual graph

# Identifying the Min Cut: Saturated Outflow

(iii) ⇒ (i):

**Claim:** If there is no augmenting path w.r.t. $f$, there is a cut $(A, B)$ s.t. $v(f) = c(A, B)$.

**Proof of Claim:** Let $f$ be a flow with no augmenting paths.

Let $A$ be the set of vertices reachable from $s$ in residual graph $G_f$.

- By definition of $A$, $s \in A$.
- Since no augmenting path ($s$-$t$ path in $G_f$), $t \notin A$.

Then

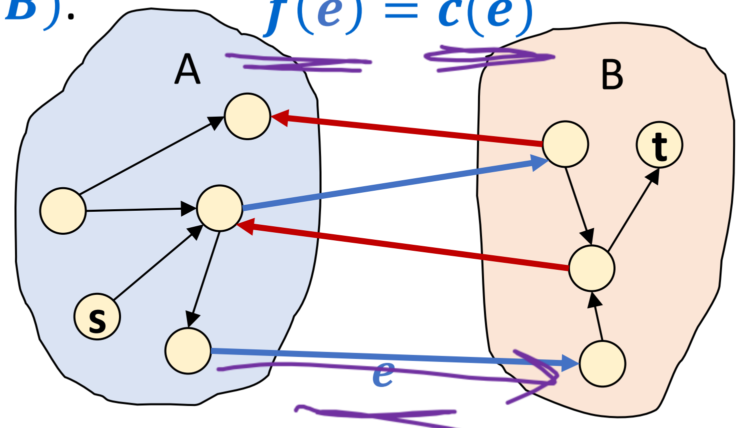
$$v(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$

$$= \sum_{e \text{ out of } A} f(e)$$

$$= \sum_{e \text{ out of } A} c(e)$$

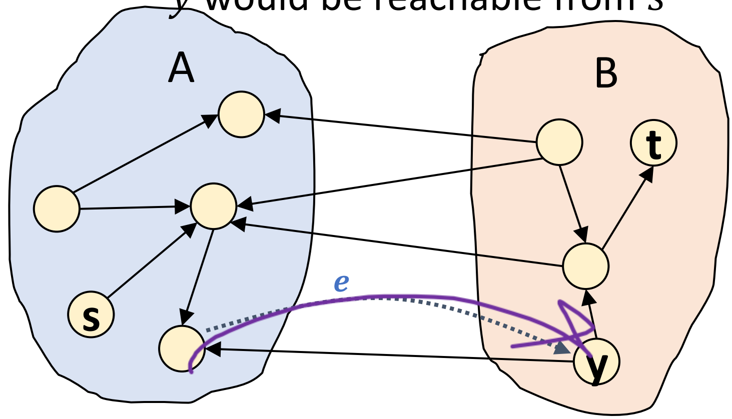
(**By contradiction:** If an edge going out of $A$ had unused capacity then the forward edge would be in the residual graph, so the edge should not cross the cut)

"$e$ is saturated"
No unused capacity on $e$
$f(e) = c(e)$




original network

$e^R$ can't exist because then $y$ would be reachable from $s$




residual graph

# Identifying the Min Cut: Conclusion

(iii) ⇒ (i):

**Claim:** If there is no augmenting path w.r.t. $\boldsymbol{f}$, there is a cut $(\boldsymbol{A}, \boldsymbol{B})$ s.t. $\boldsymbol{v}(\boldsymbol{f}) = \boldsymbol{c}(\boldsymbol{A}, \boldsymbol{B})$.

**Proof of Claim:** Let $\boldsymbol{f}$ be a flow with no augmenting paths.

Let $\boldsymbol{A}$ be the set of vertices reachable from $\boldsymbol{s}$ in residual graph $\boldsymbol{G_f}$.

