CSE 421 Winter 2025
Lecture 12: Dynamic Programming

Nathan Brunelle

http://www.cs.uw.edu/421

Algorithmic Paradigms

Greedy: Build up a solution incrementally, myopically optimizing some local criterion.

Divide-and-conquer: Break up a problem into sub-problems (each typically a constant factor smaller), solve each sub-problem *independently*, and combine solution to sub-problems to form solution to original problem.

Dynamic programming: Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems.

Algorithm Design Techniques

Dynamic Programming:

- Technique for making building solution to a problem based on solutions to smaller subproblems (recursive ideas).
- The subproblems just have to be smaller, but don't need to be a constant-factor smaller like divide and conquer.
- Useful when *the same subproblems show up over and over again*
- The final solution is simple iterative code when the following holds:
  - *The parameters to all the subproblems are predictable in advance*

Dynamic Programming History

Bellman. [1950s] Pioneered the systematic study of dynamic programming.

Etymology

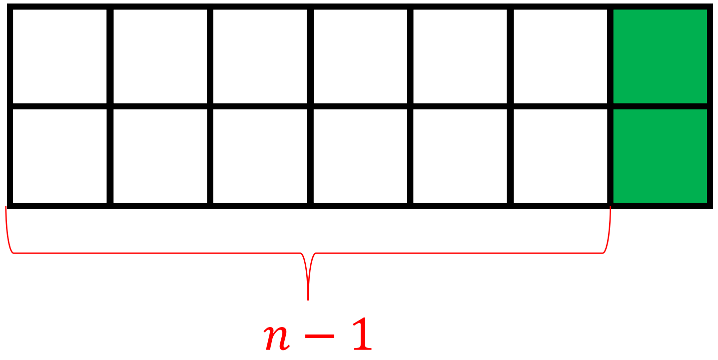
- Dynamic programming = planning over time.
- Secretary of Defense was hostile to mathematical research.
- Bellman sought an impressive name to avoid confrontation.

> "it's impossible to use dynamic in a pejorative sense"
> "something not even a Congressman could object to"

Reference: Bellman, R. E. *Eye of the Hurricane, An Autobiography.*

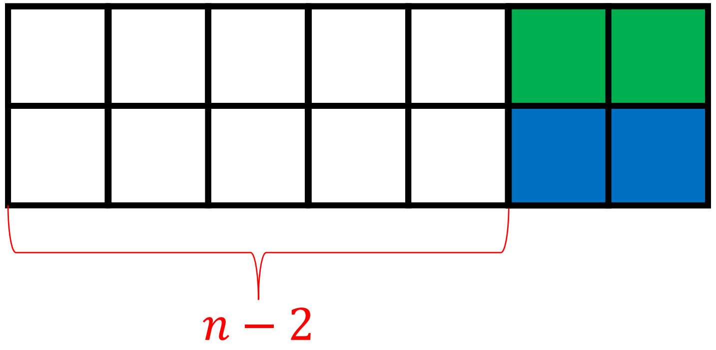
How many ways are there to tile a $2 \times n$ board with dominoes?

Two ways to fill the final column:



$$Tile(n) = Tile(n-1) + Tile(n-2)$$

$$Tile(0) = Tile(1) = 1$$

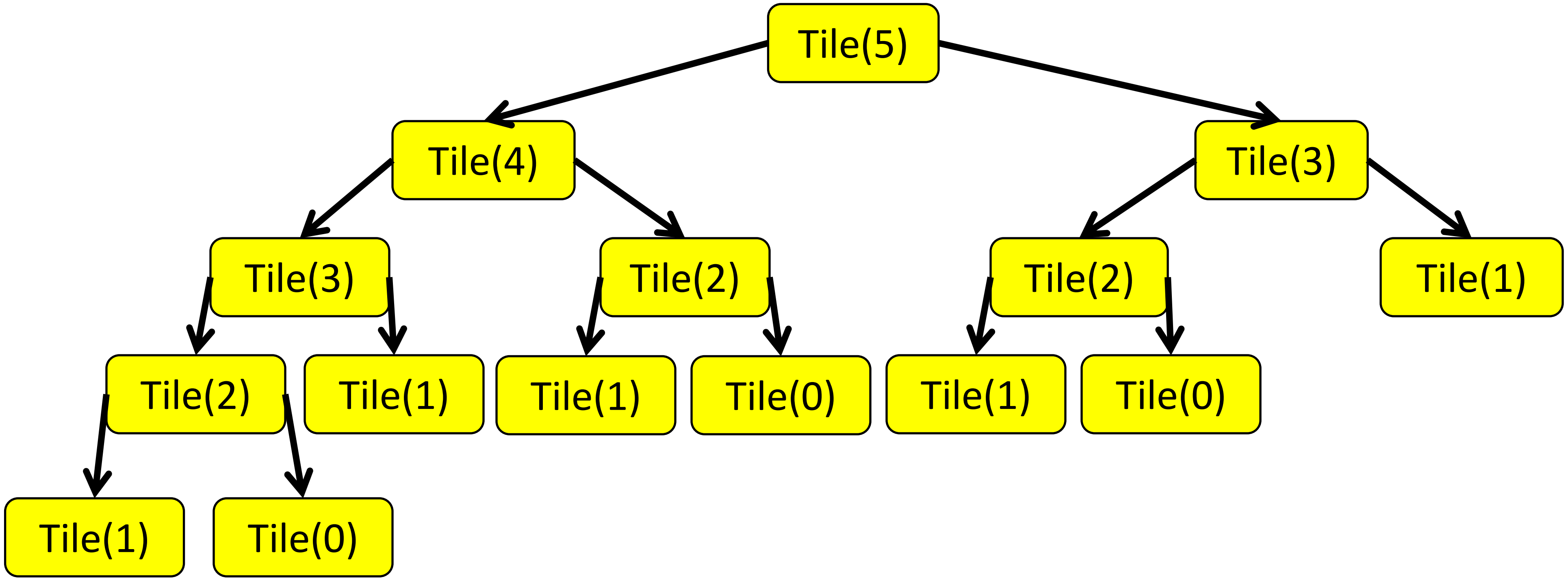


How to compute $Tile(n)$?

```
Tile(n):
    if n < 2:
        return 1
    return Tile(n-1)+Tile(n-2)
```

Running Time?

Recursion Tree

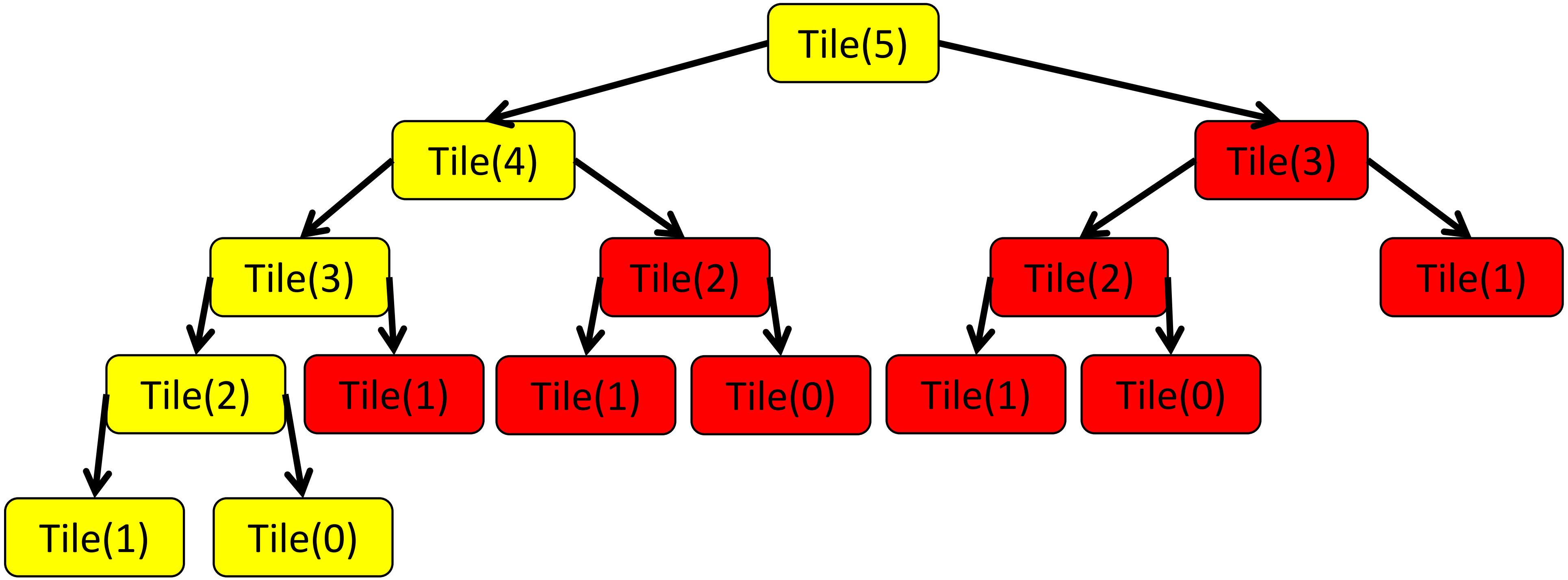


Many redundant calls!

Run time: $\Omega(2^n)$

Better way: Use Memory!

Recursion Tree



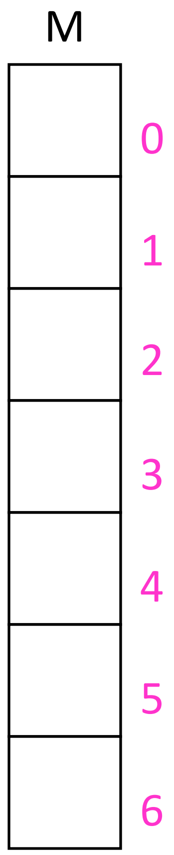
Many redundant calls!

Run time: $\Omega(2^n)$

Better way: Use Memory!

Computing $Tile(n)$ with Memory

```
Initialize Memory M
Tile(n):
    if n < 2:
        return 1
    if M[n] is filled:
        return M[n]
    M[n] = Tile(n-1)+Tile(n-2)
    return M[n]
```



Technique: “memoization” (note no “r”)

Computing $Tile(n)$ with Memory - “Top Down”

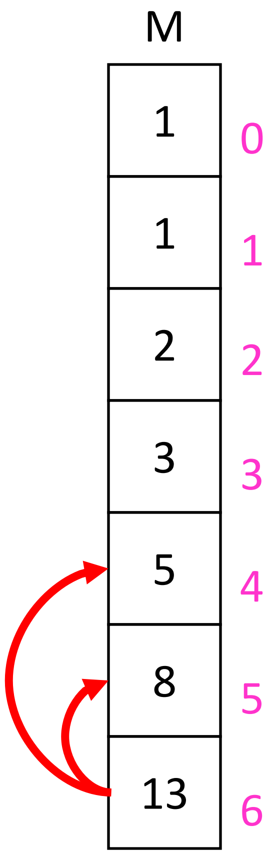
```
Initialize Memory M
Tile(n):
    if n < 2:
        return 1
    if M[n] is filled:
        return M[n]
    M[n] = Tile(n-1)+Tile(n-2)
    return M[n]
```

M

M	
1	0
1	1
2	2
3	3
5	4
8	5
13	6

Computing $Tile(n)$ with Memory - "Top Down"

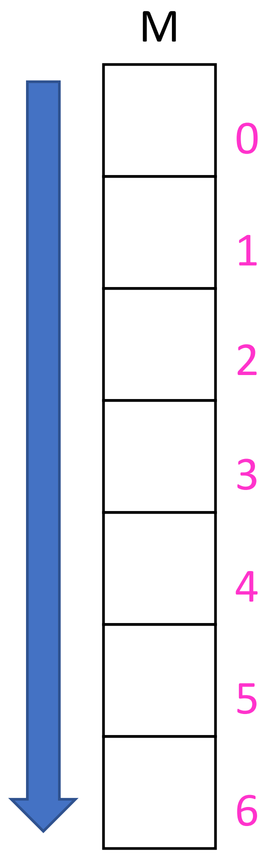
```
Initialize Memory M
Tile(n):
	if n < 2:
		return 1
	if M[n] is filled:
		return M[n]
	M[n] = Tile(n-1)+Tile(n-2)
	return M[n]
```



Recursive calls happen in a predictable order

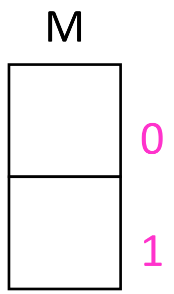
$Tile(n)$ with Memory - "Bottom Up"

```
Tile(n):
    Initialize Memory M
    M[0] = 1
    M[1] = 1
    for i = 2 to n:
        M[i] = M[i-1] + M[i-2]
    return M[n]
```



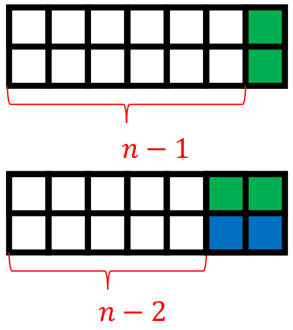
Better $Tile(n)$ with Memory - “Bottom Up”

```
Tile(n):
    M[0] = 1
    M[1] = 1
    answer = -1
    for i = 2 to n:
        answer = M[0]+M[1]
        M[0] = M[1]
        M[1] = answer
    return M[1]
```

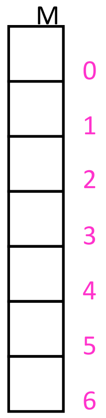


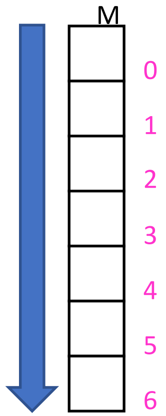
Observation: We only need to remember the last two subproblems!

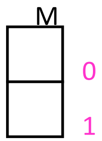
Four Steps to Dynamic Programming



Conclusion: a 1-dimensional memory of size n







1. Formulate the answer with a recursive structure
   - What are the options for the last choice?
   - For each such option, what does the subproblem look like? How do we use it?
2. Choose a memory structure.
   - Figure out the possible values of all parameters in the recursive calls.
   - How many subproblems (options for last choice) are there?
   - What are the parameters needed to identify each?
   - How many different values could there be per parameter?
3. Specify an order of evaluation.
   - Want to guarantee that the necessary subproblem solutions are in memory when you need them.
   - With this step: a "Bottom-up" (iterative) algorithm
   - Without this step: a "Top-down" (recursive) algorithm
4. See if there's a way to save space
   - Is it possible to reuse some memory locations?

Top-Down DP Idea

```
def myDPalgo(problem):
    if mem[problem] not blank:    // Check if we've solved this already
        return mem[problem]
    if baseCase(problem):    // Check if this is a base case
        solution = solve(problem)
        mem[problem] = solution    // Always save your solution before returning
        return solution
    for subproblem of problem:
        subsolutions.append(myDPalgo(subproblem)) // solve each subproblem
    solution = selectAndExtend(subsolutions) // Pick the subproblem to use
    mem[problem] = solution    // Always save your solution before returning
    return solution
```

Bottom-Up DP Idea

```
def myDPalgo(problem):
    for each baseCase:    // Identify which subproblems are base cases
        solution = solve(baseCase)
        mem[baseCase] = solution   // Save the solution for reuse
    for each subproblem in bottom-up order:
        // The order should be chosen so that every subsolution is
        // guaranteed to already be in memory when it's needed
        solution = selectAndExtend(subsolutions)
        mem[subproblem] = solution // Save the solution for reuse
    return mem[problem]
```

Weighted Interval Scheduling

Input: Like interval scheduling each request $\boldsymbol{i}$ has start and finish times $\boldsymbol{s_i}$ and $\boldsymbol{f_i}$.
Each request $\boldsymbol{i}$ also has an associated value or weight $\boldsymbol{v_i}$

$\boldsymbol{v_i}$ might be

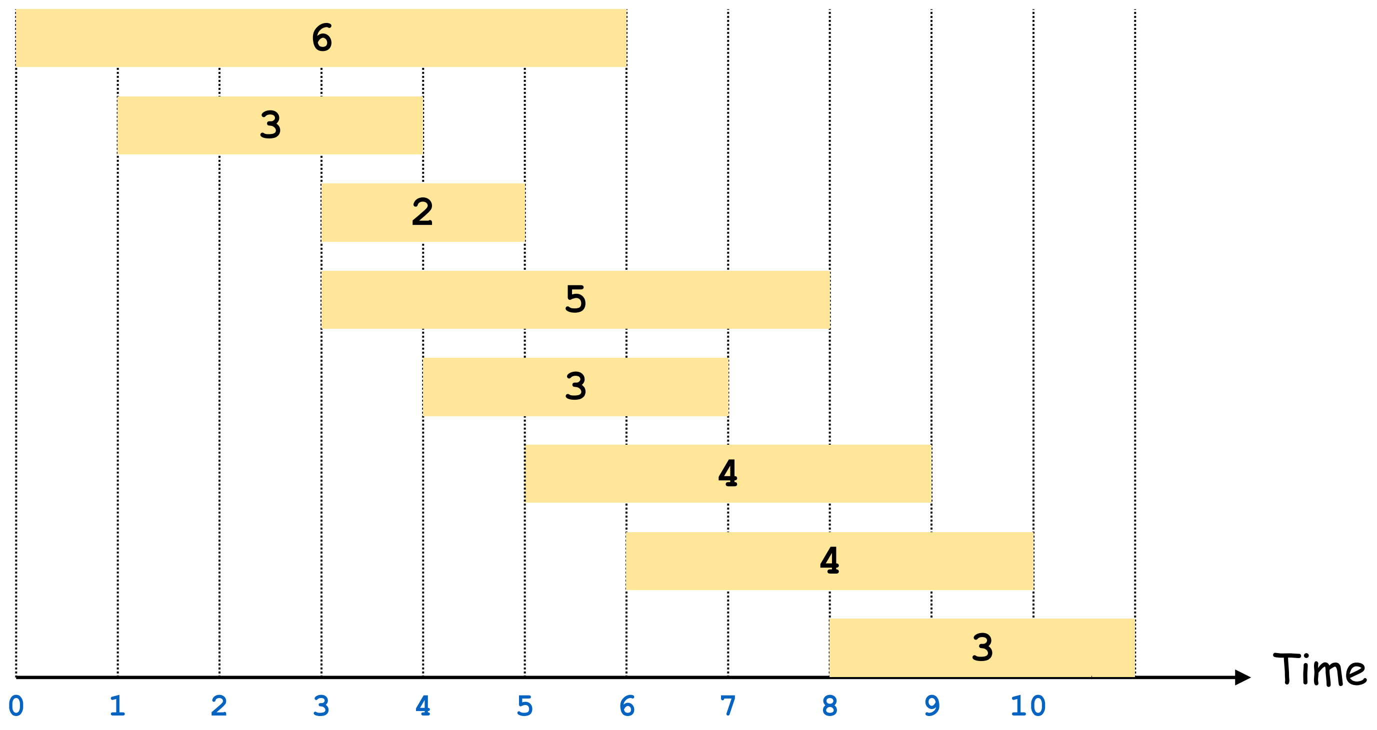
- the amount of money we get from renting out the resource
- the amount of time the resource is being used ($\boldsymbol{v_i = f_i - s_i}$)

Find: A maximum-weight compatible subset of requests.

Weighted Interval Scheduling

Input: Set of jobs with start times, finish times, and weights

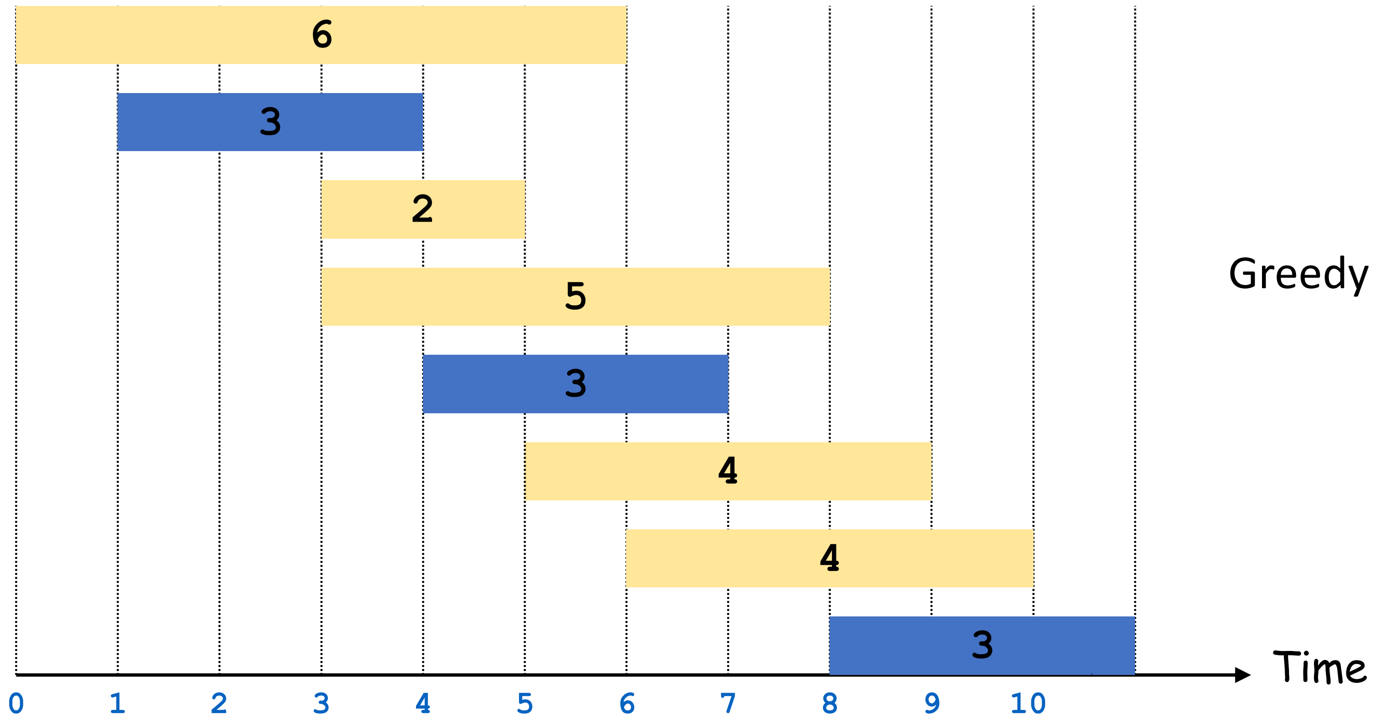
Goal: Find maximum weight subset of mutually compatible jobs.



Weighted Interval Scheduling

Input: Set of jobs with start times, finish times, and weights

Goal: Find maximum weight subset of mutually compatible jobs.

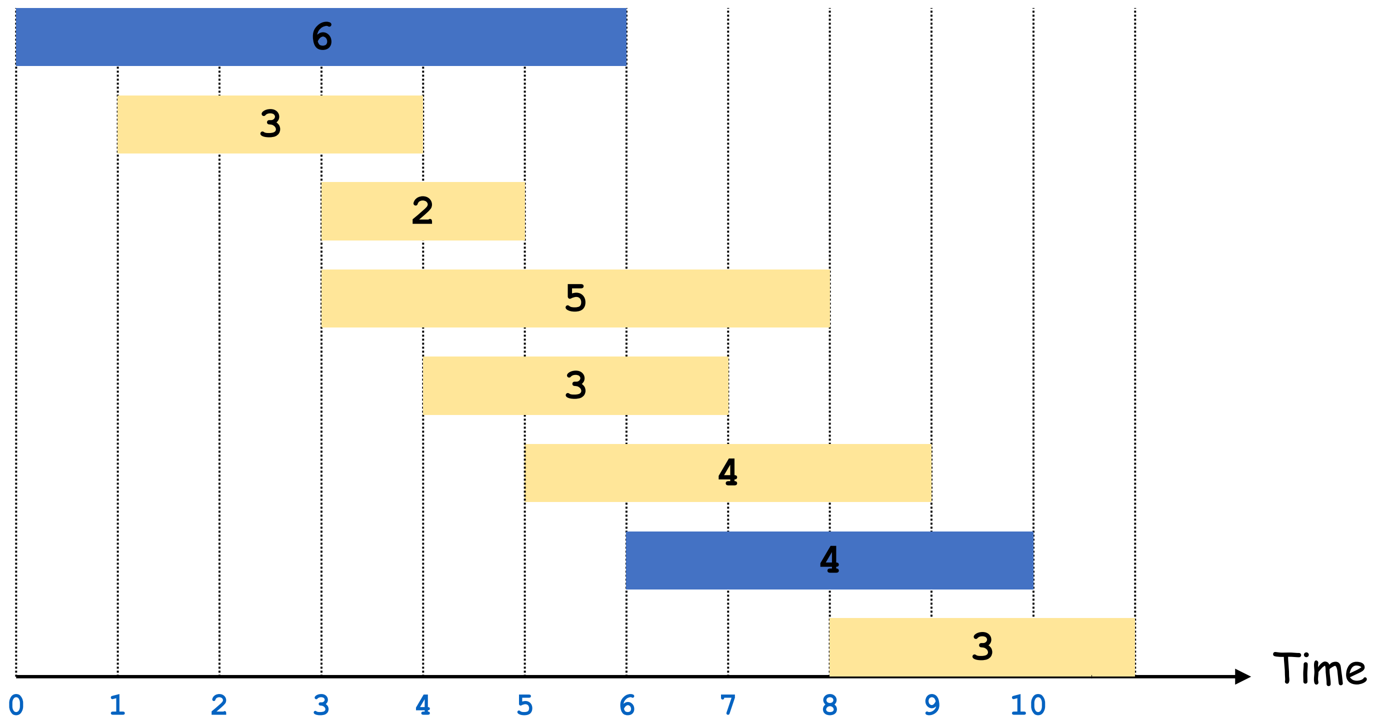


Greedy by finish times would give 9

Weighted Interval Scheduling

Input: Set of jobs with start times, finish times, and weights

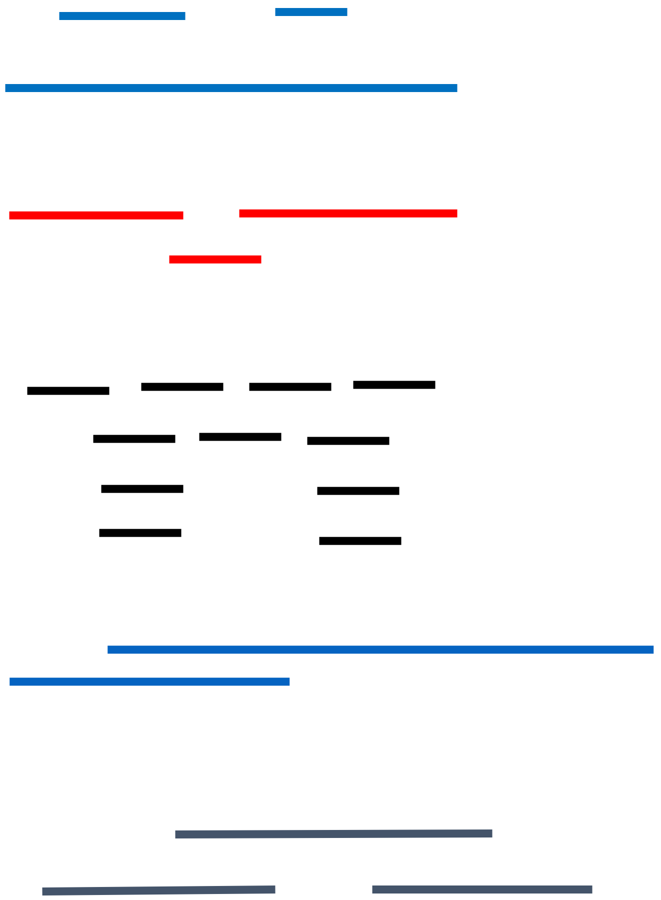
Goal: Find maximum weight subset of mutually compatible jobs.



Optimal yields 10

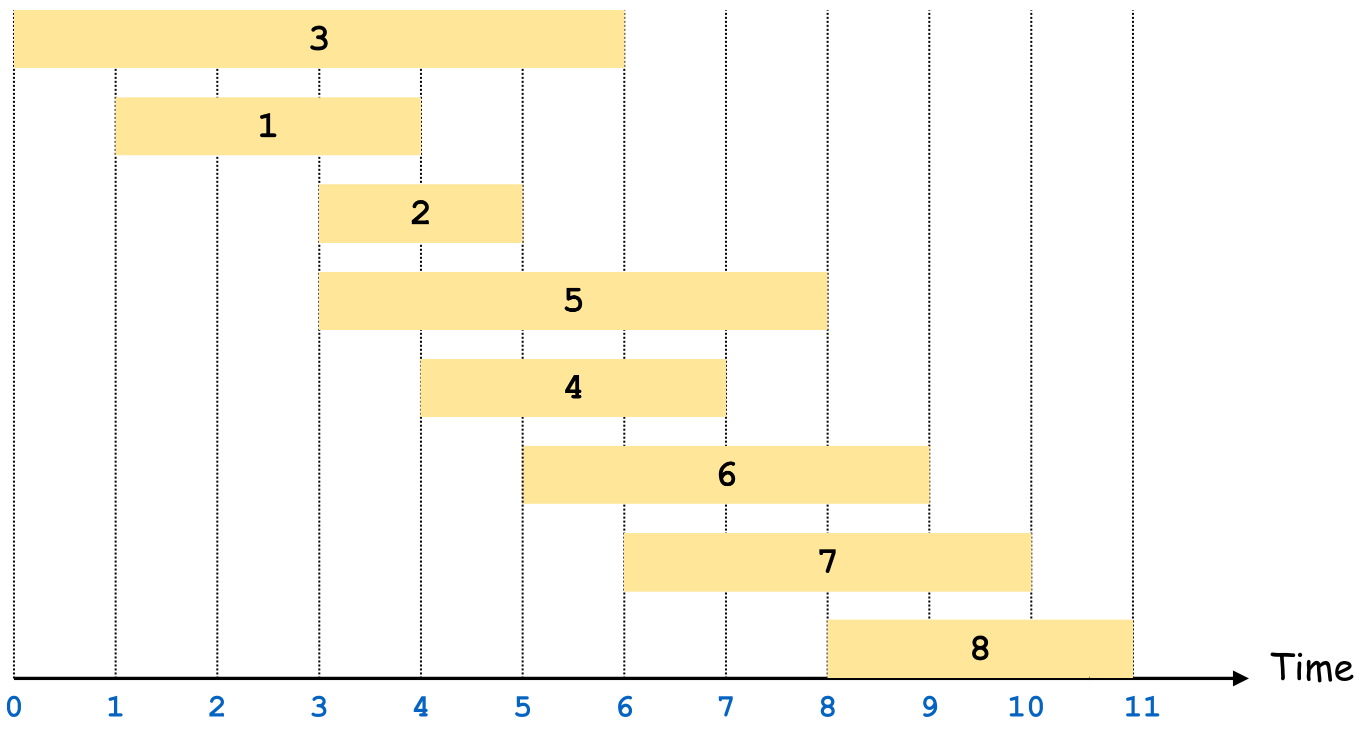
Greedy Algorithms for Weighted Interval Scheduling?