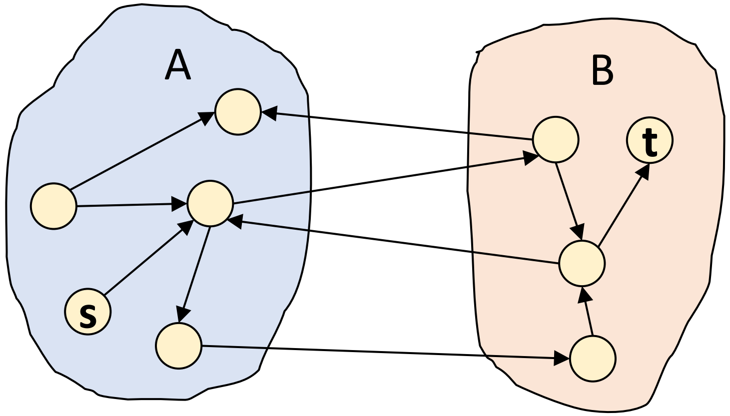
- By definition of $\boldsymbol{A}$, $\boldsymbol{s} \in \boldsymbol{A}$.
- Since no augmenting path ($\boldsymbol{s}$-$\boldsymbol{t}$ path in $\boldsymbol{G_f}$), $\boldsymbol{t} \notin \boldsymbol{A}$.

Then

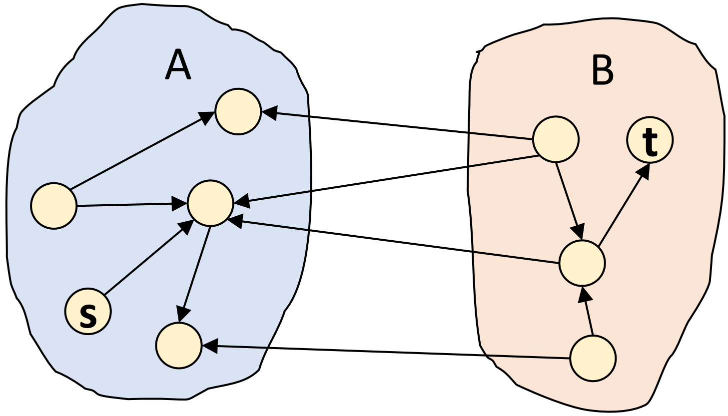
$$\boldsymbol{v}(\boldsymbol{f}) = \sum_{\boldsymbol{e} \text{ out of } \boldsymbol{A}} \boldsymbol{f}(\boldsymbol{e}) - \sum_{\boldsymbol{e} \text{ into } \boldsymbol{A}} \boldsymbol{f}(\boldsymbol{e})$$

$$= \sum_{\boldsymbol{e} \text{ out of } \boldsymbol{A}} \boldsymbol{f}(\boldsymbol{e})$$

$$= \sum_{\boldsymbol{e} \text{ out of } \boldsymbol{A}} \boldsymbol{c}(\boldsymbol{e}) \quad = \boldsymbol{c}(\boldsymbol{A}, \boldsymbol{B}) \qquad \text{(by Definition)}$$



original network



residual graph

# Fork Fulkerson Algorithm

```
FordFulkerson(G, s, t, c){
    for each e ∈ E{
        set f(e) = 0
    }
    calculate residual graph G_f
    while G_f has an s − t path P{
        augment(f, c, P)
        update G_f
    }
    return f
}
```

```
augment(f, c, P){
    b = bottleneck(P)
    for each e ∈ P{
        f(e) += b
        f(e^R) −= b
    }
    return f
}
```

# MaxFlow/MinCut & Ford-Fulkerson Algorithm

**Augmenting Path Theorem:** Flow $f$ is a max flow $\Leftrightarrow$ there are no augmenting paths wrt $f$

**Max-Flow Min-Cut Theorem:** The value of the max flow equals the value of the min cut.
[Elias-Feinstein-Shannon 1956, Ford-Fulkerson 1956] **"MaxFlow = MinCut"**

**Flow Integrality Theorem:** If all capacities are integers then there is a maximum flow with all-integer flow values.

**Ford-Fulkerson Algorithm:** $O(m)$ per iteration. With integer capacities each at most $C$ need at most **MaxFlow** $< nC$ iterations for a total of $O(mnC)$ time.