- What criterion should we try?
  - Earliest start time $\boldsymbol{s_i}$
    - Doesn't work
  - Shortest request time $\boldsymbol{f_i - s_i}$
    - Doesn't work
  - Fewest conflicts
    - Doesn't work
  - Earliest finish time $\boldsymbol{f_i}$
    - Doesn't work
  - Largest value/weight $\boldsymbol{v_i}$
    - Doesn't work



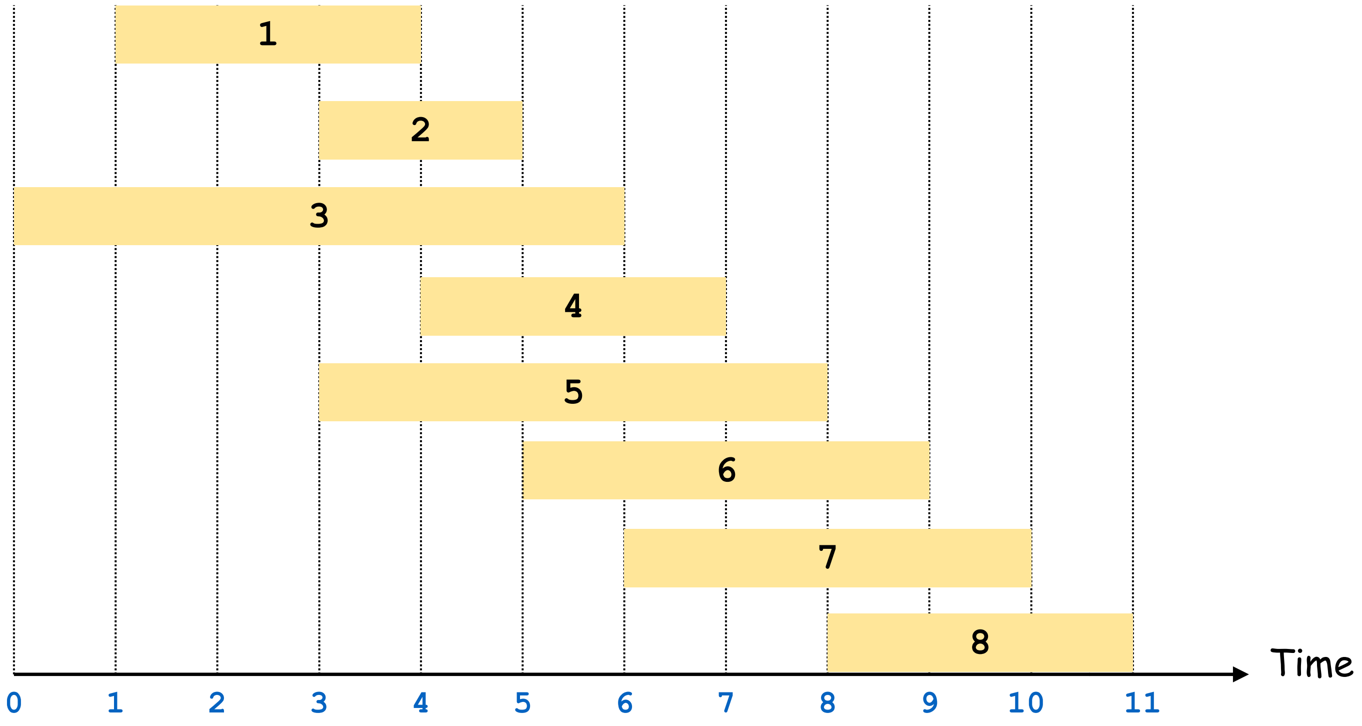
Weighted Interval Scheduling

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.



Weighted Interval Scheduling

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.



Weighted Interval Scheduling – Four Steps

1. Formulate the answer with a recursive structure
   - What are the options for the last choice?
   - For each such option, what does the subproblem look like? How do we use it?
2. Choose a memory structure.
   - Figure out the possible values of all parameters in the recursive calls.
   - How many subproblems (options for last choice) are there?
   - What are the parameters needed to identify each?
   - How many different values could there be per parameter?
3. Specify an order of evaluation.
   - Want to guarantee that the necessary subproblem solutions are in memory when you need them.
   - With this step: a “Bottom-up” (iterative) algorithm
   - Without this step: a “Top-down” (recursive) algorithm
4. See if there’s a way to save space
   - Is it possible to reuse some memory locations?

Towards Dynamic Programming: Step 1 – Recursive Algorithm

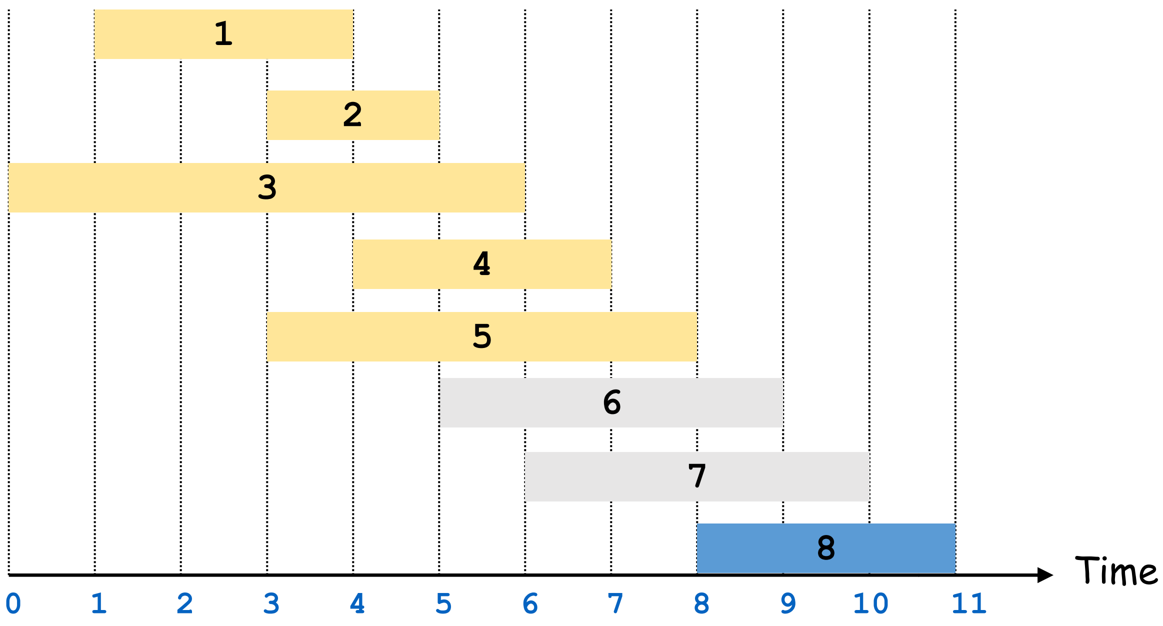
Suppose that we first sort the requests by finish time $\boldsymbol{f_i}$ so $\boldsymbol{f_1 \leq f_2 \leq \ldots \leq f_n}$.

We now want

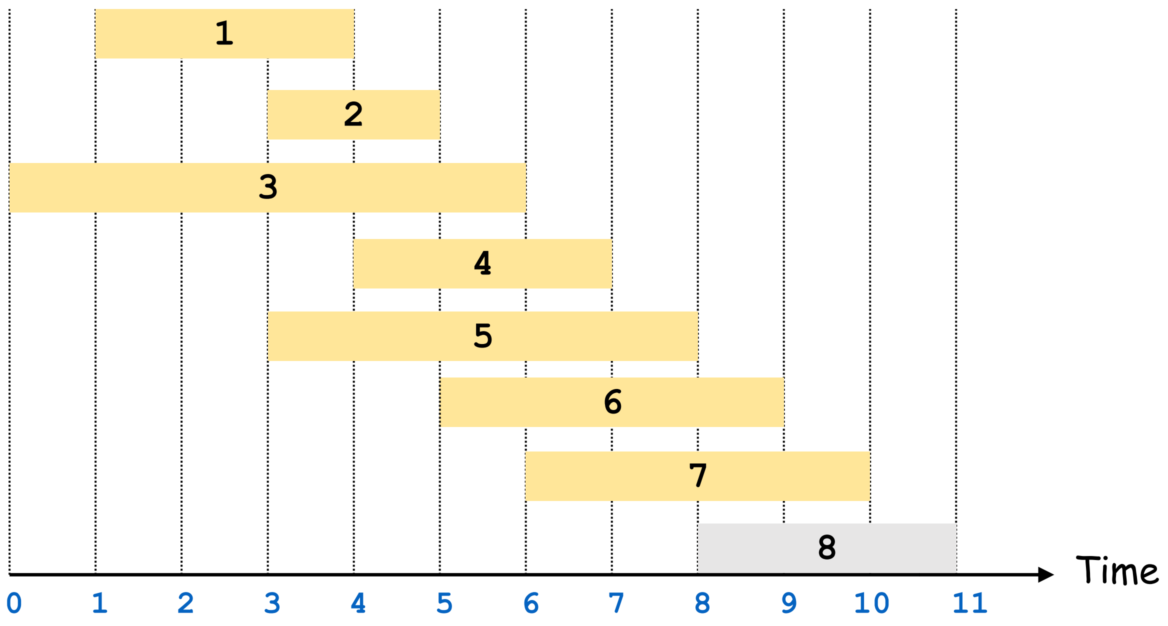
- a recursive solution that makes calls to smaller problems and
- the indices for those smaller problems to be convenient,

so we first focus on the options for the *last* request, request $\boldsymbol{n}$.

Option 1: Include the last request

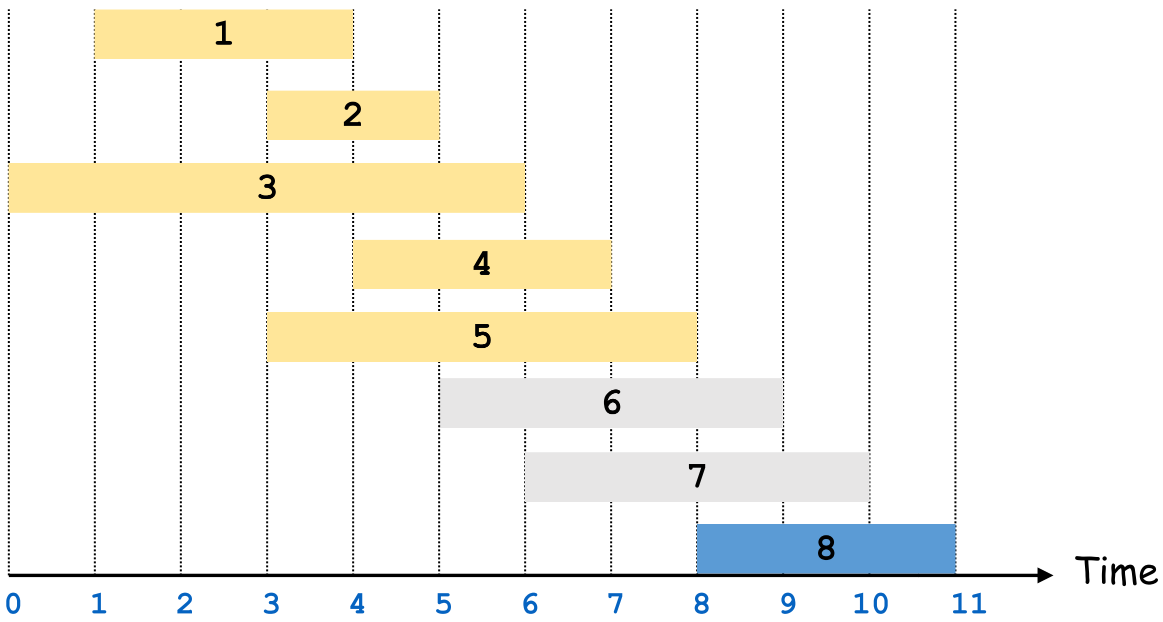


Option 2: Exclude the last request

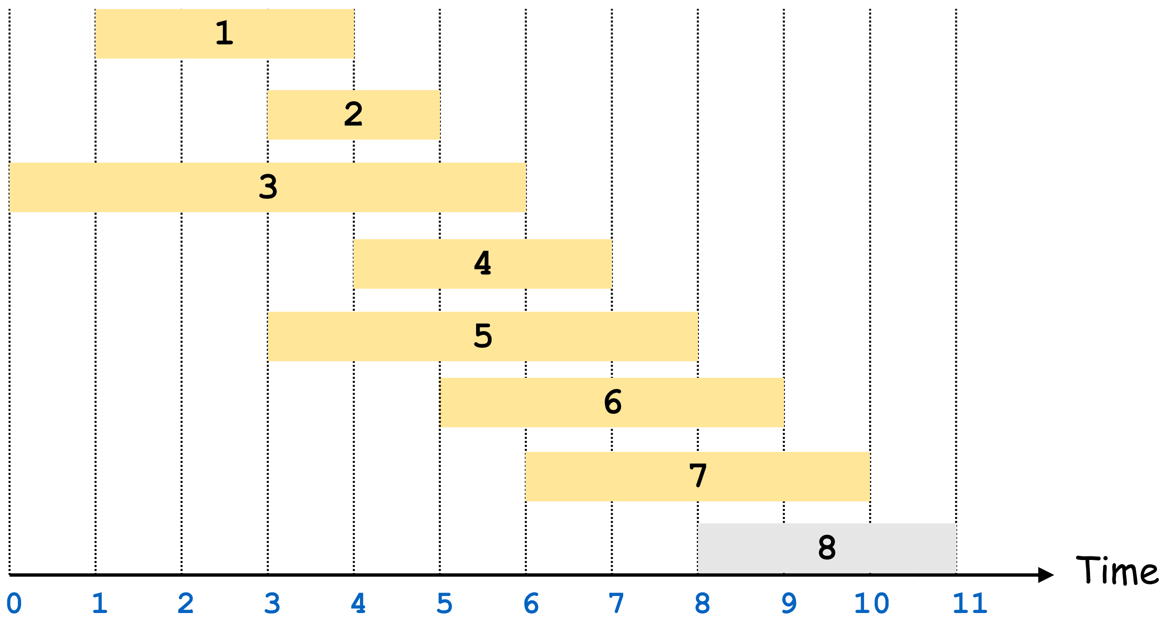


Towards Dynamic Programming: Step 1 – Recursive Algorithm

Option 1: Include the last request



Option 2: Exclude the last request



After making this choice, the best solution possible is given by:

- The value of the solution to subproblem consisting of everything compatible
- Plus the value of the last request

After making this choice, the best solution possible is given by:

- The value of the solution to subproblem consisting of everything except the last request

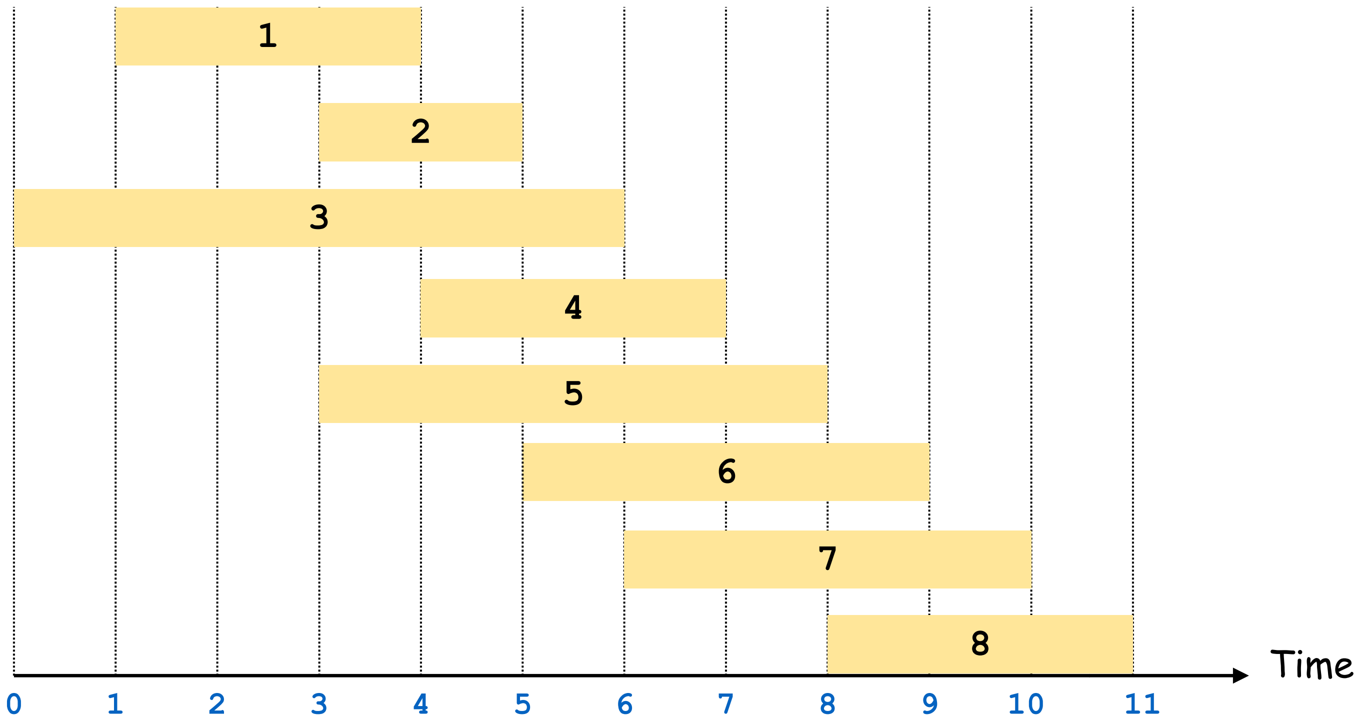
It will be convenient to be able to prune incompatible requests quickly…

Weighted Interval Scheduling

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

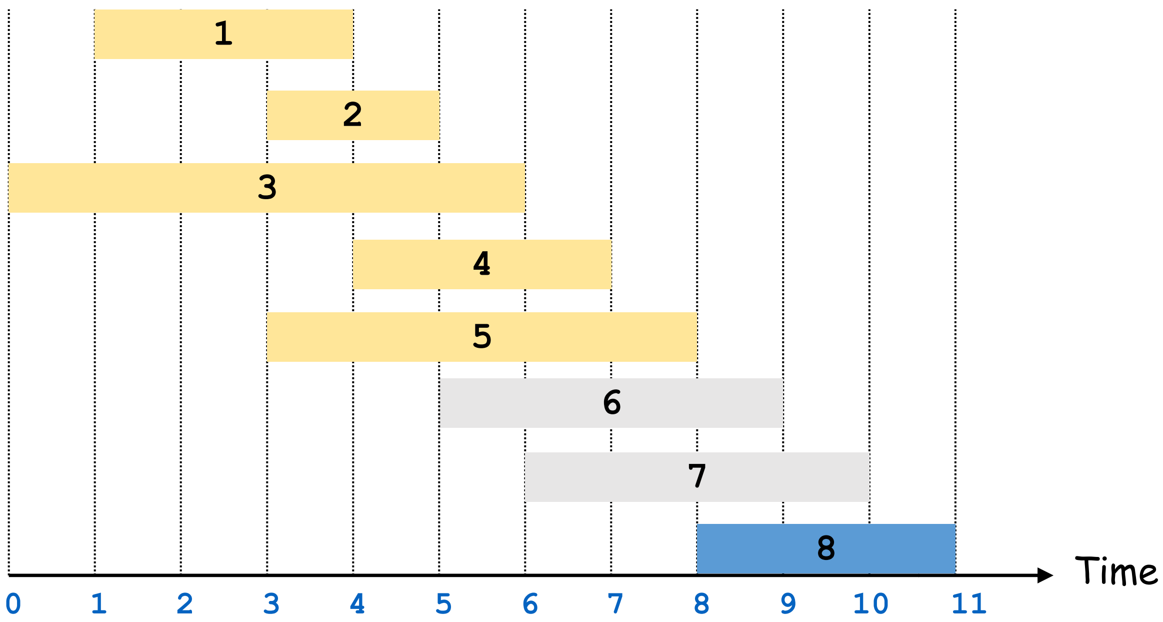
Example: $p(8) = 5, p(7) = 3, p(2) = 0$



j	$p(j)$
1	0
2	0
3	0
4	1
5	0
6	2
7	3
8	5

Towards Dynamic Programming: Step 1 – Recursive Algorithm

Option 1: Include the last request

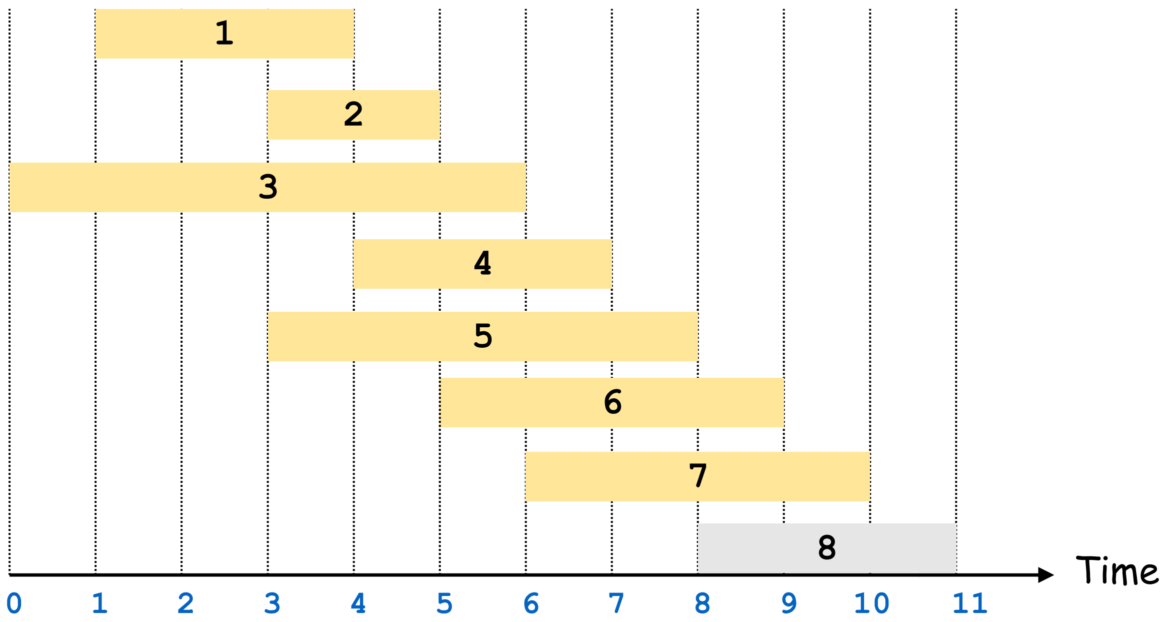


After making this choice, the best solution possible is given by:

- The value of the solution to subproblem consisting of everything compatible
- Plus the value of the last request

$$OPT\big(p(j)\big) + v_j$$

Option 2: Exclude the last request



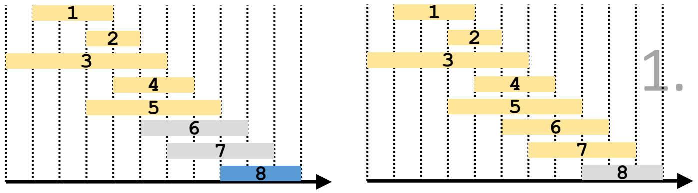
After making this choice, the best solution possible is given by:

- The value of the solution to subproblem consisting of everything except the last request

$$OPT(j-1)$$

$$OPT(j) = \max\big\{OPT\big(p(j)\big) + v_j, OPT(j-1)\big\}$$

Weighted Interval Scheduling – Four Steps



$\max\{OPT(p(j)) + v_j, \mathrm{OPT}(j-1)\}$

1. Formulate the answer with a recursive structure
   - What are the options for the last choice?
   - For each such option, what does the subproblem look like? How do we use it?
2. Choose a memory structure.
   - Figure out the possible values of all parameters in the recursive calls.
   - How many subproblems (options for last choice) are there?
   - What are the parameters needed to identify each?
   - How many different values could there be per parameter?
3. Specify an order of evaluation.
   - Want to guarantee that the necessary subproblem solutions are in memory when you need them.
   - With this step: a "Bottom-up" (iterative) algorithm
   - Without this step: a "Top-down" (recursive) algorithm
4. See if there's a way to save space
   - Is it possible to reuse some memory locations?

Towards Dynamic Programming: Step 2 – Memory Structure

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$

Subproblems are identified by a single parameter

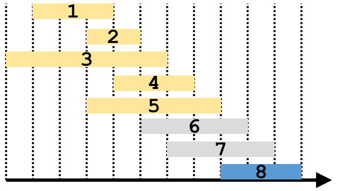
1-dimensional array

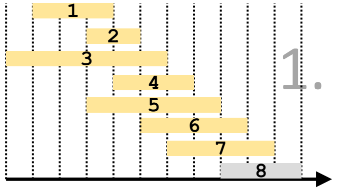
That parameter is the last-ending compatible request

length is the number of requests

j	**OPT**[j]
0	0
1	
2	
3	
4	
5	
6	
7	
8	

Weighted Interval Scheduling – Four Steps





$$\max\{OPT(p(j)) + v_j, \mathrm{OPT}(j-1)\}$$

j	OPT[j]
0	0
1	
2	
3	
4	
5	
6	
7	
8	

1. Formulate the answer with a recursive structure
   - What are the options for the last choice?
   - For each such option, what does the subproblem look like? How do we use it?
2. Choose a memory structure.
   - Figure out the possible values of all parameters in the recursive calls.
   - How many subproblems (options for last choice) are there?
   - What are the parameters needed to identify each?
   - How many different values could there be per parameter?
3. Specify an order of evaluation.
   - Want to guarantee that the necessary subproblem solutions are in memory when you need them.
   - With this step: a “Bottom-up” (iterative) algorithm
   - Without this step: a “Top-down” (recursive) algorithm
4. See if there’s a way to save space
   - Is it possible to reuse some memory locations?

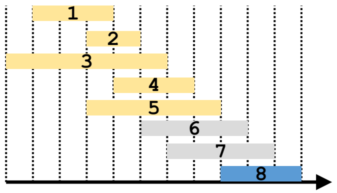
Top-Down DP Idea

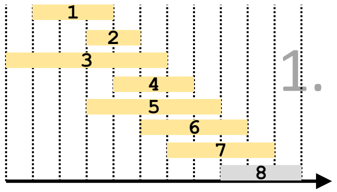
```
def myDPalgo(problem):
    if mem[problem] not blank:    // Check if we've solved this already
        return mem[problem]
    if baseCase(problem):    // Check if this is a base case
        solution = solve(problem)
        mem[problem] = solution    // Always save your solution before returning
        return solution
    for subproblem of problem:
        subsolutions.append(myDPalgo(subproblem)) // solve each subproblem
    solution = selectAndExtend(subsolutions) // Pick the subproblem to use
    mem[problem] = solution    // Always save your solution before returning
    return solution
```

Weighted Interval Scheduling Top-Down DP

```
WIS(j):
    if OPT[j] not blank:    // Check if we’ve solved this already
        return OPT[j]
    if j==0:    // Check if this is a base case
        mem[j] = 0    // Always save your solution before returning
        return mem[j]
    includej = WIS(p(j)) // Solve each subproblem
    excludej = WIS(j -1) // Solve each subproblem
    solution = max(includej+value[j], excludej) // Pick the subproblem to use
    mem[j] = solution // Always save your solution before returning
    return solution
```

Weighted Interval Scheduling – Four Steps





$$\max\{OPT(p(j)) + v_j, \mathrm{OPT}(j-1)\}$$

j	OPT[j]
0	0
1	
2	
3	
4	
5	
6	
7	
8	

1. Formulate the answer with a recursive structure
   - What are the options for the last choice?
   - For each such option, what does the subproblem look like? How do we use it?
2. Choose a memory structure.
   - Figure out the possible values of all parameters in the recursive calls.
   - How many subproblems (options for last choice) are there?
   - What are the parameters needed to identify each?
   - How many different values could there be per parameter?
3. Specify an order of evaluation.
   - Want to guarantee that the necessary subproblem solutions are in memory when you need them.
   - With this step: a “Bottom-up” (iterative) algorithm
   - Without this step: a “Top-down” (recursive) algorithm
4. See if there’s a way to save space
   - Is it possible to reuse some memory locations?

Towards Dynamic Programming: Step 3 – Order of Evaluation

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$

For any given cell j, which other cells might I need?