$$m = |E|$$
$$n = |V|$$



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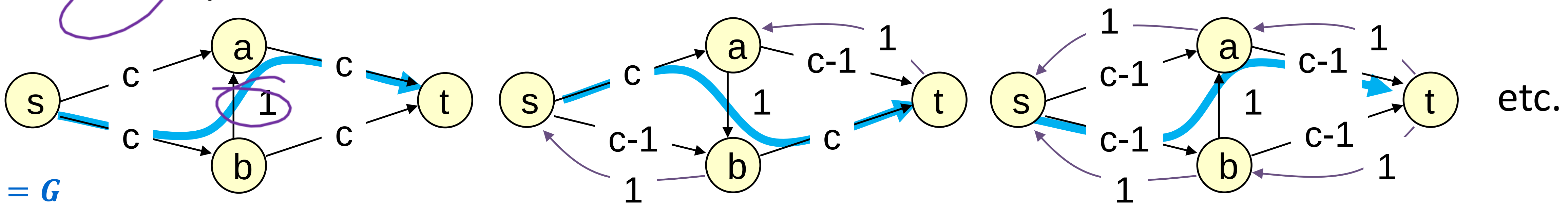
# Ford-Fulkerson Efficiency

Worst case runtime $O(mnC)$ with integer capacities $\leq C$.

- $O(m)$ time per iteration.
- At most $nC$ iterations.
- This is "pseudo-polynomial" running time.

- May take exponential time, even with integer capacities:

c $= 10^9$, say




$G_f = G$

# Polynomial-Time Variant of Ford-Fulkerson

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.
- If capacities are irrational, algorithm not guaranteed to terminate!

**Goal:** Choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

Choose augmenting paths with fewest number of edges. [Edmonds-Karp 1972 , Dinitz 1970]

- Just run BFS to find an augmenting path!

# Edmonds-Karp Algorithm (Ford-Fulkerson with BFS)

Use Breadth First Search as the search algorithm to find an $\boldsymbol{s}$-$\boldsymbol{t}$ path in $\boldsymbol{G_f}$.

- Using any shortest augmenting path

**Theorem**: Ford-Fulkerson using BFS terminates in $\boldsymbol{O(m^2 n)}$ time. [Edmonds-Karp, Dinitz]

"One of the most obvious ways to implement Ford-Fulkerson is always polynomial time"

Why might this be good intuitively?

- Longer augmenting paths involve more edges so may be more likely to hit a low residual capacity one which would limit the amount of flow improvement.

The proof uses a completely different idea…

# Edmonds-Karp Algorithm (Ford-Fulkerson with BFS)

**Analysis Focus:**

For any edge $e$ that could be in the residual graph $G_f$, (either an edge in $G$ or its reverse)

count # of iterations that $e$ is the first bottleneck edge on the augmenting path chosen by the algorithm.

**Claim:** This can't happen in more than $n/2$ iterations.

**Proof:** Write $e = (u, v)$.

Show that each time it happens, the distance from $s$ to $u$ in the residual graph $G_f$ is at least $2$ more than it was the last time.

This would be enough since the distance is $< n$
(or infinite and hence $u$ isn't reachable) so this can happen at most $n/2$ times.

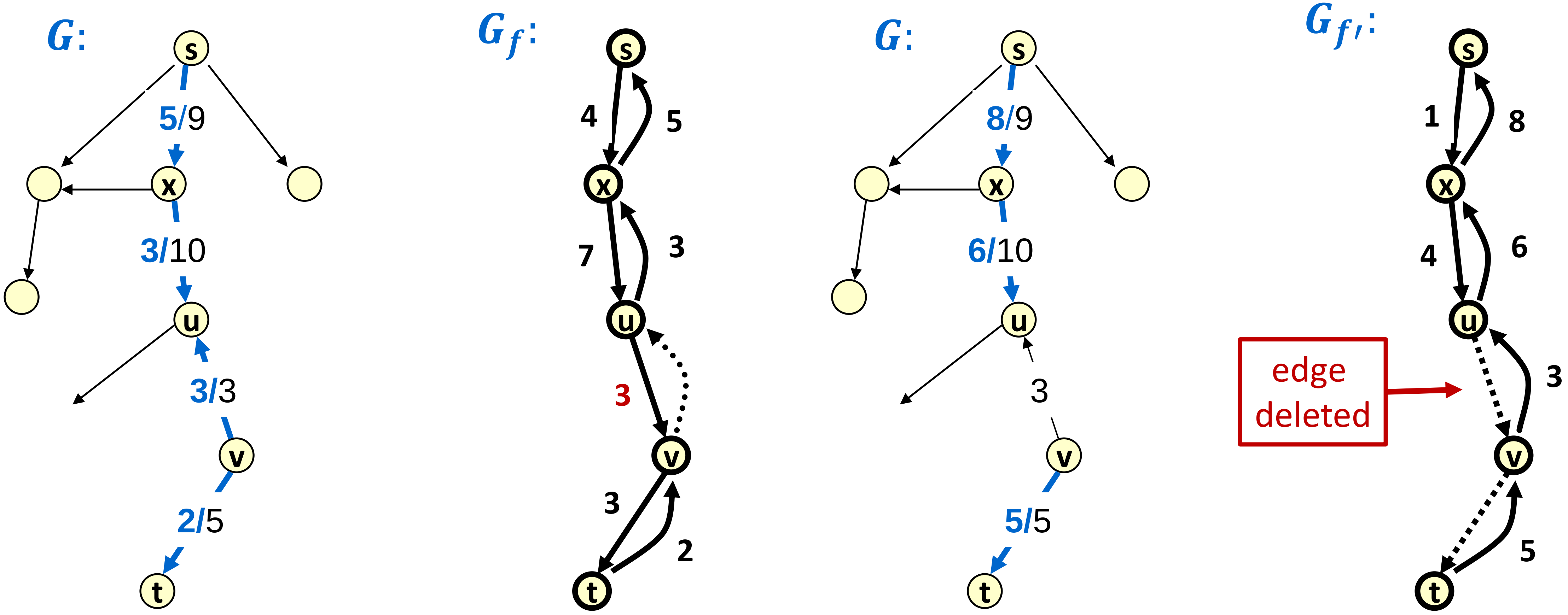
# Distances in the Residual Graph

**Key Lemma:** Let $\boldsymbol{f}$ be a flow, $\boldsymbol{G_f}$ the residual graph, and $\boldsymbol{P}$ be a shortest augmenting path. No vertex is closer to $\boldsymbol{s}$ in the residual graph after augmenting along $\boldsymbol{P}$.

**Proof:** Augmenting along $\boldsymbol{P}$ can only change the edges in $\boldsymbol{G_f}$ by either:

1. Deleting a forward edge
   - Deleting any edge can never reduce distances
2. Add a backward edge $(\boldsymbol{v}, \boldsymbol{u})$ that is the reverse of an edge $(\boldsymbol{u}, \boldsymbol{v})$ of $\boldsymbol{P}$
   - Since $\boldsymbol{P}$ was a shortest path in $\boldsymbol{G_f}$, the distance from $\boldsymbol{s}$ to $\boldsymbol{v}$ in $\boldsymbol{G_f}$ is already more than the distance from $\boldsymbol{s}$ to $\boldsymbol{u}$. Using the new backward edge $(\boldsymbol{v}, \boldsymbol{u})$ to get to $\boldsymbol{u}$ would be an even longer path to $\boldsymbol{u}$ so it is never on a shortest path to any node in the new residual graph.

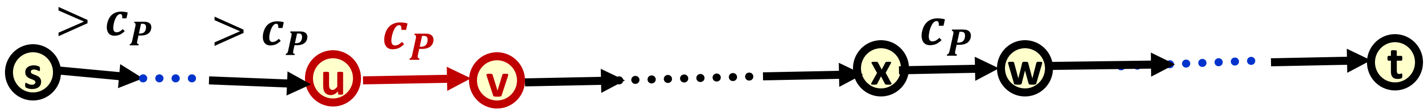
# Augmentation vs BFS



# First Bottleneck Edges in $G_f$

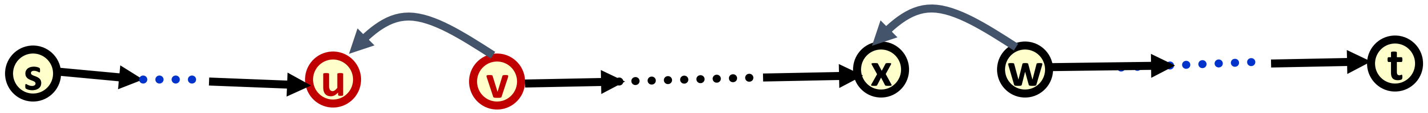
Shortest $s$-$t$ path $P$ in $G_f$

Write $c_P = \textbf{bottleneck}(P)$




$d_f(s,v) = d_f(s,u) + 1$ since $P$ is a shortest path.