- $j - 1$
- $p(j)$

It's hard to know in advance what $p(j)$ might be, but certainly $p(j) < j$

Order: increasing order of j will work

j	**OPT[*j*]**
0	0
1	
2	
3	
4	
5	
6	
7	
8	

Bottom-Up DP Idea

```
def myDPalgo(problem):
    for each baseCase:    // Identify which subproblems are base cases
        solution = solve(baseCase)
        mem[baseCase] = solution   // Save the solution for reuse
    for each subproblem in bottom-up order:
        // The order should be chosen so that every subsolution is
        // guaranteed to already be in memory when it's needed
        solution = selectAndExtend(subsolutions)
        mem[subproblem] = solution // Save the solution for reuse
    return mem[problem]
```

Weighted Interval Scheduling Bottom-Up DP

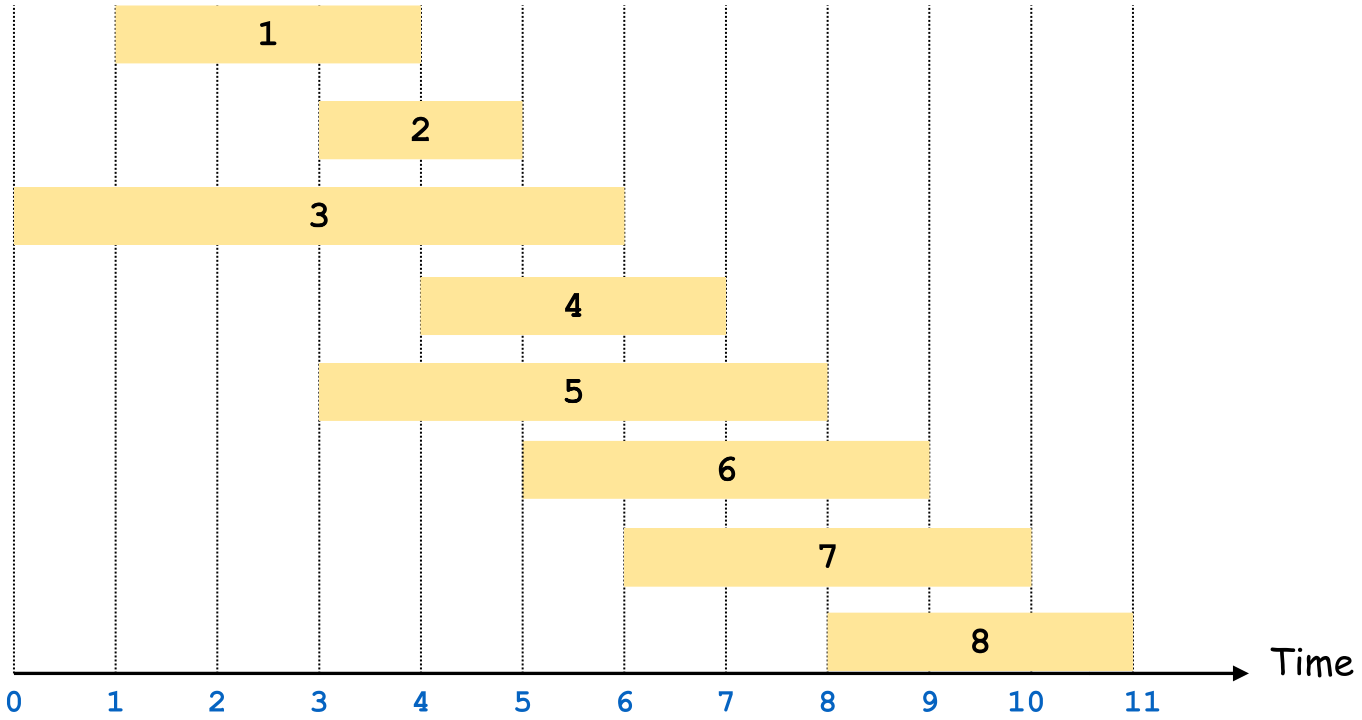
```
WIS(j):
    OPT[0] = 0   // Save the solution for the base case
    for each i = 1 up to j:
        // The order should be chosen so that every subsolution is
        // guaranteed to already be in memory when it's needed
        solution = max(OPT[p(i)]+value[i], OPT[i − 1])
        mem[i] = solution // Save the solution for reuse
    return OPT[j]
```

Example Execution (iterative)

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$



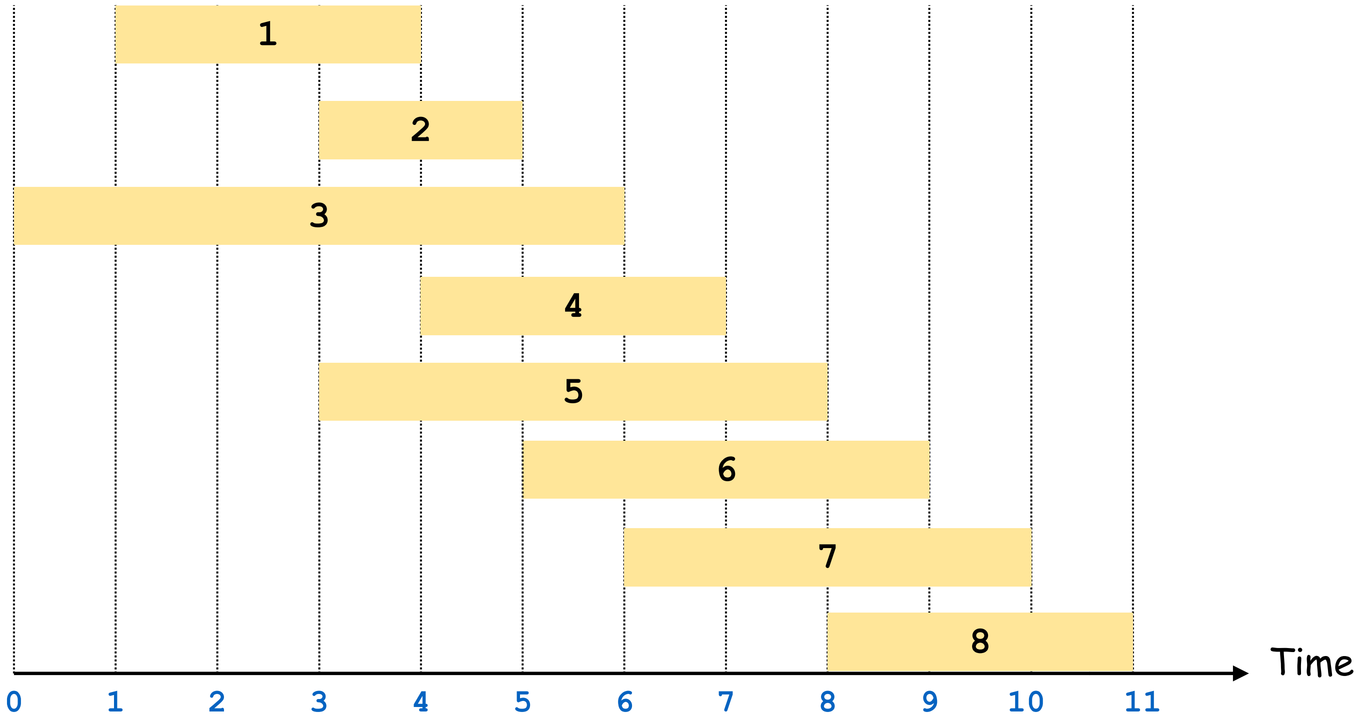
j	v_j	$p(j)$	OPT$[j]$
0	-	-	0
1	3	0	
2	2	0	
3	6	0	
4	3	1	
5	5	0	
6	4	2	
7	4	3	
8	3	5	

Example Execution (iterative)

Notation: Label jobs by finishing time: $f_1 \le f_2 \le \cdots \le f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$



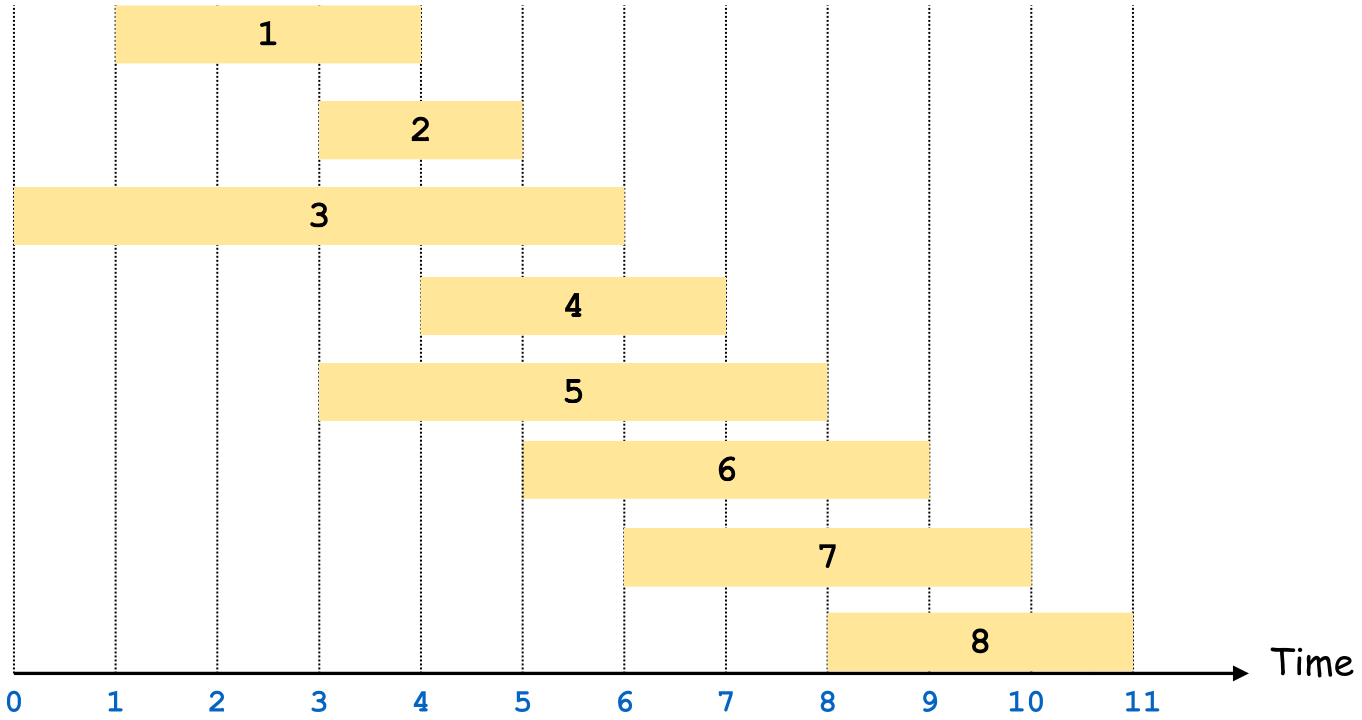
j	v_j	$p(j)$	OPT$[j]$
0	-	-	0
1	3	0	3
2	2	0	
3	6	0	
4	3	1	
5	5	0	
6	4	2	
7	4	3	
8	3	5	

Example Execution (iterative)

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$



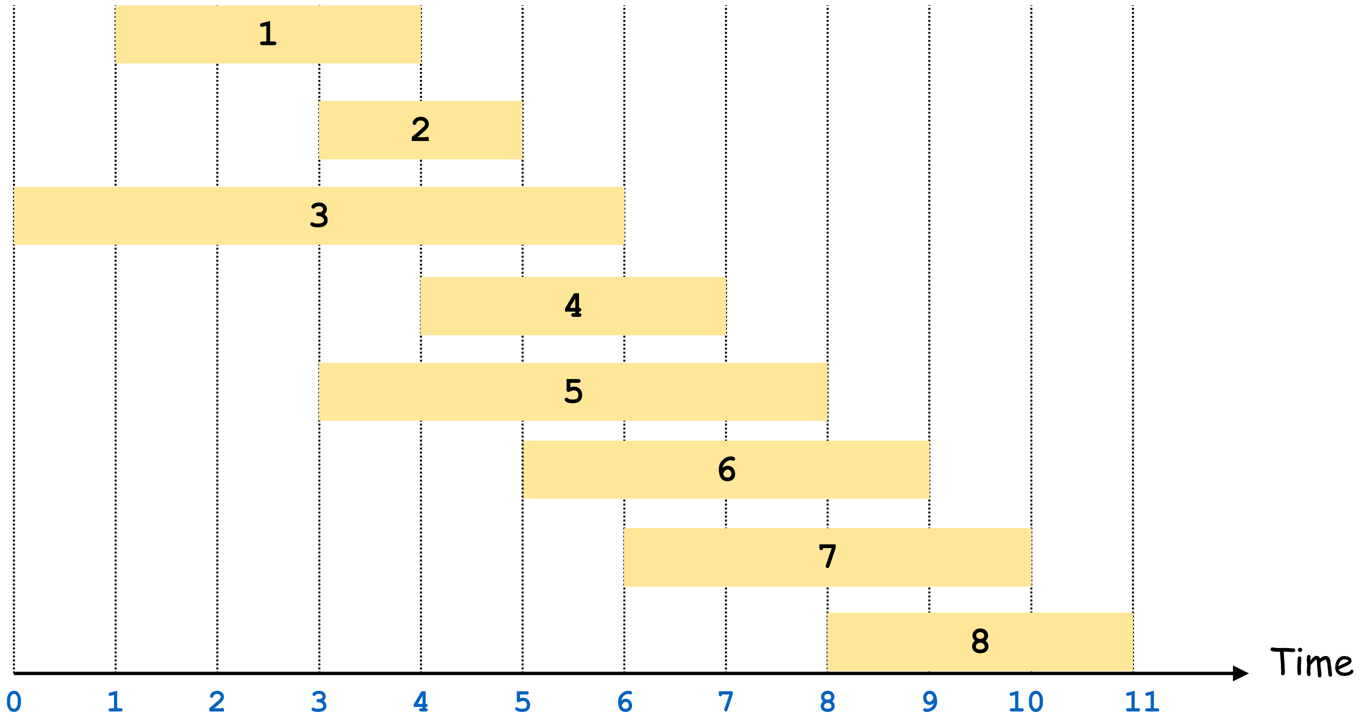
j	v_j	$p(j)$	OPT[j]
0	-	-	0
1	3	0	3
2	2	0	
3	6	0	
4	3	1	
5	5	0	
6	4	2	
7	4	3	
8	3	5	

Example Execution (iterative)

Notation: Label jobs by finishing time: $f_1 \le f_2 \le \cdots \le f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$



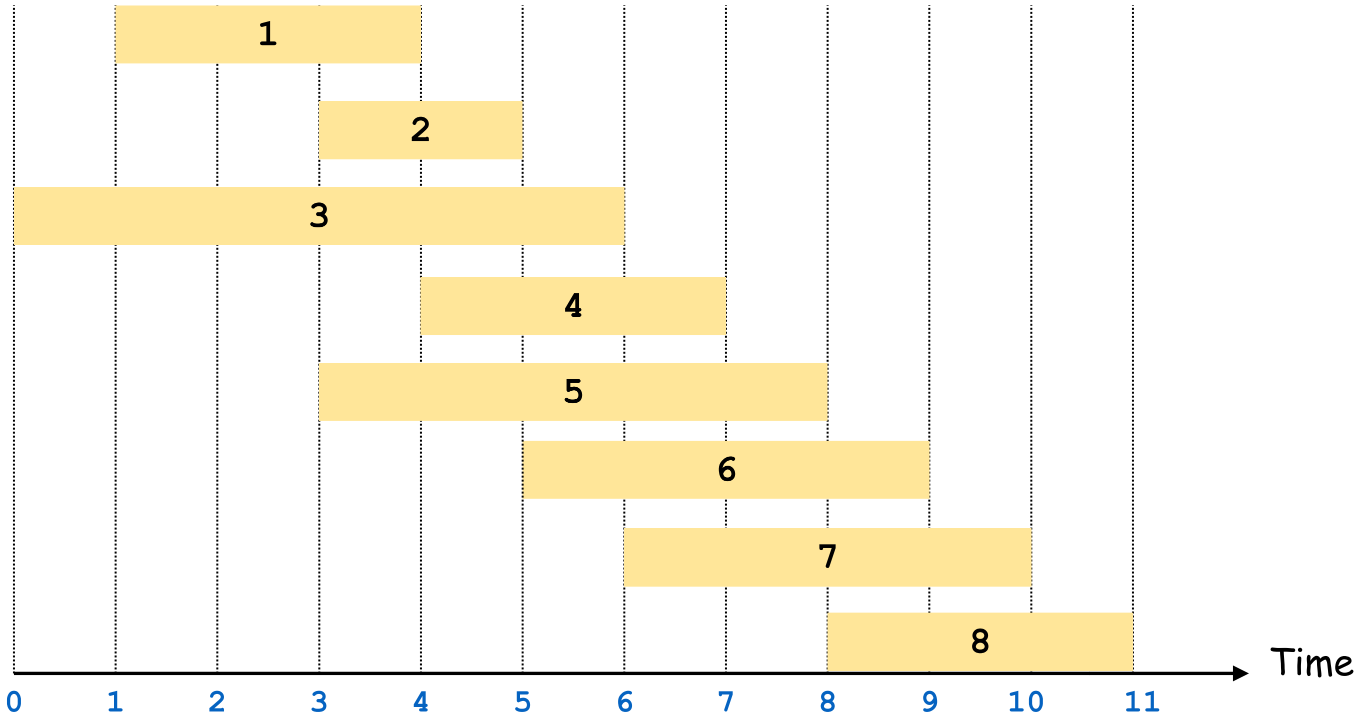
j	v_j	$p(j)$	OPT[j]
0	-	-	0
1	3	0	3
2	2	0	3
3	6	0	
4	3	1	
5	5	0	
6	4	2	
7	4	3	
8	3	5	

Example Execution (iterative)

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$



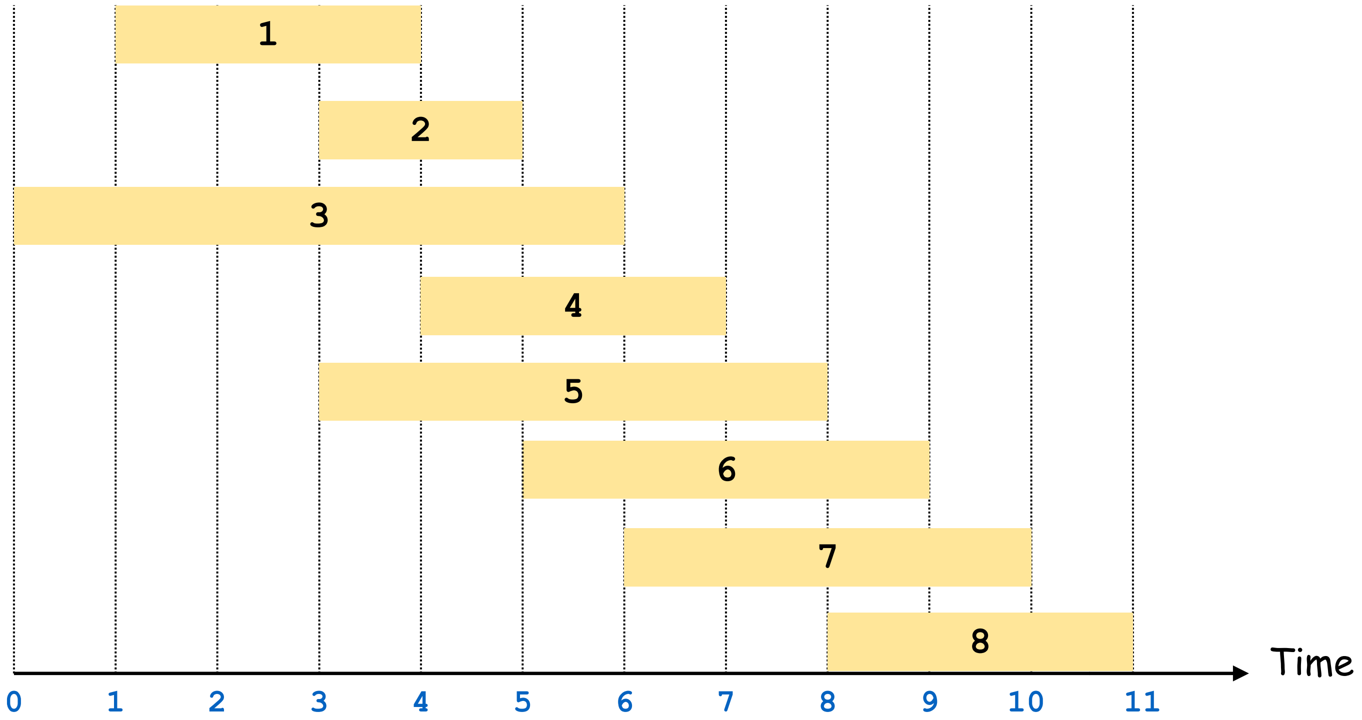
j	v_j	$p(j)$	OPT$[j]$
0	-	-	0
1	3	0	3
2	2	0	3
3	6	0	
4	3	1	
5	5	0	
6	4	2	
7	4	3	
8	3	5	

Example Execution (iterative)

Notation: Label jobs by finishing time: $f_1 \le f_2 \le \cdots \le f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$



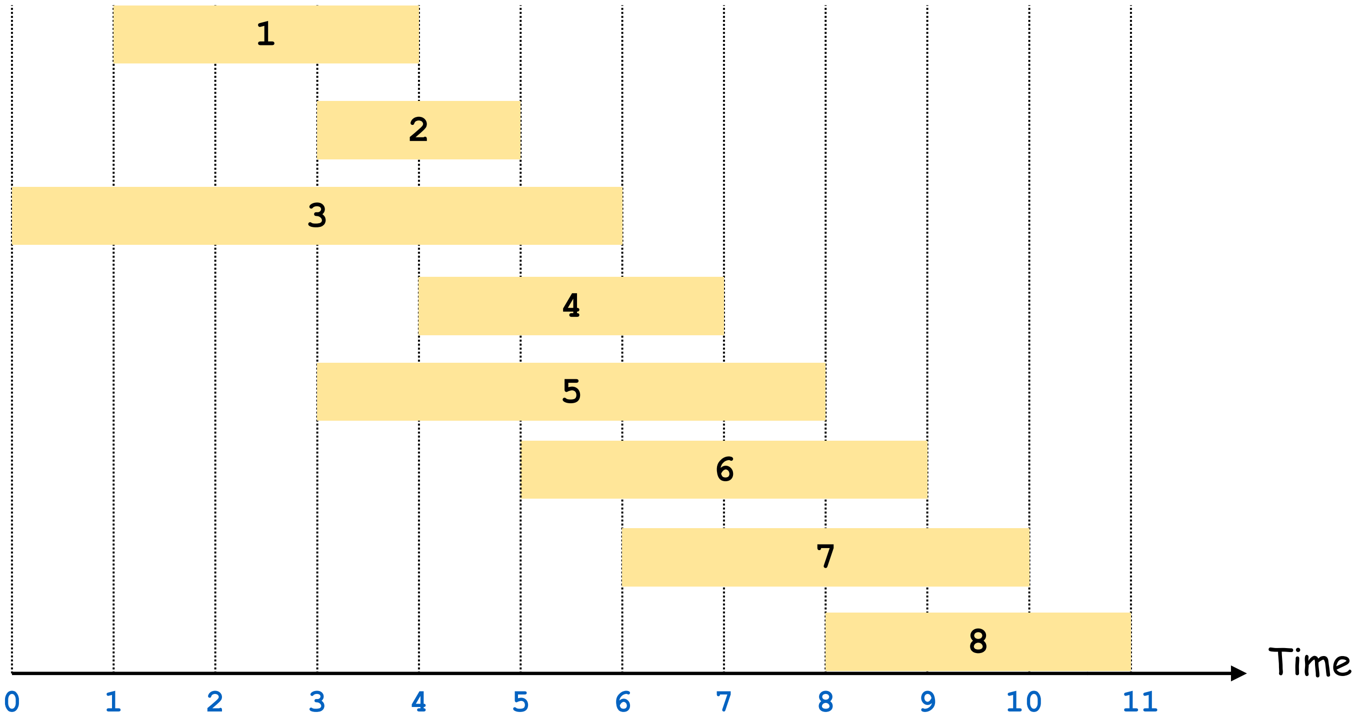
j	v_j	$p(j)$	OPT[j]
0	-	-	0
1	3	0	3
2	2	0	3
3	6	0	6
4	3	1	
5	5	0	
6	4	2	
7	4	3	
8	3	5	

Example Execution (iterative)

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$



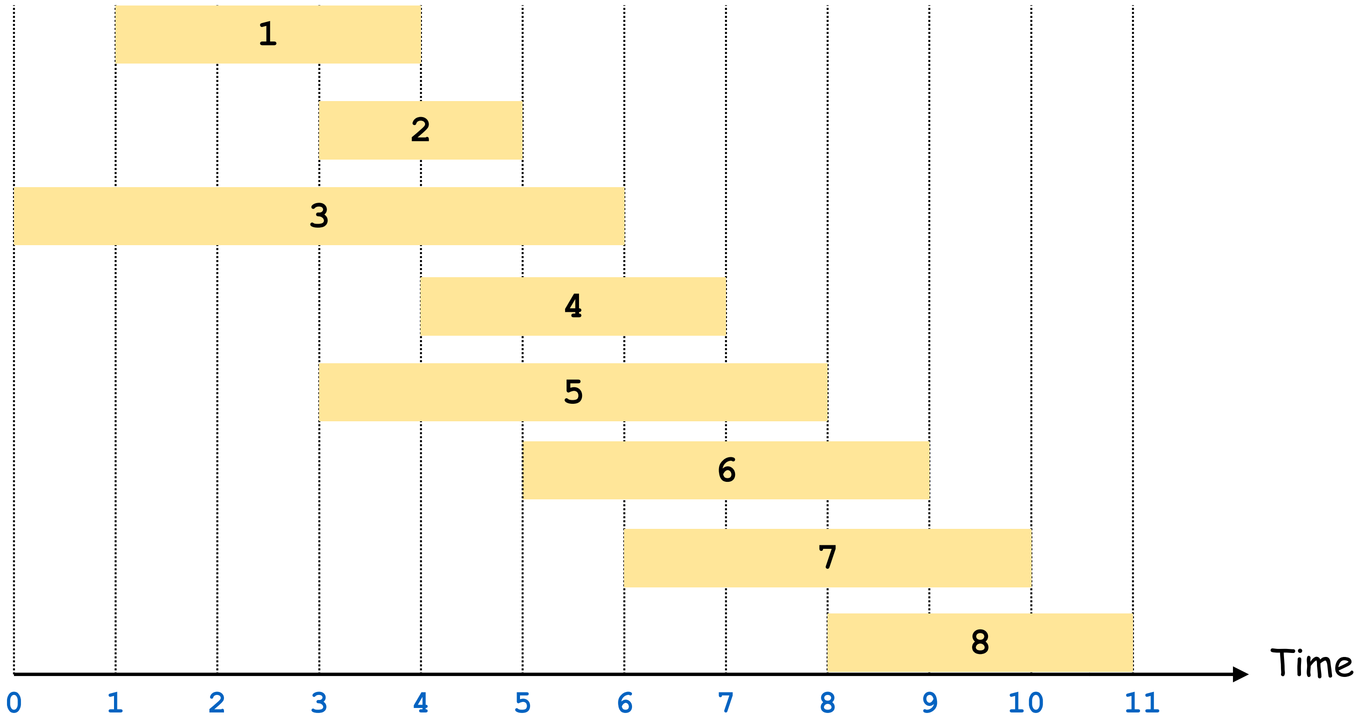
j	v_j	$p(j)$	OPT[j]
0	-	-	0
1	3	0	3
2	2	0	3
3	6	0	6
4	3	1	
5	5	0	
6	4	2	
7	4	3	
8	3	5	

Example Execution (iterative)

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$



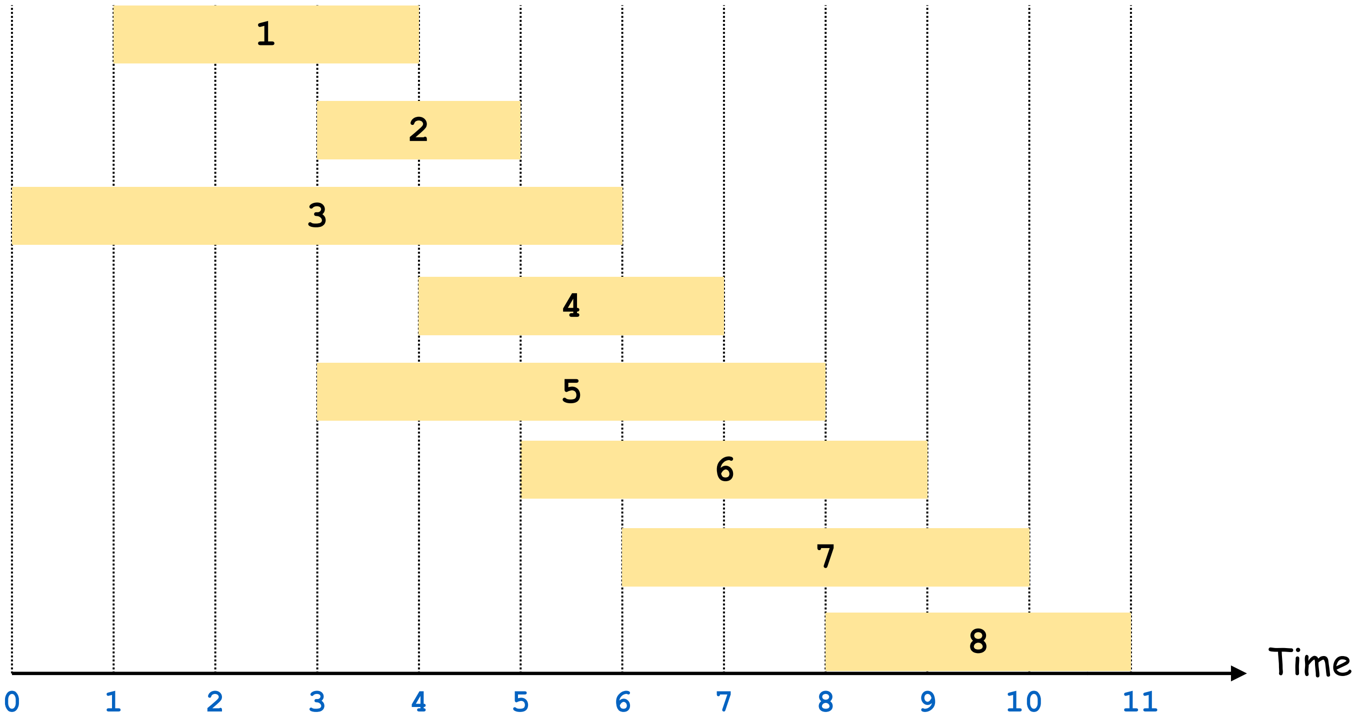
j	v_j	$p(j)$	OPT[j]
0	-	-	0
1	3	0	3
2	2	0	3
3	6	0	6
4	3	1	6
5	5	0	
6	4	2	
7	4	3	
8	3	5	