After augmenting along $P$, edge $(u,v)$ disappears; but will have edge $(v,u)$




distance is $\geq 2$ larger than before

For $(u,v)$ to be a first bottleneck edge later, it must get added back to the residual graph by augmenting along a shortest path $P'$ containing $(v,u)$ in $G_{f'}$ for some flow $f'$

Since $P'$ is shortest $d_{f'}(s,u) = d_{f'}(s,v) + 1 \geq d_f(s,v) + 1 = d_f(s,u) + 2$

The next time that $(u,v)$ is first bottleneck edge is even later so distance is at least as large!

# Edmonds-Karp Algorithm (Ford-Fulkerson with BFS)

**Analysis Focus:**

For any edge $\boldsymbol{e}$ that could be in the residual graph $\boldsymbol{G_f}$, (either an edge in $\boldsymbol{G}$ or its reverse)
count # of iterations that $\boldsymbol{e}$ is the first bottleneck edge on the augmenting path chosen by the algorithm.

**Claim:** This can't happen in more than $\boldsymbol{n/2}$ iterations

**Claim ⇒ Theorem:**

Only $\boldsymbol{2m}$ edges and $O(\boldsymbol{m})$ time per iteration so $O(\boldsymbol{m^2 n})$ time overall.

# History & State of the Art for MaxFlow Algorithms

| # | year | discoverer(s) | bound |
|---|---|---|---|
| 1 | 1951 | Dantzig | $O(n^2mU)$ |
| 2 | 1955 | Ford & Fulkerson | $O(nmU)$ |
| 3 | 1970 | Dinitz<br>Edmonds & Karp | $O(nm^2)$ |
| 4 | 1970 | Dinitz | $O(n^2m)$ |
| 5 | 1972 | Edmonds & Karp<br>Dinitz | $O(m^2 \log U)$ |
| 6 | 1973 | Dinitz<br>Gabow | $O(nm \log U)$ |
| 7 | 1974 | Karzanov | $O(n^3)$ |
| 8 | 1977 | Cherkassky | $O(n^2\sqrt{m})$ |
| 9 | 1980 | Galil & Naamad | $O(nm \log^2 n)$ |
| 10 | 1983 | Sleator & Tarjan | $O(nm \log n)$ |
| 11 | 1986 | Goldberg & Tarjan | $O(nm \log(n^2/m))$ |
| 12 | 1987 | Ahuja & Orlin | $O(nm + n^2 \log U)$ |
| 13 | 1987 | Ahuja et al. | $O(nm \log(n\sqrt{\log U}/(m+2))$ |
| 14 | 1989 | Cheriyan & Hagerup | $E(nm + n^2 \log^2 n)$ |
| 15 | 1990 | Cheriyan et al. | $O(n^3/\log n)$ |
| 16 | 1990 | Alon | $O(nm + n^{8/3} \log n)$ |
| 17 | 1992 | King et al. | $O(nm + n^{2+\epsilon})$ |
| 18 | 1993 | Phillips & Westbrook | $O(nm(\log_{m/n} n + \log^{2+\epsilon} n))$ |
| 19 | 1994 | King et al. | $O(nm \log_{m/(n \log n)} n)$ |
| 20 | 1997 | Goldberg & Rao | $O(m^{3/2} \log(n^2/m) \log U)$<br>$O(n^{2/3}m \log(n^2/m) \log U)$ |

Source: Goldberg & Rao, FOCS '97

| | | | |
|---|---|---|---|
| 21 | 2013 | Orlin | $O(mn)$ |
| 22 | 2014 | Lee & Sidford | $m\sqrt{n} \log^{O(1)} n \log U$ |
| 23 | 2016 | Madry | $m^{10/7} U^{1/7} \log^{O(1)} n$ |
| 24 | 2021 | Gao, Liu, & Peng | $m^{3/2-1/328} \log^{O(1)} n \log U$ |
| 25 | 2022 | van den Brand et al. | $m^{3/2-1/58} \log^{O(1)} n \log U$ |
| 26 | 2022 | Chen et al. | $m^{1+o(1)} \log U$ |

Tables use $U$ instead of $C$ for the upper bound on capacities

Methods:

Augmenting Paths – increase flow to capacity

Preflow-Push – decrease flow to get flow conservation

Linear Programming – randomized, high probability of optimality

2012 Orlin + King et al. $O(nm)$