Example Execution (iterative)

Notation: Label jobs by finishing time: $f_1 \le f_2 \le \cdots \le f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$



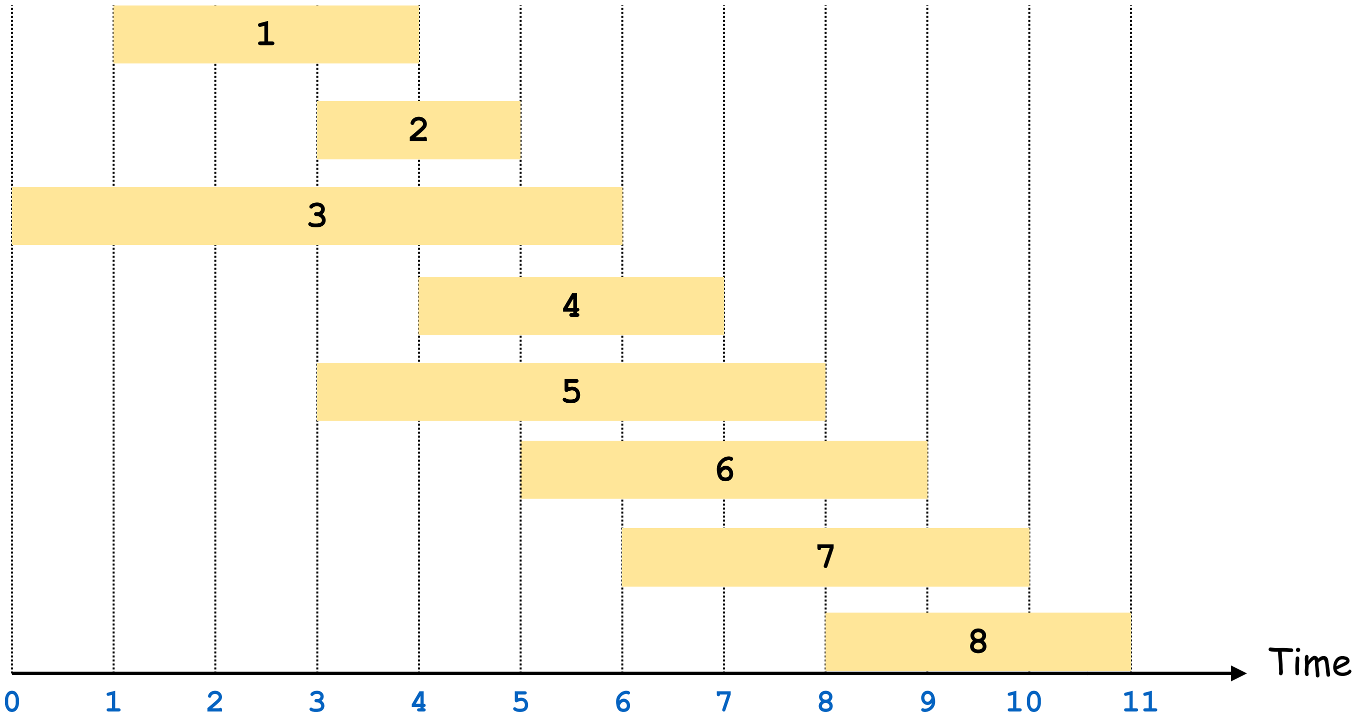
j	v_j	$p(j)$	OPT[j]
0	-	-	0
1	3	0	3
2	2	0	3
3	6	0	6
4	3	1	6
5	5	0	5
6	4	2	
7	4	3	
8	3	5	

Example Execution (iterative)

Notation: Label jobs by finishing time: $\boldsymbol{f_1 \leq f_2 \leq \cdots \leq f_n}$.

Defn: $\boldsymbol{p(j)}$ = largest index $\boldsymbol{i < j}$ s.t. job $\boldsymbol{i}$ is compatible with $\boldsymbol{j}$.

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$



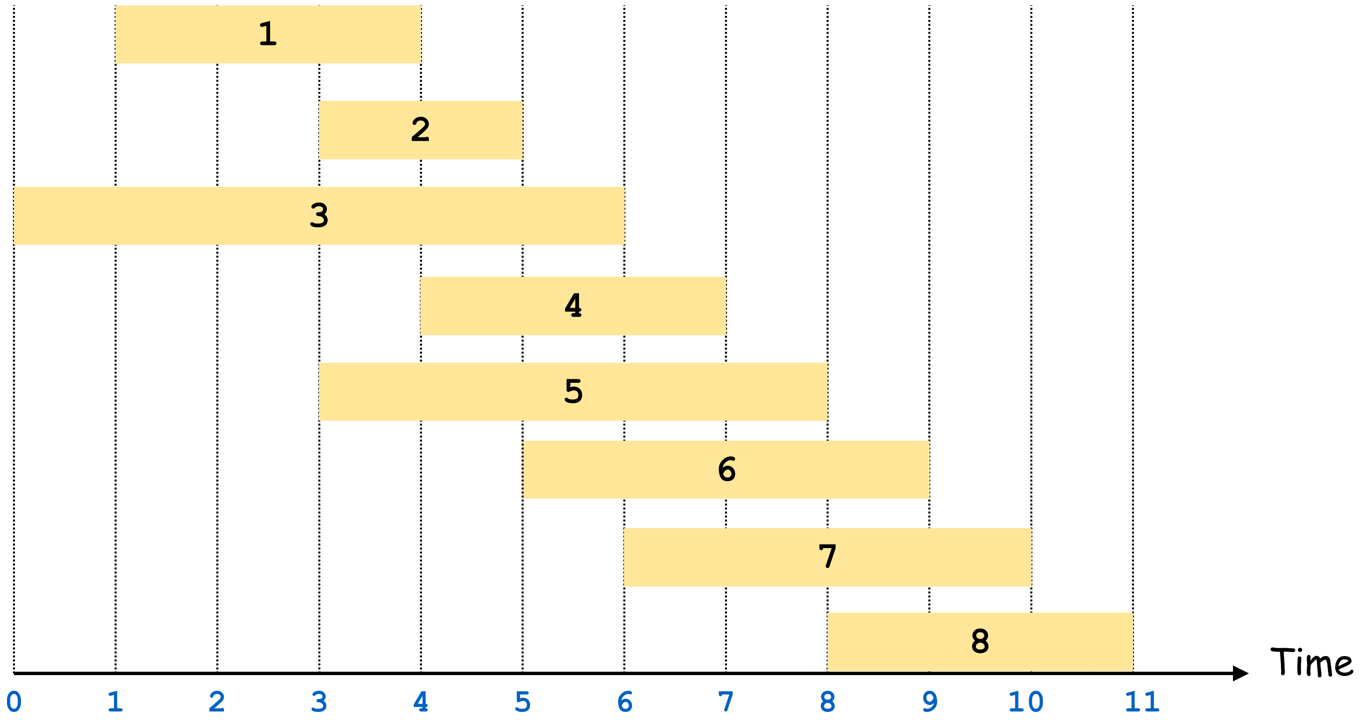
j	v_j	$p(j)$	OPT[j]
0	-	-	0
1	3	0	3
2	2	0	3
3	6	0	6
4	3	1	6
5	5	0	6
6	4	2	7
7	4	3	
8	3	5	

Example Execution (iterative)

Notation: Label jobs by finishing time: $\boldsymbol{f_1 \leq f_2 \leq \cdots \leq f_n}$.

Defn: $\boldsymbol{p(j)}$ = largest index $\boldsymbol{i < j}$ s.t. job $\boldsymbol{i}$ is compatible with $\boldsymbol{j}$.

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$



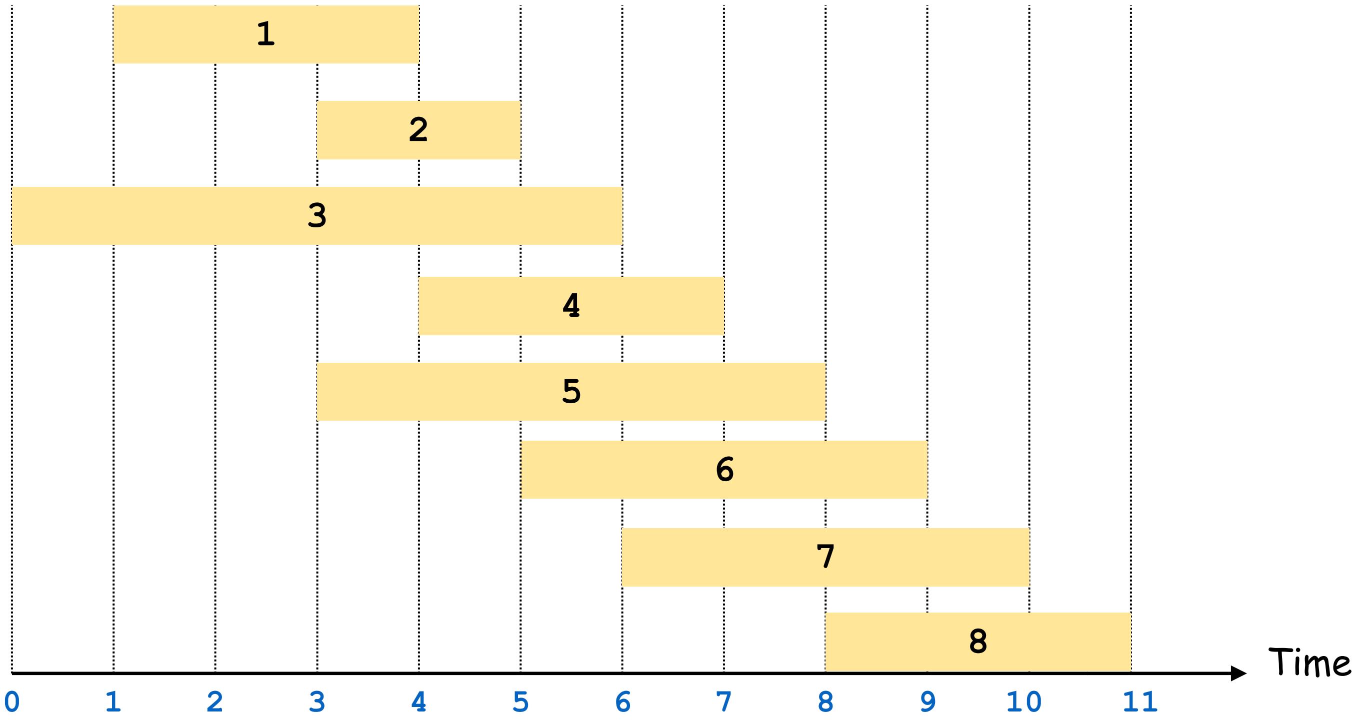
j	v_j	$p(j)$	OPT[j]
0	-	-	0
1	3	0	3
2	2	0	3
3	6	0	6
4	3	1	6
5	5	0	6
6	4	2	7
7	4	3	
8	3	5	

Example Execution (iterative)

Notation: Label jobs by finishing time: $f_1 \le f_2 \le \cdots \le f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$



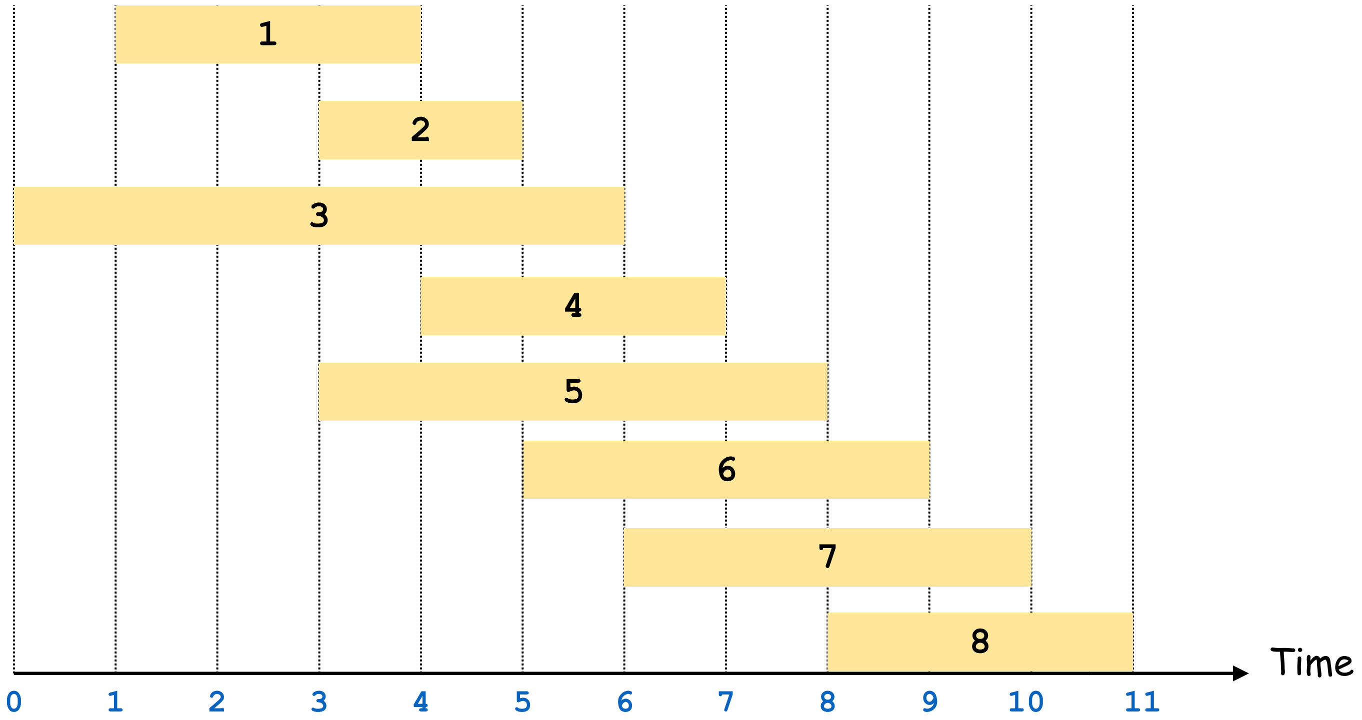
j	v_j	$p(j)$	OPT$[j]$
0	-	-	0
1	3	0	3
2	2	0	3
3	6	0	6
4	3	1	6
5	5	0	6
6	4	2	7
7	4	3	10
8	3	5	

Example Execution (iterative)

Notation: Label jobs by finishing time: $f_1 \le f_2 \le \cdots \le f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$



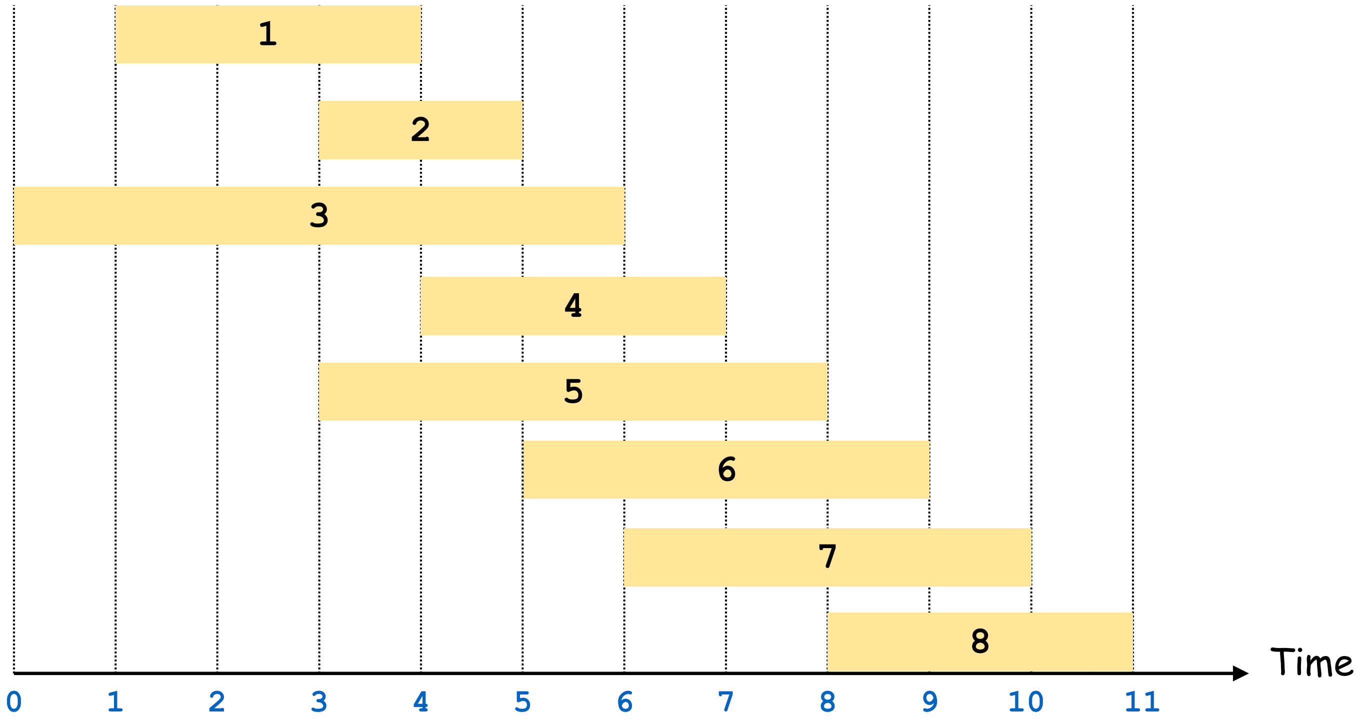
j	v_j	$p(j)$	OPT$[j]$
0	-	-	0
1	3	0	3
2	2	0	3
3	6	0	6
4	3	1	6
5	5	0	6
6	4	2	7
7	4	3	10
8	3	5	

Example Execution (iterative)

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$



j	v_j	$p(j)$	OPT$[j]$
0	-	-	0
1	3	0	3
2	2	0	3
3	6	0	6
4	3	1	6
5	5	0	6
6	4	2	7
7	4	3	10
8	3	5	10

Weighted Interval Scheduling: Finding the Solution

So far we have computed the value **OPT**$(\boldsymbol{n})$ but we probably want to know what that solution **OPT** actually is!

We can do this, too, by keeping track of which option was better at each step.

Define **Used**$[\boldsymbol{j}]$ = $\begin{cases} \mathbf{1} & \text{solution with value } \mathbf{OPT}(\boldsymbol{j}) \text{ includes request } \boldsymbol{j} \\ \mathbf{0} & \text{otherwise} \end{cases}$

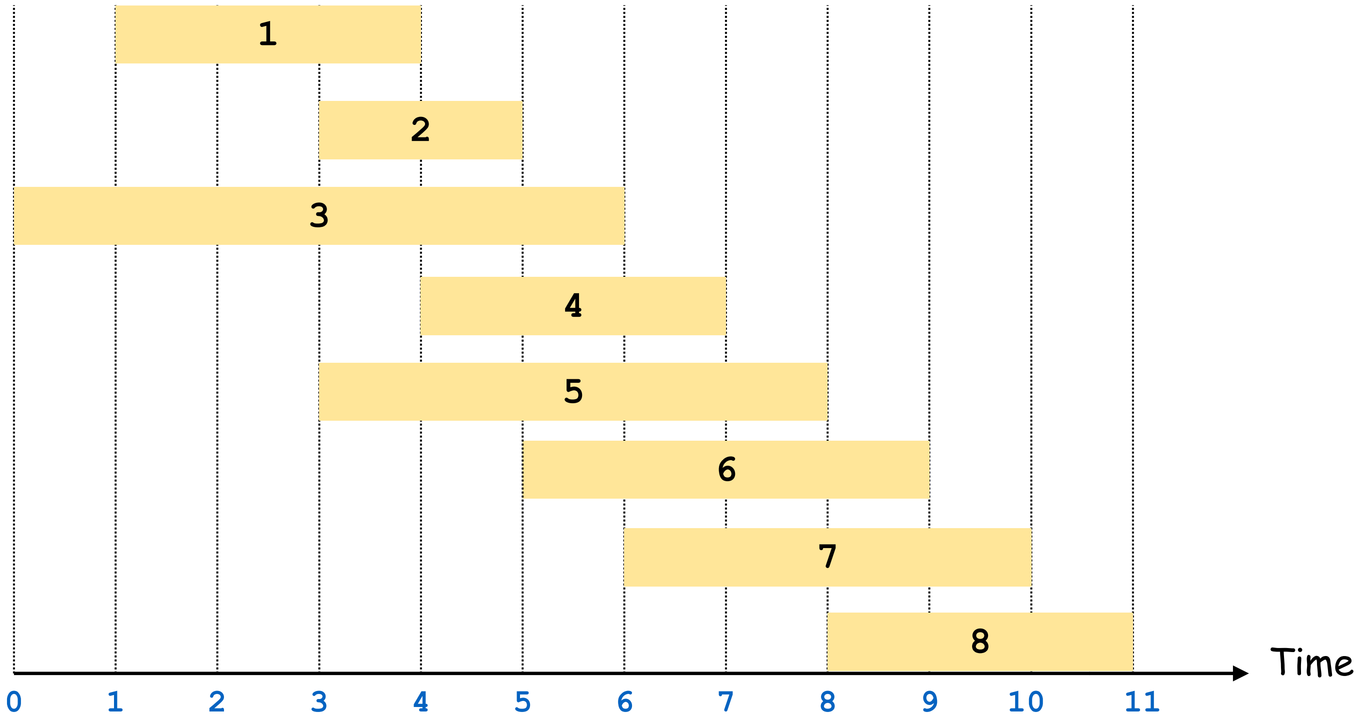
This gives a “pointer” that leads the way along a path to the optimal solution…

Weighted Interval Scheduling: Finding the Solution

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$



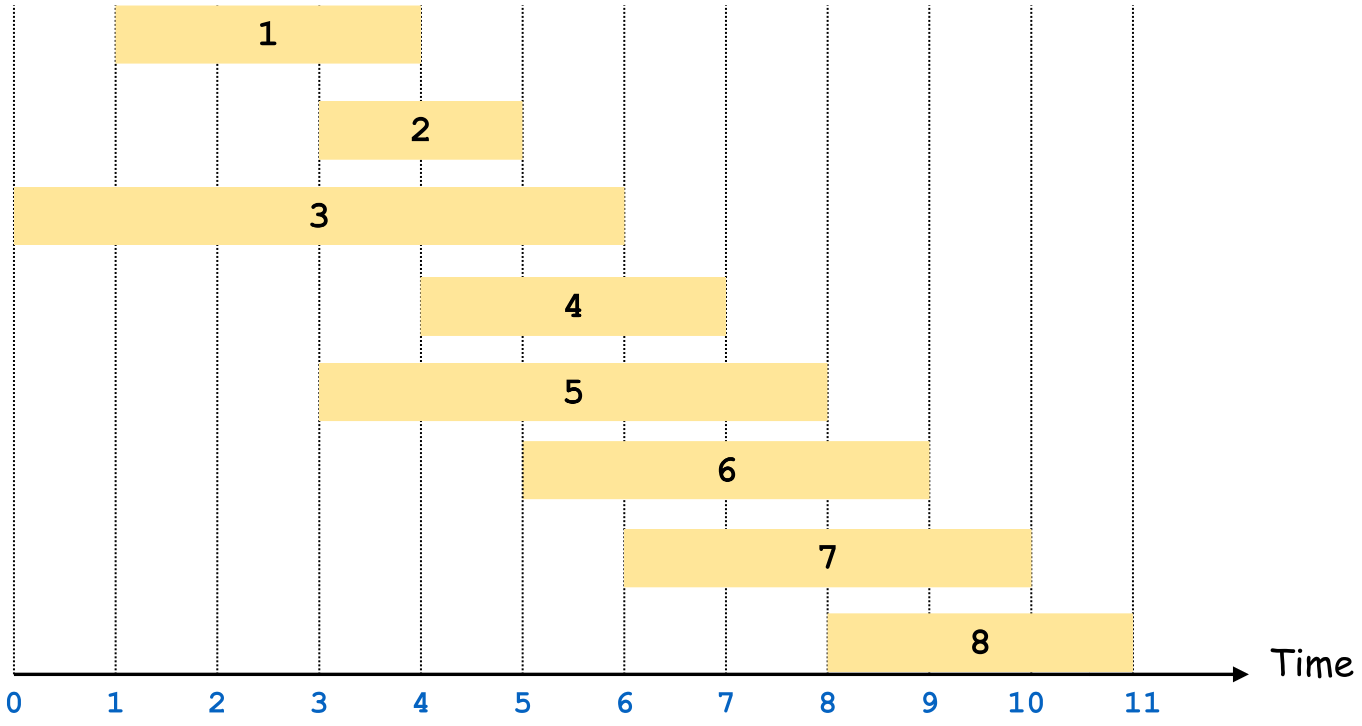
j	v_j	$p(j)$	OPT[j]	Used[j]
0	-	-	0	-
1	3	0	3	1
2	2	0	3	0
3	6	0	6	1
4	3	1	6	1
5	5	0	6	0
6	4	2	7	1
7	4	3	10	1
8	3	5	10	0

Weighted Interval Scheduling: Iterative Solution

Notation: Label jobs by finishing time: $f_1 \le f_2 \le \cdots \le f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

$$OPT(j) = \max\{OPT(p(j)) + v_j, OPT(j-1)\}$$



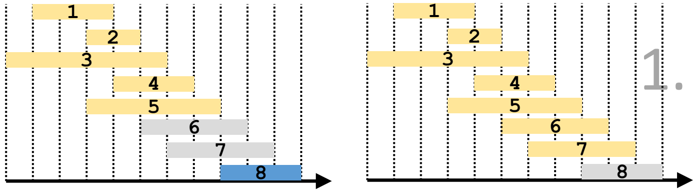
j	v_j	$p(j)$	OPT[j]	Used[j]
0	-	-	0	-
1	3	0	3	1
2	2	0	3	0
3	6	0	6	1
4	3	1	6	1
5	5	0	6	0
6	4	2	7	1
7	4	3	10	1
8	3	5	10	0

Weighted Interval Scheduling - Complete

Sort requests by finish time
Compute each p(i)
WIS(j):
  OPT[0] = 0
  for each $i = 1$ up to j:
    includei = OPT[$p(i)$]+value[i]
    excludei = OPT[$i - 1$]
    if includei > excludei:
      OPT[i] = includei
      used[i] = 1
    else:
      OPT[i] = excludei
      used[i] = 0
  return find_opt(used);

find_opt(used):
  j= n
  intervals = {}
  while $j > 0$:
    if used[j]==0:
      j = $j - 1$
    else:
      intervals.add(j)
      $j = p(j)$
  return intervals

Weighted Interval Scheduling – Four Steps



$$\max\{OPT(p(j)) + v_j, \mathrm{OPT}(j-1)\}$$

j	OPT[*j*]
0	0
1	
2	
3	
4	
5	
6	
7	
8	

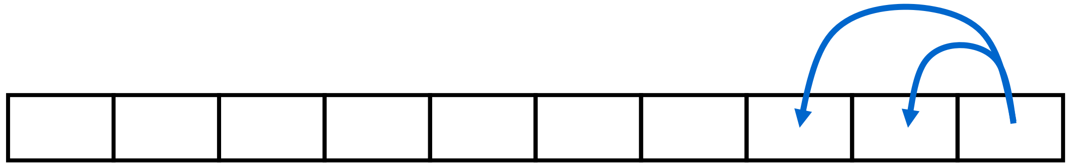
j	OPT[*j*]
0	0
1	
2	
3	
4	
5	
6	
7	
8	

1. Formulate the answer with a recursive structure
   - What are the options for the last choice?
   - For each such option, what does the subproblem look like? How do we use it?
2. Choose a memory structure.
   - Figure out the possible values of all parameters in the recursive calls.
   - How many subproblems (options for last choice) are there?
   - What are the parameters needed to identify each?
   - How many different values could there be per parameter?
3. Specify an order of evaluation.
   - Want to guarantee that the necessary subproblem solutions are in memory when you need them.
   - With this step: a "Bottom-up" (iterative) algorithm
   - Without this step: a "Top-down" (recursive) algorithm
4. See if there's a way to save space
   - Is it possible to reuse some memory locations?

Dynamic Programming Patterns

Fibonacci pattern:

- 1-dimensional, $O(1)$ values immediately prior
- Space saving possible



Weighted interval scheduling pattern:

- 1-dimensional, $O(1)$ values arbitrarily far back
- No space saving possible